A NEW CHARACTERIZATION OF SIMPLE JANKO-GROUPS

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Abstract. It is proved that simple Janko-groups J_1, J_2, J_3 and J_4 can be determined by their order and one irreducible character degree.
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1. Introduction and preliminary results

Let G be a finite group and $cd(G)$ the set of all complex irreducible character degrees of G. Characters of a finite group can give some important information of the group's structure.

In [1], Huppert posed the following conjecture: if H is a finite non-abelian simple group such that $cd(G) = cd(H)$, then $G \cong H \times A$, where A is an abelian group. And it was verified that the conjecture holds for many non-ableian simple

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groups. Also it was proved that a finite simple group can be uniquely determined by its character table in [2]. In this paper, we manage to characterize the finite simple groups by less character degrees. In fact, we shall prove the four simple Janko-groups, J_1, J_2, J_3 and J_4 can be unique determined by the group order and one irreducible character degree. The following theorems will be proved:

Theorem A. *Simple Janko-groups J_1, J_3 and J_4 can be uniquely determined by their orders and the largest irreducible character degrees.*

Theorem B. *Janko-group J_2 is uniquely determined by its order and the second largest irreducible character degree.*

For convenience we denote the largest irreducible character degree of G as $L(G)$, and the second largest irreducible character degree of G as $S(G)$. We use $A\ .\ B$ to denote any group having a normal subgroup isomorphic to, for which the corresponding quotient group isomorphic to B, and use Z_n to denote the cyclic group of order n. All further unexplained symbols and notations are standard and can be found, for instance, in [3] and [4].

In order to prove the above theorems, we need the following lemmas:

Lemma 1. *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$.*

Proof. Let H be a normal subgroup of G, which is maximal such that G/H is non-solvable. Then if K/H is a chief factor, it is a direct product of isomorphic non-abelian simple groups, and also it is the unique minimal normal subgroup of G/H, and from this, the desired conclusion is immediate.

Lemma 2. *Suppose that a Sylow $p-$subgroup of a solvable group G is not normal. Then some prime power dividing $|G|$ and exceeding 1 is congruent to 1 mod p.*

Proof. Let P be a Sylow $p-$subgroup of G. If P is not normal, let N be its normalizer, and let $N \leq M$, where M is maximal in G. Then $|G : M|$ is a prime power and $|G : N| = |G : M| \cdot |M : N|$. By Sylow's theorem, $|G : N|$ and $|M : N|$ are congruent to 1 mod p, so $|G : M|$ is congruent to 1.

Lemma 3. *Let G be a finite solvable group and $|G| = 2^{\alpha} \cdot 3^{\beta} \cdot 5^2 \cdot 7^{\gamma}$, where $\alpha \leq 7$, $\beta \leq 3$, $\gamma \leq 1$. Then $O_5(G) \neq 1$.*

Proof. If $O_5(G) = 1$, then a Sylow $5-$subgroup P acts faithfully on the Fitting subgroup F of G for G is solvable. Consequently, P acts nontrivially on some Sylow p'-subgroup of F, which implies that there exists a power of 2, 3 or 7 dividing $|G|$ that is congruent to 1 mod 5. Since the only prime power q dividing $|G|$ that is congruent to 1 mod 5 is $q = 2^4$, it follows that P acts faithfully on the Sylow 2−subgroup of F, which means some 2−group of order dividing 2^7 having an automorphism group of order divisible by 25. But the order of $GL_7(2)$ cannot be divided by 25, this is a contradiction.

Lemma 4. *Let G be a non-solvable group. Suppose that G has a normal series $1 \trianglelefteq K \trianglelefteq G$ such that K is abelian and G/K is a non-abelian simple group. If $Aut(K)$ does not contain any simple section isomorphic to G/K. Then $K = Z(G)$.*

Proof. It follows straightforward from N/C theorem by observing.

Lemma 5. *Let $H \trianglelefteq G$ such that $G/H \cong A_5$ and let $\sigma \in Irr(H)$, $\varphi \in Irr(G)$ with $[\varphi_H, \sigma] \neq 0$. Suppose σ is invariant in G and $5 \mid \varphi(1)/\sigma(1)$. Then φ is the unique irreducible constituent of σ^G with degree divided by $5 \cdot \sigma(1)$. Moreover, $\varphi(1) = 5 \cdot \sigma(1)$.*

Proof. Write $\sigma^G = e_i\varphi_i$, where $\varphi_i \in \mathrm{Irr}(G)$ and $\varphi_1 = \varphi$, e_i are positive integers. Since σ is invariant in G, we have $(\varphi_i)_H = e_i\sigma$. Therefore, $\varphi_i(1) = e_i \cdot \sigma(1)$ and

$$|G : H|\sigma(1) = \sigma^G(1) = \sum e_i \cdot \varphi_i(1) = \sum e_i^2 \cdot \sigma(1),$$

so that $\sum e_i^2 = |G : H| = 60$. By the assumption, $5 \mid e_1$. Thus the fact $e_1^2 \leq \sum e_i^2 = 60$ forces that $e_1 = 5$. If some $e_i = 1$, then σ is extendible to φ_i. Hence $\varphi_i\lambda$ for $\lambda \in \mathrm{Irr}(G/H)$ are all of the irreducible constituents of σ^G. Since $H/A \cong A_5$, we have that φ is the unique irreducible constituent of σ^G with degree divided by $5 \cdot \sigma(1)$. If all of $e_i > 1$, it follows by $\sum e_i^2 = 60$ and $e_1 = 5$ that $e_i \neq 5$ for $i \geq 2$. Therefore φ is the unique irreducible constituent of σ^G with degree divided by $5 \cdot \sigma(1)$. Moreover, $\varphi(1) = e_1 \cdot \sigma(1) = 5 \cdot \sigma(1)$.

2. Proof of Theorems

Proof of Theorem A. We write the proof in several cases.

Case 1. Let G be a group having the same order of J_1 and having an irreducible character χ with $\chi(1) = L(J_1)$, we prove that $G \cong J_1$.

By [3], $|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and $\chi(1) = L(J_1) = 11 \cdot 19$. If $O_{19}(G) \neq 1$, then $O_{19}(G)$ is abelian and thus $\xi(1) \mid |G/O_{19}(G)|$ for every $\xi \in \mathrm{Irr}(G)$. But $\chi(1) = 11 \cdot 19 \nmid |G/O_{19}(G)|$, a contradiction. Therefore $O_{19}(G) = 1$. Hence G is non-solvable by Lemma 2. Therefore by Lemma 1 G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$. As $|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, we have K/H is isomorphic to one of A_5, $L_2(7)$, $L_2(11)$ and J_1.

Suppose $K/H \cong A_5$. Since $|Out(A_5)| = 2$, we have $|G/K| = 1$ or 2, and get $|H| = 2^t \cdot 7 \cdot 11 \cdot 19$, where $t = 0$ or 1. Clearly H is solvable for 15 does not divides $|H|$, thus Sylow 19−subgroups of H is normal in H by Lemma 2, which of course is normal in G too, a contradiction. Similarly we can prove that $K/H \ncong L_2(7)$ and $L_2(11)$.

Now, we have $K/H \cong J_1$, which concludes $G \cong J_1$ eventually.

Case 2. Let G be a group having the same order of J_3 and having an irreducible character χ with $\chi(1) = L(J_3)$, we prove that $G \cong J_3$.

In this case, $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, and $\chi(1) = L(J_3) = 2 \cdot 3^4 \cdot 19$. Then $O_{19}(G) = 1$ by the similar reason as in Case 1. Further, G is non-solvable by Lemma 2 and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$ by Lemma 1. As $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, we have that K/H is isomorphic to one of A_5, A_6, $L_2(17), L_2(19), L_2(16), U_4(2)$ and J_3.

Suppose $K/H \cong A_5$. Since $|Out(A_5)| = 2$, we have $|G/K| \mid 2$ and $|H| = 2^\alpha \cdot 3^4 \cdot 17 \cdot 19$ where $\alpha = 4$ or 5. If H is solvable, then $O_{19}(H) \neq 1$ by Lemma 2, a contradiction to $O_{19}(G) = 1$. Now, we have H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that B/A is a direct product of isomorphic non-abelian simple groups and $|H/B| \mid |Out(B/A)|$. As $|H| = 2^\alpha \cdot 3^4 \cdot 17 \cdot 19$, one has that $B/A \cong L_2(17)$. Since $|Out(B/A)| = 2$, it follows that $|A| = 2^\beta \cdot 3^2 \cdot 19$ where $\beta = 1$ or 2. Clearly A is solvable, hence $O_{19}(A) \neq 1$ by Lemma 2. Consequently $O_{19}(G) \neq 1$, a contradiction.

If $K/H \cong A_6$ or $L_2(16)$, we get $O_{19}(G) \neq 1$ by the same arguments as before, a contradiction too.

Now, assume that $K/H \cong L_2(17)$. Since $|Out(L_2(17))| = 2$, we have $|H| = 2^\alpha \cdot 3^3 \cdot 5 \cdot 19$ where $\alpha = 2$ or 3. Let θ be an irreducible constituent of χ_H. Since $\chi(1)/\theta(1) \mid |G/H| = 2^\beta \cdot 3^2 \cdot 17$, where $\beta = 4$ or 5, we have that $3^2 \cdot 19 \mid \theta(1)$. Thus $\theta(1)^2 \geq 3^4 \cdot 19^2 > |H|$, a contradiction.

Suppose $K/H \cong L_2(19)$. Since $|Out(L_2(19))| = 2$, we have $|H| = 2^\alpha \cdot 3^3 \cdot 17$ where $\alpha = 4$ or 5. Let P be a Sylow $17-$subgroup of H. If H is solvable, then $P \trianglelefteq H$ by Lemma 2, and thus $P \trianglelefteq G$. Let φ be an irreducible constituent of χ_P. Since P is abelian, $\varphi(1) = 1$. Noticing $|Aut(P)| \cong Z_{16}$ and $t = |G : I_P(\varphi)|$ divides both of $|Aut(P)|$ and $\chi(1)$, we have that $t \leq 2$, which implies that $e = [\chi_P, \varphi] = 2 \cdot 3^4 \cdot 19$ or $3^4 \cdot 19$, and thus $[\chi_P, \chi_P] = e^2 \cdot t \geq 2 \cdot 3^8 \cdot 19^2 > |G : P|$, a contradiction. Hence H is non-solvable. Then H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that $B/A \cong L_2(17)$. Since $|Mult(L_2(17))| = |Out(L_2(17))| = 2$, we have that H is isomorphic to one of $Z_3 \times L_2(17)$, $(Z_3 \times L_2(17)) \;.\; Z_2$, $Z_3 \times SL_2(17)$ and $Z_6 \times L_2(17)$. Let θ be an irreducible constituent of χ_H. Since $\chi(1)/\theta(1) \mid |G/H|$, we have $3^2 \mid \theta(1)$. By the structure of H, we know that $\theta(1) = 3^2$ or $2 \cdot 3^2$ and H has at most 27 irreducible characters of degree $\theta(1)$, and thus $t = |G : I_G(\theta)| \leq 27$. Let U be a maximal subgroup of G containing $I_G(\theta)$. Then $1 \leq |G : U| \mid |G : I_G(\theta)|$. By checking maximal subgroups of $L_2(19)$ (see [3]), we have that $t = 1$, which implies that $e = [\chi_H, \theta] = 2 \cdot 3^2 \cdot 19$ or $3^2 \cdot 19$, and thus $[\chi_H, \chi_H] = e^2 \cdot t \geq 3^4 \cdot 19^2 > |G : H|$, a contradiction.

Therefore $K/H \cong J_3$, which concludes $G \cong J_3$ as we want.

Case 3. Let G be a group having the same order of J_4 and having an irreducible character χ with $\chi(1) = L(J_4)$, we prove that $G \cong J_4$.

Since the approach used in this case is the same as previous cases, we just write the idea of the proof. In this case, $|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$, and $\chi(1) = L(J_4) = 3 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37$. Since $29 \cdot 37 | \chi(1)$, we have that $O_{29}(G) = O_{37}(G) = 1$, and thus G is non-solvable by Lemma 2. Hence, by Lemma 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$. As $|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$, we have $K/H \cong A_5$, $L_3(2)$, A_6, $L_2(8)$, $L_2(11)$, A_7, $U_3(3)$, $L_2(23)$, M_{11}, $L_2(29)$, $L_2(31)$, A_8, $L_3(4)$, $L_2(32)$, M_{12}, M_{22}, $L_5(2)$, M_{23}, $U_3(11)$, M_{24} or J_4.

Except that K/H is not isomorphic to J_4, we can use the same approach to show that $O_{29}(G)$ or $O_{37}(G)$ is nontrivial by Lemma 2 and come to contradictions. Therefore we have that $K/H \cong J_4$, which concludes $G \cong J_4$.

Proof of Theorem B. By assumption and [3], we have that $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ and G has an irreducible character β with $\beta(1) = S(J_2) = 2^2 \cdot 3 \cdot 5^2$. If $O_5(G) \neq 1$, then $O_5(G)$ is abelian of order 5 or 5^2, and thus $\xi(1) \mid |G/O_5(G)|$ for every $\xi \in \mathrm{Irr}(G)$. But $\beta(1) = 2^2 \cdot 3 \cdot 5^2 \nmid |G/O_5(G)|$, a contradiction. Hence $O_5(G) = 1$, it follows that G is non-solvable by Lemma 3. In the following, we write the proof step by step.

Step 1. to prove that $O_7(G) = 1$.

Let $N = O_7(G) \neq 1$. Then $|N| = 7$. Since $G/C_G(N) \lesssim Aut(N)$, we have that $|G/C_G(N)| = 1, 2, 3$ or 6. Obviously, $O_7(G)$ is the Sylow 7-subgroup of G, so $C_G(N)$ has a normal subgroup M such that $C_G(N) \cong N \times M$. Notice that G is non-solvable, we have by Lemma 1 that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq M \trianglelefteq C_G(N) \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|M/K| \mid |Out(K/H)|$. By $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, one has that $K/H \cong A_5,\ A_6,\ L_2(8),\ U_3(3)$ or $A_5 \times A_5$. We go on discussing the $G/C_G(N)$ case by case.

Case 1. to prove that $|G/C_G(N)| = 1$ is impossible.

Otherwise, $|M| = 2^7 \cdot 3^3 \cdot 5^2$. Let $\theta \in \mathrm{Irr}(M)$ such that $[\beta_M, \theta] \neq 0$. Then $\beta(1) = \theta(1) = 2^2 \cdot 3 \cdot 5^2$ and thus $\theta(1)^2 > |M|$, a contradiction.

Case 2. to prove that $|G/C_G(N)| = 2$ is impossible.

Otherwise, $|M| = 2^6 \cdot 3^3 \cdot 5^2$. Let $\theta \in \mathrm{Irr}(M)$ such that $[\beta_M, \theta] \neq 0$. Then $\theta(1) = 2 \cdot 3 \cdot 5^2$ or $2^2 \cdot 3 \cdot 5^2$. If $\theta(1) = 2^2 \cdot 3 \cdot 5^2$, then $\theta(1)^2 > |M|$, a contradiction. Hence $\theta(1) = 2 \cdot 3 \cdot 5^2$.

Suppose that $K/H \cong A_5$. Since $|Out(A_5)| = 2$, we have $|H| = 2^\alpha \cdot 3^2 \cdot 5$, where $\alpha = 3$ or 4. We claim that H is non-solvable. Suppose that H is solvable. If $\alpha = 3$, then $O_5(H) \neq 1$ by Lemma 2, which implies that $O_5(G) \neq 1$, a contradiction. Therefore $\alpha = 4$. We assert that H has a normal series $1 \trianglelefteq R \trianglelefteq S \trianglelefteq H$ such that

$|R| = 2^4$, $|S/R| = 5$, R is elementary abelian and S/R acts fixed-point-freely on R. Since $O_5(H) \leq O_5(G)$ and $O_5(G)$=1, we have that $O_5(H) = 1$. Thus H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq C \trianglelefteq H$, such that $|B/A| = 2^4$, $|C/B| = 5$, B/A is elementary abelian and C/B acts fixed-point-freely on B/A.

Suppose first that $|A| = 3$. Because $H/C_H(A) \lesssim Aut(A)$, we see that $|C_H(A)| = 2^a \cdot 3^2 \cdot 5$, where $a = 3$ or 4. If $|C_H(A)| = 2^3 \cdot 3^2 \cdot 5$, then $O_5(H) \neq 1$ by Lemma 2, a contradiction. So $|C_H(A)| = 2^4 \cdot 3^2 \cdot 5$, that is $C_H(A) = H$. Let T be the Hall {2,5}-subgroup of C. Then T is characteristic in $C \trianglelefteq H$, and thus $T \trianglelefteq H$. Since $O_5(T) \leq O_5(G)$, we have $O_5(T) = 1$. It follows that $|O_2(T)| = 2^4$ and $O_2(T) \trianglelefteq H$. Therefore, $1 \trianglelefteq O_2(T) \trianglelefteq T \trianglelefteq H$ is the series as we want.

If $|A| = 3^2$, then by the same reason as above, $C_H(A) = H$, so $T \trianglelefteq H$, and thus $1 \trianglelefteq O_2(T) \trianglelefteq T \trianglelefteq H$ is the series as wanted.

If $|A| = 1$, then the normal series $1 \trianglelefteq B \trianglelefteq C \trianglelefteq H$ is what we want.

Since $GL(4, 2)$ has no subgroup of order $3^2 \cdot 5$, we have that $3 \mid |C_H(R)|$. It follows that $H \cong (S \times Z_3) \rtimes Z_3$ or $H \cong S \times E$, where $|E| = 9$. Let $\varphi \in Irr(H)$ such that $[\theta_H, \varphi] \neq 0$. Suppose that $H \cong (S \times Z_3) \rtimes Z_3$. Then H has exactly 45 linear characters and 3 irreducible characters of degree 15. Thus $\varphi(1) = 15$ and $t = |M : I_M(\varphi)| \leq 3$. Let U be a maximal subgroup of M containing $I_M(\varphi)$. Then $1 \leq |M : U| \mid |M : I_M(\varphi)|$. By checking maximal subgroups of A_5 (see [3]), we have that $t = 1$, which forces that $[\theta_H, \theta_H] = 2^2 \cdot 5^2 > |M : H|$, a contradiction. Suppose that $H \cong S \times E$, where $|E| = 9$. It is clear that S has exactly 3 non-linear irreducible characters of degree 5, and then H has exactly 45 linear characters and 27 irreducible characters of degree 5. Then $\varphi(1)$=5. Since both of S and E are the Hall subgroups of H, we have S and E are both normal in M. By checking the maximal subgroups of A_5, we know that the conjugate action of M/H on $\mathrm{Irr}_1(S)$ is trivial, and the orbit lengths under the conjugate action of M/E on $\mathrm{Irr}(E)$ is 1 or 6. Hence $t = |M : I_M(\varphi)| = 1$ or 6, which implies that $[\theta_H, \theta_H] \geq 2 \cdot 3 \cdot 5^2 > |M : H|$, a contradiction.

Now we have proved that H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that $B/A \cong A_5$ or A_6 and $|H/B| \mid |Out(B/A)|$. Let $\varphi \in \mathrm{Irr}(H)$ and $\sigma \in \mathrm{Irr}(A)$ such that $e_1 = [\theta_H, \varphi] \neq 0$ and $e_2 = [\varphi_A, \sigma] \neq 0$.

Suppose that $B/A \cong A_5$. Since $|Out(A_5)| = 2$, we have that $|A| = 2^b \cdot 3$, where $b \leq 2$. Suppose that $|A| = 2^2 \cdot 3$. By checking the order of G, we know that $K = M$ and $B = H$. If A is abelian, then $A = Z(H)$ by Lemma 4, and thus $A \trianglelefteq M$. By $\sigma(1) = 1$ and $|Aut(A)| \leq 6$, we have that $t = |M : I_M(\sigma)| \leq 6$, thus $[\theta_A, \theta_A] \geq 2 \cdot 3 \cdot 5^4 > |M : A|$, a contradiction. If A is non-abelian, then the first column of the character table of A is one of sequences {1, 1, 1, 3} and {1, 1, 1, 1, 2, 2}. If $\sigma(1) = 1$, then $t_2 = |H : I_H(\sigma)| \leq 3$. By checking maximal subgroups of A_5, it is easy to get that $t_2 = 1$. Hence $e_2 = 5$, φ is the unique irreducible constituent of σ^H of degree 5 by Lemma 5. Therefore H has at most 4 irreducible characters of degree 5, which implies that $t_1 = |M : I_M(\varphi)| \leq 4$. By checking properties of maximal subgroups of A_5, it is easy to get that $t_1 = 1$, so that $[\theta_H, \theta_H] = 2^2 \cdot 3^2 \cdot 5^2 > |M : H|$, a contradiction. By the same discussion, we have that if $\sigma(1) \neq 1$, then $t_1 = 1$, and thus $[\theta_H, \theta_H] \geq 2^2 \cdot 5^2 > |M : H|$, a contradiction. Moreover we can get contradictions by the same arguments for the rest possibilities of $|A|$.

Now we consider the case that $B/A \cong A_6$. Since $|Out(A_6)| = 4$, we have that $|H/B|| 4$, which implies that $|A| = 1$ or 2. If $|A| = 2$, then $H \cong SL_2(9)$ or $H \cong Z_2 \times A_6$, which has at most 4 irreducible characters of degree 5 and at most 3 irreducible characters of degree 10. Then $\varphi(1) = 5$ or 10, and $|M : I_M(\varphi)| = 1$ by checking properties of maximal subgroups of A_6, which forces that $[\theta_H, \theta_H] \geq 3^2 \cdot 5^2 > |M : H|$, a contradiction. If $|A| = 1$, then $H \cong A_6$ or $H \cong A_6 \,.\, Z_2$. A contradiction appears through the same arguments.

While K/H is isomorphic to A_6 or $A_5 \times A_5$, we get contradiction by the same approach as above.

Case 3. To prove $|G/C_G(N)| = 3$ is impossible.

Otherwise, $|M| = 2^7 \cdot 3^2 \cdot 5^2$. Let $\theta \in$Irr(M) such that $[\beta_M, \theta] \neq 0$. Then $\theta(1) = 2^2 \cdot 3 \cdot 5^2$ or $2^2 \cdot 5^2$. If $\theta(1) = 2^2 \cdot 3 \cdot 5^2$, then $\theta(1)^2 > |M|$, a contradiction. Hence $\theta(1) = 2^2 \cdot 5^2$.

Suppose that $K/H \cong A_5$. Since $|Out(A_5)| = 2$, we have that $|K/H|| 2$, which implies that $|H| = 2^\alpha \cdot 3 \cdot 5$, where $\alpha = 4$ or 5. We claim that H is non-solvable. Suppose that H is solvable. Since $O_5(H) \leq O_5(G) = 1$, we have that $O_5(H) = 1$, which implies that $|O_2(H)| = 2^4$ or 2^5 and the elements of order 5 in H act nontrivially on $O_2(H)/O_2(H)^{'}$. Furthermore $O_2(H)$ is elementary abelian while $|O_2(H)| = 2^4$, and $O_2(H)$ is extra special or elementary abelian while $|O_2(H)| = 2^5$. Since $O_2(H)$ *char* $H \trianglelefteq M$, we have $O_2(H) \trianglelefteq M$.

Suppose first that $|O_2(H)| = 2^5$. Then M has a normal series: $1 \trianglelefteq O_2(H) \trianglelefteq B \trianglelefteq H \trianglelefteq M$, such that $|B/O_2(H)| = 5$, $|H/B| = 3$ and $M/H \cong A_5$. By Lemma 4, $H/B = Z(M/B)$. Since $|Mult(A_5)| = 2$, one has that $M/B = H/B \times A_5$. Therefore M has a normal series $B \trianglelefteq H_1 \trianglelefteq M$ such that $H_1/B \cong A_5$ and $M/H_1 \cong H/B$. Let $\varphi \in$Irr(H_1) such that $[\theta_{H_1}, \varphi] \neq 0$. Then $\varphi(1) = 2^2 \cdot 5^2$ by Clifford Theorem, which means that $\varphi(1)^2 > |H_1|$, a contradiction. Now we have $|O_2(H)| = 2^4$. Let $D = O_2(H)$. Because $M/C_M(D) \leq Aut(D)$ and $5^2 \nmid |Aut(D)|$, we have that $5 \mid |C_M(D)|$. Recall that the elements of order 5 in H are not contained in $C_M(D)$, so $5^2 \nmid |C_M(D)|$. If $C_M(D)$ is solvable, then $O_5(C_M(D)/D) \neq 1$ by Lemma 2. Hence $O_5(C_M(D)) \neq 1$, so that $O_5(M) \neq 1$, a contradiction. Therefore $C_M(D)$ is not solvable. By Jordan-Hölder theorem, $C_M(D)/D$ has a section isomorphic to A_5. Meanwhile $|C_M(D)/D|$ equals one of $2^3{\cdot}3^2{\cdot}5$, $2^3{\cdot}3{\cdot}5$, $2^2{\cdot}3^2{\cdot}5$ and $2^2{\cdot}3{\cdot}5$. Let $\varphi \in$Irr$(C_M(D))$ such that $[\theta_{C_M(D)}, \varphi] \neq 0$. If $|C_M(D)/D| = 2^3{\cdot}3^2{\cdot}5$ or $2^3{\cdot}3{\cdot}5$, then $\varphi(1) = 2^2{\cdot}5$ by Clifford theorem. Let λ be an irreducible constituent of φ_D. Since λ is linear and invariant in $C_M(D)$, we have $[\varphi_D, \varphi_D] = 2^4{\cdot}5^2 > |C_M(D) : D|$, a contradiction. If $|C_M(D)/D| = 2^2{\cdot}3{\cdot}5$, then $\varphi(1) = 2^2 \cdot 5$ or $2 \cdot 5$, we still have $[\varphi_D, \varphi_D] \geq 2^2 \cdot 5^2 > |C_M(D) : D|$, a contradiction too. At last $|C_M(D)/D| = 2^2 \cdot 3^2 \cdot 5$ and $\varphi(1) = 2^2 \cdot 5$ or $2 \cdot 5$. Clearly $C_M(D)/D \cong Z_3 \times A_5$. Let $E/D = A_5$. Then φ_E is irreducible. By the same arguments as above, we get $[\varphi_D, \varphi_D] \geq 2^2 \cdot 5^2 > |E : D|$, a contradiction.

Now we have proved that H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that $B/A \cong A_5$ and $|H/B| \mid |Out(B/A)|$. By $|Out(A_5)| = 2$, $|H/B|| \ 2$, so $|A| = 2^b$, where $b = 1$, 2 or 3. Let $\varphi \in \mathrm{Irr}(H)$, $\sigma \in \mathrm{Irr}(A)$ such that $[\theta_H, \varphi] \neq 0$ and $[\varphi_A, \sigma] \neq 0$.

Suppose that $|A| = 2^3$. By checking the order of M, we know that $K = M$ and $B = H$. By Clifford theorem, $5 \mid \varphi(1)/\sigma(1)$ and $\sigma(1) = 1$ or 2. Assume first that $\sigma(1) = 1$. By Lemma 4 and 5, $A/A^{'} \leq Z(H/A^{'})$, and for each linear character λ of A, there is at most one irreducible constituent of λ^H with degree 5, which means that $\varphi(1) = 5$ and H has at most 8 irreducible characters of degree 5. Hence $|M : I_M(\varphi)| < 8$, which menas that $[\theta_H, \theta_H] \geq 2^4 \cdot 5 > |M : H|$, a contradiction. If $\sigma(1) = 2$, by the fact that A has exactly one irreducible character with degree 2, we have that σ is invariant in H. Hence by the same reasoning as above, we get $\varphi(1) = 10$, $|M : I_M(\varphi)| = 1$, and thus $[\theta_H, \theta_H] = 2^2 \cdot 5^2 > |M : H|$, a contradiction. For the rest possibilities of $|A|$, we can get contradictions by the same arguments as above.

By the same way, we get $K/H \ncong A_5 \times A_5$.

Now suppose $K/H \cong A_6$. Since $|Out(A_6)| = 4$, we have that $|K/H| \mid 4$, which implies that $|H| = 2^\alpha \cdot 5$, where $\alpha = 4$, 3 or 2. If $\alpha = 3$ or 2, then $O_5(H) \neq 1$ by Lemma 2, it follows that $O_5(M) \neq 1$, a contradiction. Now we have $\alpha = 4$. By $O_5(H) = 1$, it follows that $|O_2(H)| = 2^4$, meanwhile $O_2(H)$ is elementary abelian and $H/O_2(H)$ act fixed-point-freely on $O_2(H)$. Hence H has exactly 3 irreducible character of degree 5. Let φ be an irreducible constituent of θ_H such that $e = [\theta_H, \varphi] \neq 0$ and $t = |M : I_M(\varphi)|$. Then $\varphi(1)$=5 and $t = |M : I_M(\varphi)| \leq 3$ by Clifford theorem. Since $K/H \cong A_6$, we see that $t = 1$, which forces that $[\theta_H, \theta_H] = 2^4 \cdot 5^2 > |M : H|$, a contradiction.

Case 4. to prove that $|G/C_G(N)| = 6$ is impossible.

Otherwise, $|M| = 2^6 \cdot 3^2 \cdot 5^2$. Let $\theta \in$Irr(M) such that $[\beta_M, \theta] \neq 0$. Then $\theta(1) = 2^2 \cdot 3 \cdot 5^2$, $2^2 \cdot 5^2$, $2 \cdot 3 \cdot 5^2$ or $2 \cdot 5^2$. If $\theta(1) = 2^2 \cdot 3 \cdot 5^2$ or $2 \cdot 3 \cdot 5^2$, then $\theta(1)^2 > |K|$, a contradiction. Hence $\theta(1) = 2^2 \cdot 5^2$ or $2 \cdot 5^2$.

Suppose that $K/H \cong A_5$. Since $|Out(A_5)| = 2$, we have that $|H| = 2^\alpha \cdot 3 \cdot 5$, where $\alpha = 3$ or 4. We claim that H is non-solvable. Otherwise if H is solvable. Then by $O_5(H) = 1$ we have $\alpha = 4$ by Lemma 2. By the same arguments as in Case 2, we can show that M has a normal series $1 \trianglelefteq O_2(H) \trianglelefteq B \trianglelefteq H_1 \trianglelefteq M$, such that $O_2(H)$ is elementary abelian with order 2^4, $|B/O_2(H)| = 5$, $H_1/B \cong A_5$, $|M/H_1| = 3$ and $B/O_2(H)$ act fixed-point-freely on $O_2(H)$. Let φ be an irreducible constituent of θ_{H_1} and σ an irreducible constituent of φ_B. By Clifford Theorem, $\varphi(1) = \theta(1)$ and $\sigma(1) = 5$. By the structure of B, it is to show that B has exactly 3 irreducible characters of degree 5, which implies that $t = |H_1 : I_{H_1}(\sigma)| \leq 3$. But $H_1/B \cong A_5$, we have that $t = 1$, so that $[\varphi_B, \varphi_B] \geq 2^2 \cdot 5^2 > |H_1 : B|$, a contradiction.

Now we have shown that H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that $B/A \cong A_5$ and $|H/B| \mid 2$. Let φ be an irreducible constituent of θ_H and σ an irreducible constituent of φ_A.

Suppose that $|A| = 2^2$. By Lemma 4, $A = Z(H)$, and thus $A \trianglelefteq M$. Since $A = Z(H)$, $\sigma(1) = 1$ and $t_2 = |H : I_H(\sigma)| = 1$. By Lemma 5, φ is the unique irreducible constituent of σ^H with degree 5. Then $t_1 = |M : I_M(\varphi)| = 1$, thus $[\theta_H, \theta_H] \geq 2^2 \cdot 5^2 > |M : H|$, a contradiction. Now suppose that $|A| = 2$. Then H is isomorphic to one of $SL_2(5)\ .\ Z_2$, $Z_2 \times S_5$, $Z_2 \times A_5$ and $SL_2(5)$. Hence $\varphi(1) = 5$ and H has at most 4 irreducible characters with degree 5 respectively. But $K/H \cong A_5$, it is easy to show that $t = |M : I_M(\varphi)| = 1$. If $H \cong SL_2(5)\ .\ Z_2$ or $Z_2 \times S_5$, then $[\theta_H, \theta_H] \geq 2^2 \cdot 5^2 > |M : H|$, a contradiction. If $H \cong Z_2 \times A_5$ or $SL_2(5)$, let ψ be an irreducible constituent of θ_K. Again by Lemma 5 we have $\psi(1) = 5^2$ and $\theta_K = \psi$. But $\theta_K(1) = \theta(1) = 2^2 \cdot 5^2$ or $2 \cdot 5^2$, which does not equal to $\psi(1)$, a contradiction. If $|A| = 1$, then $M \cong S_6 \times S_6$, which has no irreducible character of degree $2^2 \cdot 5^2$ and $2 \cdot 5^2$, a contradiction.

By the same way as above, we can show $K/H \ncong A_5 \times A_5$.

Now suppose that $K/H \cong A_6$. By $|Out(A_6)| = 4$, we see that $|H| = 2^\alpha \cdot 5$, where $1 \leq \alpha \leq 3$. Clearly, H is solvable, and thus $O_5(H) \neq 1$ by Lemma 2. Hence $O_5(G) \neq 1$, a contradiction.

Step 2. to prove that $G \cong J_2$.

By G is non-solvable, we get by Lemma 1 that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$. As $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, we have $K/H \cong A_5$, $L_2(7)$, A_6, $L_2(8)$, A_7, $U_3(3)$, $A_5 \times A_5$ or J_2.

Suppose that $K/H \cong A_5$. We claim that H is non-solvable. Assume that H is solvable. Because $O_5(H) \leq O_5(G) = 1$ and $O_7(H) \leq O_7(G) = 1$, we have that $O_5(H) = O_7(H) = 1$. Hence H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq C \trianglelefteq D \trianglelefteq H$, such that $|B/A| = 2^\alpha$, $|C/B| = 5$, $|D/C| = 7$ and B/A is elementary abelian and C/B acts fixed-point freely on B/A, where $\alpha = 4$ or 5. Since $GL(4,2)$ and $GL(5,2)$ have no subgroup of order $5 \cdot 7$, we have that $|C_{H/A}(B/A)|$ is divided by 5 or 7. If $5 \mid |C_{H/A}(B/A)|$, then $O_5(H/A) \neq 1$. Let $E/A = O_5(H/A)$. By $|A| = 2^{5-\alpha} \cdot 3^\gamma$, where $1 \leq \gamma \leq 2$, we have $O_5(E) \neq 1$ by Lemma 2. Noticing that $O_5(E)$ is characteristic in $E \trianglelefteq H$, one has that $O_5(E) \leq O_5(H) \leq O_5(G)$, a contradiction to $O_5(G) = 1$. By the same way, we can show $O_7(G) \neq 1$ while $7 \mid |C_{H/A}(B/A)|$.

Now H is non-solvable. By Lemma 1, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$, such that $B/A \cong A_5$, A_6, A_7 or $L_2(7)$, and $|H/B| \mid |Out(B/A)|$. Since $O_5(G) = O_7(G) = 1$, it follows that $G \cong L_2(7) \times A_5 \times A_5$ or $B/A \cong A_i, (i = 5, 7)$ and A is solvable. For the first case, G has no irreducible character of degree $S(J_2)$, a contradiction. If $B/A \cong A_5$ and that A is solvable. Since $O_7(A) = 1$, A has a normal series $1 \trianglelefteq D \trianglelefteq E \trianglelefteq A$ such that $|D| = 2^3$, $|E/D| = 7$, $|A/E| = 3$ and D is elementary abelian and E/D acts nontrivially on D. Clearly that $|A/A'| = 3$ or 21, and then the first column of the character table of A is one of the sequences {1, 1, 1, 3, 3, 7, 7, 7} or {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 7, 7, 7 } respectively. Let φ be an irreducible constituent of β_H and σ be an irreducible constituent of φ_A. Suppose first that $\sigma(1) = 1$. By

Lemma 4, $A/A^{'} = Z(H/E)$, which implies that every linear character of A is invariant in H. By Lemma 5, it follows that $\varphi(1) = 5$, and for each $\lambda \in$Irr(A) there is at most one irreducible character of H with degree 5 lying over λ, this means that H has at most 21 irreducible character of degree 5. Hence $t = |G : I_G(\varphi)| \leq 20$, then $[\beta_H, \beta_H] \geq 2^2 \cdot 3^2 \cdot 5 > |G : H|$, a contradiction. Suppose that $\sigma(1) = 3$. Since A has exactly 2 irreducible characters of degree 3, we have that $t_1 = |H : I_H(\sigma)| \leq 2$. But $B/A \cong A_5$, by checking maximal subgroups of A_5, we have that $t_1 = 1$. Hence $\varphi(1) = 15$ and H has at most 2 irreducible characters of degree 15 by Lemma 5. Therefore, $|G : I_G(\varphi)| \leq 2$, which forces that $[\beta_H, \beta_H] = 2^4 \cdot 5^2 > |G : H|$, a contradiction.

If $B/A = A_7$, then $|H|$ divides $4|B/A|$. If $|G/K| = 2$, then $G/H = S_5$, $H \cong Z_2.A_7$, $Z_2 \times A_7$ or $A_7.Z_2$. If $G/K = 1$, then $G/H = A_5$, $H \cong A \times A_7$ ($|A| = 4$), $Z_2 \times Z_2.A_7$ or $Z_2.A_7.Z_2$. Let $\theta \in Irr(K)$ such that $[\beta_K,\ \theta] \neq 0$, and $\sigma \in Irr(H)$ such that $[\theta_H,\ \sigma] \neq 0$. Since $5|\sigma(1)$, by checking the character table of H, we get that $\sigma(1) = 10,\ 15$ or 20 and $t = |K : I_K(\sigma)| < 5$. But $K/H \cong A_5$, by checking maximal subgroup of A_5, we get $t = 1$, which force that $[\theta_H,\ \theta_H] > |K : H|$, a contradiction. Similarly, we can show that $K/H \neq A_7$.

Since $O_5(G) = O_7(G) = 1$, we have that K/H is not isomorphic to $L_2(8)$, A_6, $U_3(3)$ and $A_5 \times A_5$. And if K/H is isomorphic to $L_2(7)$ or A_6, then $G \cong L_2(7) \times A_5 \times A_5$, which has no irreducible character of degree $S(J_2)$, a contradiction.

Now we have proved that $K/H \cong J_2$, which concludes $G \cong J_2$. This ends the proof of Theorem B.

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