INTEGRAL EQUATION FOR THE NUMBER OF INTEGER POINTS IN A CIRCLE

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Abstract. The problem is to obtain the most accurate upper estimate for the absolute value of the difference between the number of integer points in a circle and its area (when the radius tends to infinity). In this paper we obtain an integral equation for the function expressing the dependence of the number of integer points in a circle on its radius. The kernel of the equation contains the Bessel functions of the first kind, and the equation itself is a kind of the Hankel transform.
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1. The problem and calculations

The Gauss circle problem is the problem of determining how many integer lattice points there are in a circle centered at the origin and with given radius. Let us consider the circle $K(R) : x^2 + y^2 \leq R$ and let $A(\sqrt{R})$ be the number of

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points with integer coordinates within this circle. As R increases, $A(\sqrt{R})$ is approximately equal to the area inside the circle πR . Let us define $\Delta(R) = |A(\sqrt{R}) - \pi R|$. The Gauss circle problem consists in estimating the upper bound for $\Delta(R)$ when $R \to \infty$ as much precisely as possible.

Similarly, the Dirichlet divisor problem consists in finding the number of integer points under the hyperbola $K_1(R) : xy \leq R$, $0 < x \leq R$, $0 < y \leq R$. The problems of Gauss and Dirichlet were investigated by many authors. Gauss himself found the estimation $O(R^{1/2})$ for$\Delta(R)$. Voronoy [1] obtained the result $O(R^{1/3} lnR)$ for the Dirichlet problem, while Serpinksij found the estimation $O(R^{1/3})$ in 1903 and Hua $O(R^{13/40})$ in 1942 [2] for the circle. Hardy and Littlewood [3] proved that it is impossible to get a better estimation than $O(R^{1/4} ln^2 R)$. Probably the most precise estimation for the circle up to now is $O(R^{131/208})$ obtained by Huxley in the early 2000s [4]. To give the readers more information about the problem, they can take a look to the references [5, 6]. Our aim is to derive an integral equation for the number of integer points in a circle.

Let $\delta_\epsilon(x) = 1 + \sum_{n=1}^{\infty} \frac{sin\, 2\pi n\epsilon}{2\pi n\epsilon} \, cos\, 2\pi nx$ be a periodic function with period 1, which is tending to the periodic delta-function of Dirac when $\epsilon \to 0$. Let us consider the integral

$$A_\epsilon(\sqrt{R}) = \int_K \int_{(R)} \delta_\epsilon(x)\delta_\epsilon(y) \, dx \, dy. \tag{1}$$

We can see that $\lim_{\epsilon \to 0} A_\epsilon(\sqrt{R}) = A(\sqrt{R})$. Let us calculate the integral in the right hand side of (1). The function $\delta_\epsilon(x)$ may be written as follows: $\delta_\epsilon(x) = \sum_{-\infty}^{\infty} c_n e^{2\pi i n x}$, where $c_n = \frac{sin\, 2\pi n\epsilon}{2\pi n\epsilon}$ for $n \neq 0$ and $c_n = 1$ for $n = 0$. Then

$$\begin{aligned} A_\epsilon(\sqrt{R}) &= \int_K \int_{(R)} \delta_\epsilon(x)\delta_\epsilon(y) \, dx \, dy \\ &= \int_0^{\sqrt{R}} r \, dr \int_0^{2\pi} \sum_{n,m=-\infty}^{+\infty} c_n c_m e^{2\pi i r \, (n \cos\phi + m \sin\phi)} \, d\phi. \end{aligned}$$

After changing the order of integration and summation and transforming the function $n \cos\phi + m \sin\phi$, the expression takes the form:

$$\begin{aligned} A_\epsilon(\sqrt{R}) &= \sum_{n,m=-\infty}^{+\infty} c_n c_m \int_0^{\sqrt{R}} r \, dr \int_0^{2\pi} e^{2\pi i r \, \sqrt{n^2+m^2} cos(\phi + \phi_0)} \, d\phi \\ &= \sum_{n,m=-\infty}^{+\infty} c_n c_m \int_0^{\sqrt{R}} r \, dr \int_0^{2\pi} e^{2\pi i r \, \sqrt{n^2+m^2} cos(\phi)} \, d\phi. \end{aligned}$$

Taking into account that $J_0(x) = \frac{1}{\pi} \int_0^\pi e^{ix \cos\phi} \, d\phi$ is the Bessel function of the first kind and zero order, we get:

$$A_\epsilon(\sqrt{R}) = 2\pi \sum_{n,m=-\infty}^{+\infty} c_n c_m \int_0^{\sqrt{R}} r \, J_0(2\pi r \sqrt{n^2 + m^2}) \, dr. \tag{2}$$

Using the relation

$$xJ_0(x) = \frac{d}{dx}\left(xJ_1(x)\right), \tag{3}$$

where $J_1(x)$ is the Bessel function of the first kind, we get by integrating from (2):

$$A_\epsilon(\sqrt{R}) = \sqrt{R}\sum_{n,m=-\infty}^{+\infty} c_n c_m \frac{J_1(2\pi\sqrt{R(n^2+m^2)})}{\sqrt{n^2+m^2}}. \tag{4}$$

Since C_n tend to 1 while $\epsilon \to 0$ we get:

$$\begin{aligned} A_\epsilon(\sqrt{R}) &= \sqrt{R}\sum_{n,m=-\infty}^{+\infty} \frac{J_1(2\pi\sqrt{R(n^2+m^2)})}{\sqrt{n^2+m^2}} \\ &= \pi R + \sqrt{R}\sum_{n^2+m^2\neq 0} \frac{J_1(2\pi\sqrt{R(n^2+m^2)})}{\sqrt{n^2+m^2}}. \end{aligned}$$

The series for $A(\sqrt{R})$ does not converge absolutely, but it sums up to the number of integer points in a round area with radius $R^{1/2}$ (if R is an integer, the points on the circumference are counted with the coefficient 1/2). The expression $A(\sqrt{R})$ may be rewritten in the following way:

$$\begin{aligned} A(\sqrt{R}) &= \lim_{\epsilon\to 0}\sqrt{R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{J_1(2\pi\sqrt{R}\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\delta_\epsilon(x)\,\delta_\epsilon(y)dxdy \\ &= \lim_{\epsilon\to 0}\sqrt{R}\int_0^{\infty} J_1(2\pi r\sqrt{R})\,dr\int_0^{2\pi}\delta_\epsilon(r\,cos\,\phi)\,\delta_\epsilon(r\,sin\,\phi)\,d\phi. \end{aligned}$$

Since (see above)

$$\int_0^{2\pi}\delta_\epsilon(r\,cos\,\phi)\,\delta_\epsilon(r\,sin\,\phi)\,d\phi = 2\pi\sum_{n,m=-\infty}^{+\infty} c_n c_m\,J_0(2\pi r\sqrt{n^2+m^2}),$$

we get:

$$A(\sqrt{R}) = \lim_{\epsilon\to 0} 2\pi\sqrt{R}\int_0^{+\infty} J_1(2\pi r\sqrt{R})\sum_{n,m=-\infty}^{+\infty} c_n c_m\,J_0(2\pi r\sqrt{n^2+m^2})\,dr.$$

Let us denote $\sqrt{R}=\rho$. Using the property (3) of Bessel functions and integrating by parts, we get:

$$A(\rho)=\lim_{\epsilon\to 0}\int_0^{+\infty}\left(\frac{1}{r}J_1(2\pi\rho r)-2\pi\rho J_1'(2\pi\rho r)\right)\left(\sum_{n,m=-\infty}^{+\infty} c_n c_m\frac{J_1(2\pi r\sqrt{n^2+m^2})}{\sqrt{n^2+m^2}}\right)dr.$$

In view of (4), the following expression is derived:

$$A(\rho) = \lim_{\epsilon \to 0} \int_0^{+\infty} \left(\frac{1}{r^2} J_1(2\pi\rho r) - \frac{2\pi\rho}{r} J_1'(2\pi\rho r) \right) A_\epsilon(r)\, dr.$$

Now, replacing $J_1'(2\pi\rho r)$ according to formula (3) and taking the limit $\epsilon \to 0$ we finally obtain the integral equation:

$$A(\rho) = \int_0^{+\infty} A(r)\, K(\rho,\, r)\, dr, \tag{5}$$

where the core is

$$K(\rho,\, r) = \frac{2}{r^2} J_1(2\pi\rho r) \;-\; \frac{2\pi\rho}{r} J_0(2\pi\rho r).$$

2. Conclusion

Let us note that the integral transform $F(\rho) = \int_0^{+\infty} A(r)\, K(\rho, r)\, dr$ with the core $K(\rho, r) = r\, J_s(2\pi\rho r)$, $s \geq -1/2$ is known as the Hankel transform. Thus, for the function expressing the dependence of the number of integer points in a circle on its radius, the integral equation is obtained, which is a kind of the Hankel transform. It can be used for further investigations of the Gauss circle problem. Also of interest is a possible generalization of the methodology and applying it to other similar problems (for example, the Dirichlet divisor problem).

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