P-EXPANDABLE SPACES

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Abstract. We introduce the concept of P-expandable spaces as a variation of expandable spaces. A space (X, τ) is said to be P-expandable if every locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of X there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subsets of X such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Delta$. We characterize P-expandable spaces and study their basic properties. We show that if a space (X, τ) is a quasi submaximal space, then (X, τ) is P-expandable if and only if it is expandable.

Keywords: preopen set, p-locally finite collection, expandable space, P-expandable space.

1. Introduction

By a space, we mean a topological space in which no separation axioms is assumed unless explicitly stated. Let (X, τ) be a space and A be a subset of X. The closure of A, the interior of A and the relative topology on A in (X, τ) will be denoted by $cl(A)$, $\text{int}(A)$ and τ_A, respectively. A is called a preopen subset of (X, τ) [3] if $A \subseteq \text{int}(cl(A))$. The complement of a preopen set is called a preclosed set. A is called semi-open [12] (resp. α-sets [13], regular closed) if $A \subseteq cl(\text{int}(A))$ (resp. $A \subseteq \text{int}(cl(\text{int}(A)))$, $A = cl(\text{int}(A))$). The family of all subsets of a space (X, τ) which are preopen (resp. preclosed, semi-open, regular closed) is denoted by $PO(X, \tau)$ (resp. $PC(X, \tau)$, $SO(X, \tau)$, $RC(X, \tau)$). It is known that the collection of all α-sets of (X, τ) forms a topology on X, denoted by τ^α, finer than τ and $PO(X, \tau) = PO(X, \tau^\alpha)$.

A space (X, τ) is called submaximal [11] if every dense subset of (X, τ) is open. It is known that (X, τ) is submaximal if and only if $\tau = PO(X, \tau)$. In [4], Al-Nashef introduced the notion of quasi-submaximal spaces where a space (X, τ) is quasi-submaximal if $cl(D) - D$ is nowhere dense subset for each dense subset D of (X, τ). This is equivalent to saying that $\text{int}(D)$ is dense for each dense subset D of (X, τ) [4].

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Mashhour et al. [2] used preopen sets to define P_1-paracompact and P_2-paracompact spaces. In 2007, Al-Zoubi and Al-Ghour [8] define P_3-paracompact space and the notion P-locally finite collections and study their properties. In this paper we introduce P-expandable spaces by using preopen sets and p-locally finiteness and study their topological properties. We deal with subspaces, sum, image and the inverse images of P-expandable.

Lemma 1.1. *Let A and B be subsets of a space (X, τ).*

i. If $A \in PO(X, \tau)$ and $B \in SO(X, \tau)$, then $A \cap B \in PO(B, \tau_B)$ ([6]).

ii. If $A \in PO(B, \tau_B)$ and $B \in PO(X, \tau)$, then $A \in PO(X, \tau)$ ([6]).

iii. If $A \in PO(X, \tau)$ and $B \in \tau$, then $A \cap B \in PO(X, \tau)$ ([7]).

Definition 1.2. A collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of a space (X, τ) is called locally finite (resp. p-locally finite [8]) if for each $x \in X$, there exists $W_x \in \tau$ (resp. $W_x \in PO(X, \tau)$) containing x and W_x intersects at most finitely many members of $\mathcal{F}$.

Corollary 1.3 ([8]). *Let (X, τ) be any space:*

i. Every locally finite collection subset of X is p-locally finite collection subset of X.

ii. Every p-locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of preopen subsets of a quasi- submaximal space X is locally finite.

iii. Every p-locally finite collection of open sets (α-sets, regular closed sets) is locally finite.

Definition 1.4. A function $f : (X, \tau) \to (Y, \sigma)$ is called

i. Preirresoulte [5] if and only if $f^{-1}(A) \in PO(X, \tau)$ for each $A \in PO(Y, \sigma)$.

ii. Strongly preclosed [16] if $f(A) \in PC(Y, \sigma)$ for each $A \in PC(X, \tau)$.

iii. M-preopen [2] if $f(A) \in PO(Y, \sigma)$ for each $A \in PO(X, \tau)$.

iv. Countable perfect [10] if f is continuous, closed, surjective function such that $f^{-1}(y)$ is countable compact for each y in Y.

Lemma 1.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous function.*

i. If $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ is a locally finite collection of subsets of (Y, σ), then $f^{-1}(\mathcal{F}) = \{f^{-1}(F_\alpha) : \alpha \in \Delta\}$ is a locally finite collection in (X, τ) [10].

ii. Let f be a countable perfect function. If $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ is a locally finite collection of subsets of (X, τ), then $f(\mathcal{F}) = \{f(F_\alpha) : \alpha \in \Delta\}$ is a locally finite collection in (Y, σ) ([1]).

Recall that a space (X,τ) is called strongly compact relative to X [2] if every cover of A by preopen sets of X has a finite subcover.

Theorem 1.6 ([8]). *Let $f:(X,\tau)\to(Y,\sigma)$ be a function:*

i. If f is a preirresolute function and $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ is a p-locally finite collection in (Y,σ), then $f^{-1}(\mathcal{F})=\{f^{-1}(F_\alpha):\alpha\in\Delta\}$ is a p-locally finite collection in (X,τ).

ii. If f is a strongly preclosed function such that $f^{-1}(y)$ is strongly compact relative to (X,τ) for every $y\in Y$ and $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ is a p-locally finite collection of subsets of (X,τ), then $f(\mathcal{F})=\{f(F_\alpha):\alpha\in\Delta\}$ is a p-locally finite collection in (Y,σ).

Corollary 1.7. *Let (X,τ) be a space, then the following are equivalent:*

i. (X,τ) is expandable.

ii. For every locally finite collection $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ of subsets of X there exists a p-locally finite collection $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ of open subsets of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$ [8].

iii. For every locally finite collection $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ of subset of X there exists a locally finite collection $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ of preopen subset of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$.

iv. For every locally finite collection $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ of subsets of X there exists a p-locally finite $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ collection of α-open subsets of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$.

Proof. (i→ii →iii →iv) These implication follow from definitions, Corollary 1.3 and the fact that $\tau\subseteq\tau_\alpha$.

(iv→i) Let $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ be a locally finite collection of subsets of a space (X,τ). Then, by (iv), there exists a p-locally finite collection $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ of α-open sets subsets of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$. Then, by Corollary 1.3, $\{\mathrm{int}(cl(\mathrm{int}(G_\alpha))):\alpha\in\Delta\}$ is a locally finite collection of open subset of X such that $F_\alpha\subseteq\mathrm{int}(cl(\mathrm{int}(G_\alpha)))$ for all $\alpha\in\Delta$. Hence (X,τ) is expandable. □

2. P-expandable spaces

Definition 2.1. A space (X,τ) is said to be P-expandable (resp. P_1-expandable, pre-expandable) if every locally finite (resp. p-locally finite, p-locally finite) collection $\mathcal{F}=\{F_\alpha:\alpha\in\Delta\}$ of subsets of X there exists a p-locally finite (resp. locally finite, p-locally finite) collection $\mathcal{G}=\{G_\alpha:\alpha\in\Delta\}$ of preopen (resp. open, preopen) subsets of X such that $F_\alpha\subseteq G_\alpha$ for each $\alpha\in\Delta$.

It is clear (from the fact that the closure of any locally finite collection is locally finite) that a space (X, τ) is P-expandable iff every locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of closed subsets of X, there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subsets of X such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Delta$.

The following diagram follows immediately from the definitions in which none of these implications is reversible.

$$\begin{array}{ccc} \text{Expandable} & \rightarrow & \text{P-expandable} \\ \uparrow & & \uparrow \\ \text{P}_1\text{-expandable} & \rightarrow & \text{pre-expandable} \end{array}$$

To show that none of these implications is reversible, In the above diagram, we consider the following examples.

Example 2.2. Let $X = N \cup N^-$ with the topology $\tau = \{U \subseteq X : N \subseteq U\} \cup \{\phi\}$ such that N is the set of all positive integers and N^- is the set of all negative integers. Then $PO(X, \tau) = \{A \subseteq X : A \cap N \neq \phi\}$.

(i) Note that (X, τ) is not expandable since the collection $\{\{x\} : x \in N^-\}$ is locally finite in (X, τ) and there exists no locally finite collection $\{U_x : x \in N^-\}$ of open sets in (X, τ) such that $x \in U_x$ for $x \in N^-$.

(ii) To see that (X, τ) is pre- expandable (hence P-expandable). Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a p-locally finite collection in (X, τ). Put $\Delta_1 = \{\alpha \in \Delta : U_\alpha \cap \mathbb{N} \neq \phi\}$ and $\Delta_2 = \{\alpha \in \Delta : U_\alpha \cap \mathbb{N} = \phi\}$. Now, for $\alpha \in \Delta_2$, choose $x_\alpha \in U_\alpha$ and put $U^*_\alpha = U_\alpha \cup \{-x_\alpha\}$. Put $\mathcal{U}^* = \{U_\alpha : \alpha \in \Delta_1\} \cup \{U^*_\alpha : \alpha \in \Delta_2\}$. Then, it is clear $\mathcal{U}^*$ is a collection of preopen sets in (X, τ) such that for all $\alpha \in \Delta$, there exists $H_\alpha \in \mathcal{U}^*$ such that $U_\alpha \subseteq H_\alpha$. Finally, we show that $\mathcal{U}^*$ is p-locally finite in (X, τ). Let $x \in X$. Then, there exists a preopen set P_x in (X, τ) such that $x \in P_x$ and a finite subset Δ'_1 of Δ_1 and a finite subset Δ'_2 of Δ_2 such that $P_x \cap U_\alpha = \phi$ for all $\alpha \in \Delta - (\Delta'_1 \cup \Delta'_2)$. Now, if $x \in \mathbb{N}^-$, put $P^*_x = (P_x - \{-x_\alpha : \alpha \in \Delta_2 - \Delta'_2\}) \cup \{-x\}$. Then P^*_x is a preopen set in (X, τ) such that $x \in P^*_x$ and P^*_x intersect at most finitely many members of $\mathcal{U}^*$. If $x \in \mathbb{N}$, put $P^*_x = (P_x - \{-x_\alpha : \alpha \in \Delta_2 - \Delta'_2\}) \cup \{x\}$. Then P^*_x is a preopen set in (X, τ) such that $x \in P^*_x$ and P^*_x intersect at most finitely many members of $\mathcal{U}^*$. Thus $\mathcal{U}^*$ is a p-locally finite collection of preopen sets in (X, τ) and so (X, τ) is pre-expandable.

Example 2.3. Let $X = \mathbb{R}$ with the topology $\tau = \{U : U \subseteq \mathbb{Q}\} \cup \{\mathbb{R}\}$. Note that $PO(X, \tau) = \{U : U \subseteq \mathbb{Q}\} \cup \{U : \mathbb{Q} \subseteq U\}$ and every locally finite collection is finite. Hence (X, τ) is expandable (and so P-expandable). On the other hand, (X, τ) is not pre-expandable since the collection $\{\{x\} : x \in \mathbb{R} - \mathbb{Q}\}$ is p-locally finite in X but there does not exists a p-locally finite collection of preopen set $\{G_x : x \in \mathbb{R} - \mathbb{Q}\}$ in (X, τ) such that $x \in G_x$ for each $x \in \mathbb{R} - \mathbb{Q}$. If $\mathcal{G} = \{G_x : x \in \mathbb{R} - \mathbb{Q}\}$ is p-locally finite collection of preopen sets, then $\{x\} \cup \mathbb{Q} \subseteq G_x$ for all $x \in \mathbb{R} - \mathbb{Q}$. Choose $x_\circ \in \mathbb{Q}$ and $p_\circ \in PO(X, \tau)$ such that $x_\circ \in p_\circ$. Then $p_\circ \cap G_x \neq \phi$ for all x.

Example 2.4. Let $X = \mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}\}$. Note that $PO(X, \tau) = \tau_{dis}$ and so (X, τ) is pre-expandable. On the other hand, every locally finite is finite, therefore (X, τ) is expandable. To show that (X, τ) is not P_1-expandable, we consider the collection $\mathcal{U} = \{\{x\} : x \in \mathbb{Q}\}$. $\mathcal{U}$ is p-locally finite in X but there does not exists a locally finite collection of open set $\{G_x : x \in \mathbb{Q}\}$ in (X, τ) such that $x \in G_x$ for each $x \in \mathbb{Q}$. Note that, if $G_x \in \tau$ such that $\{x\} \subseteq G_x$ then either $G_x = \mathbb{Q}$ or $G_x = \mathbb{R}$ and so $\{G_x\}$ is not locally finite.

Note that Example 2.2 and Example 2.3 shows that expandable and pre-expandable spaces are independent notions.

Proposition 2.5. *Let (X, τ) be a space, then the following are equivalent:*

i. (X, τ) *is* P_1*-expandable.*

ii. For every p-locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of X there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subset of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$.

iii. For every p-locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of X there exists locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$.

Proof. (i→ii→iii) These implication follow from the definition and Corollary 1.3..

(iii→i) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a p-locally finite collection of subsets of X. Then, by (iii) there exists a locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$. Then, $\{\mathrm{int}(cl(G_\alpha)) : \alpha \in \Delta\}$ is a locally finite collection of open subset of X such that $U_\alpha \subseteq \mathrm{int}(cl(G_\alpha))$ for all $\alpha \in \Delta$. Hence (X, τ) is P_1-expandable. □

Proposition 2.6. *Let (X, τ) be any space:*

i. If (X, τ) is a quasi-submaximal, then (X, τ) is expandable iff it is P*-expandable.*

ii. If (X, τ) is a submaximal, then (X, τ) is expandable iff it is pre-expandable.

Proof. The easy proof is left to the reader. □

In Example 2.2 and Example 2.3 show that the conditions in Proposition 2.6 are essential. Recall that a space (X, τ) is called countable P-compact [15], if every countable preopen cover of (X, τ) has a finite subcover. It is clear that every p-locally finite collection of countably P-compact space is finite [14].

Proposition 2.7. *Let (X, τ) be a countably* P*-compact space. Then (X, τ) is expandable if and only if it is* P*-expandable.*

Proof. The necessity is clear and we need only prove the sufficiency. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a locally finite collection of X. Then there exists a P-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of X such that $U_\alpha \subseteq G_\alpha$ for each $\alpha \in \Delta$, Since (X, τ) is countably P-compact then $\{\text{int}(cl(G_\alpha)) : \alpha \in \Delta\}$ is a locally finite collection of open subset of X such that $U_\alpha \subseteq \text{int}(cl(G_\alpha))$ for all $\alpha \in \Delta$. Hence (X, τ) is expandable. □

Recall that a space (X, τ) is called P_1-paracompact [2], (resp. P_2-paracompact [2], P_3-paracompact [8]) if every preopen (resp. preopen, open) cover of X has a locally finite open (resp. locally finite preopen, p-locally finite preopen) refinement.

Theorem 2.8. *Every* P_3*-paracompact space is* P*-expandable.*

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ be a locally finite collection of closed subsets of X. Let Δ' be the collection of all finite subsets of Δ. For $\beta \in \Delta'$, let $V_\beta = X - \cup\{F_\alpha : \alpha \notin \beta\}$. Because $\mathcal{F}$ is the locally finite collection, V_β is open. Also, V_β meets only finitely many elements of $\mathcal{F}$. Let $\mathcal{V} = \{V_\beta : \beta \in \Delta'\}$. Then $\mathcal{V}$ is an open cover of X. Since X is P_3-paracompact, $\mathcal{V}$ has a p-locally finite preopen refinements, say $\mathcal{W} = \{W_\gamma : \gamma \in \Delta\}$. Set $U_\alpha = \cup\{W_\gamma \in \mathcal{W} : W_\gamma \cap F_\alpha \neq \phi\}$ for each $\alpha \in \Delta$. Because arbitrary unions of preopen sets are preopen set, U_α is preopen and $F_\alpha \subseteq U_\alpha$ for each $\alpha \in \Delta$. Now, we shall try to show that $\{U_\alpha : \alpha \in \Delta\}$ is p-locally finite. Since $\mathcal{W}$ is p-locally finite, for each $x \in X$, there exists a preopen set U_x in (X, τ) containing x and U_x intersects at most finitely many members of $\mathcal{W}$. Also, by the definition of U_α, we say that $U_x \cap U_\alpha \neq \phi$ if and only if $U_x \cap W_\gamma \neq \phi$ and $W_\gamma \cap F_\alpha \neq \phi$ for some $\gamma \in \Delta$. Since $\mathcal{W}$ is refinement of $\mathcal{V}$, there is number V_β of $\mathcal{V}$ containing W_γ for each number W_γ of $\mathcal{W}$. Then W_γ meets only finitely many F_α for each $\gamma \in \Delta$. Thus, $\{U_\alpha : \alpha \in \Delta\}$ is p-locally finite. □

Corollary 2.9. *Every* P_1*-paracopmact (reps.* P_2*-paracompact) space is* P*-expandable.*

The following example shows that the converse of the above corollary need not be true.

Example 2.10. Let ω_1 denote the first uncountable ordinal and let $X = [0, \omega_1)$ with the usual order topology. Then, from [9], X is countable compact but not paracompact since the collection $\{[0, \alpha) : \alpha < \omega_1\}$ is an open cover of X which has no open locally finite refinement. Hence X is P-expandable but neither P_1-paracompact nor P_2-paracompact.

Theorem 2.11. *Let* (X, τ) *be a space:*

i. *If* (X, τ^α) *is* P*-expandable, then* (X, τ) *is* P*-expandable.*

ii. *If* (X, τ) *is* P*-expandable submaximal space, then* (X, τ^α) *is* P*-expandable.*

Proof. This follows immediately from the definitions and the facts that $\tau \subseteq \tau^{\alpha}$ and $PO(X, \tau^{\alpha}) = PO(X, \tau)$. □

The converse of part (i) of Theorem 2.11 is not true in general as the following example shows.

Example 2.12. Let X be an infinite set and $q \in X$. Let $\tau = \{\phi, X, \{q\}\}$. Then (X, τ) is P-expandable. But (X, τ^{α}) is not P-expandable and not submaximal. Since $\tau^{\alpha} = PO(X, \tau) = PO(X, \tau^{\alpha}) = \{\phi\} \cup \{U \subseteq X : q \in U\}$. Now, the collection $\{\{x\} : x \in X - \{q\}\}$ is locally finite in (X, τ^{α}) and there is no p-locally finite collection$\{G_{\alpha} : x \in X - \{q\}\}$ of preopen subset in (X, τ^{α}) such that $x \in G_{\alpha}$ and $x \in X - \{q\}$.

Definition 2.13. A space (X, τ) is said to be $\omega -$ P-expandable if every locally finite collection $\mathcal{F} = \{F_{\alpha} : \alpha \in \Delta,\ |\Delta| \leq \omega\}$ of subsets of X there exists a p-locally finite collection $\mathcal{G} = \{G_{\alpha} : \alpha \in \Delta\}$ of preopen subsets of X such that $F_{\alpha} \subseteq G_{\alpha}$ for each $\alpha \in \Delta$.

Theorem 2.14. *Let (X, τ) be a space. Then (X, τ) is $\omega - \mathrm{P}$-expandable if and only if every countable open cover of X has a p-locally finite preopen refinement.*

Proof. Sufficiency is similar to the proof of Theorem 2.8.

To prove necessity, let $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$ be a countable open cover of X. Put $A_i = \cup\{U_j : j \leq i\}$ for each $i \in \mathbb{N}$. Let $B_1 = A_1$ and $B_i = A_i - A_{i-1}$ such that $i = 2, 3, 4, ...$ Therefore $B_i \subseteq U_i$ for each $i \in \mathbb{N}$. For $x \in X$, let $i(x) = \min\{i \in \mathbb{N} : x \in U_i\}$. Then $x \in B_{i(x)}$. Put $\mathcal{A} = \{B_i : i \in \mathbb{N}\}$. Then, $\mathcal{A}$ is a refinement of $\mathcal{U}$ and $\mathcal{A}$ is locally finite since $U_i \cap B_i = \phi$ for $j > i$. Because X is $\omega -$ P-expandable, there exists a p-locally finite collection $\{G_i : i \in \mathbb{N}\}$ of preopen subsets of X such that $B_i \subseteq G_i$ for each $i \in \mathbb{N}$. Let $V_i = U_i \cap G_i$ for each $i \in \mathbb{N}$. By Lemma 1.1, V_i is preopen set in (X, τ) for each $i \in \mathbb{N}$. Let $\mathcal{V} = \{V_i : i \in \mathbb{N}\}$. Since $\{G_i : i \in \mathbb{N}\}$ is p-locally finite, $\mathcal{V}$ is p-locally finite. Because $\mathcal{A}$ is a cover of X, there exists some $i \in \mathbb{N}$ such that $x \in B_i$ for each $x \in X$. Since $B_i \subseteq V_i$, $x \in V_i$. Thus, $\mathcal{V}$ is a p-locally finite preopen refinement of $\mathcal{U}$. □

3. Operations

In this section we study some basic operation on P-expandable spaces.

Definition 3.1. A subset A of a space (X, τ) is called an:

i. α P-expandable set in (X, τ) if every locally finite (in X) collection $\mathcal{F} = \{F_{\alpha} : \alpha \in \Delta\}$ of subsets of A there exists a p-locally finite collection $\mathcal{G} = \{G_{\alpha} : \alpha \in \Delta\}$ of preopen subset of (X, τ) such that $F_{\alpha} \subseteq G_{\alpha}$ for all $\alpha \in \Delta$.

ii. β P-expandable set in (X, τ) if and only if (A, τ_A) is P-expandable.

Note that α P-expandable and β P-expandable sets are linearly independent. To see that we give the following examples.

Example 3.2. Let $X = \mathbb{R}$ with the topology $\tau = \{U : 1 \in U\} \cup \{\phi\}$. Note that the set of all $PO(X, \tau) = \tau$. Put $A = \mathbb{R} - \{1\}$. Then $A \notin PO(X, \tau)$ and $\tau_A = \tau_{dis}$. Therefore, A is β P-expandable but not α P-expandable.

Example 3.3. Let (X, τ) be as in Example 2.2 and let $A = \mathbb{N}^- \cup \{1\}$. Then $\tau_A = \{\{1\} \cup H : H \subseteq \mathbb{N}^-\} \cup \{\phi\}$ and $PO(A, \tau_A) = \tau_A$. To show that (A, τ_A) is not βP-expandable we consider the collection $\mathcal{U} = \{\{x\} : x \in \mathbb{N}^-\}$. Then $\mathcal{U}$ is a locally finite collection of subsets of A and note that if $\mathcal{U}$ is a locally finite (this equivalent p-locally finite) collection of open (preopen) subsets of (A, τ_A), then $\mathcal{U}$ is finite. Therefore, (A, τ_A) is not β P-expandable. On the other hand, A is α P-expandable. Indeed, let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a locally finite (in X) collection of subsets of A. Then $1 \notin U_\alpha$ for every $\alpha \in \Delta$. As in Example part (2) we show that there exists a p-locally finite collection $\rho = \{P_\alpha : \alpha \in \Delta\}$ of preopen subsets of X such that $U_\alpha \subseteq P_\alpha$ for all $\alpha \in \Delta$. Thus A is α P-expandable.

A subset A of a space (X, τ) is called pre-clopen if A is preopen and preclosed.

Theorem 3.4. *Let A and B be subsets of a space (X, τ) such that $A \subseteq B$.*

i. If B is pre-clopen in (X, τ) and A is α P-expandable in (B, τ_B) then A is α P-expandable in (X, τ).

ii. If B is semi-open in (X, τ) and A is α P-expandable in (X, τ), then A is α P-expandable in (B, τ_B).

Proof. i) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a locally finite collection of subsets of A. Then there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subsets of (B, τ_B) such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$. Since B is pre-clopen subset in (X, τ), then , by Lemma 1.1, $\mathcal{G}$ is p-locally finite collection of preopen subsets of (X, τ). For, let $x \in X$. Then either $x \in B$ or $x \notin B$. If $x \in B$, then there exists a preopen set W in (B, τ_B) containing x such that W intersects at most finitely many members of $\mathcal{G}$. Since B is preopen in (X, τ) then W is preopen in (X, τ), by Lemma 1.1 and hence $\mathcal{G}$ is p-locally finite collection in (X, τ). However, if $x \notin B$, then $X - B$ is preopen set in (X, τ) containing x which intersects no member of $\mathcal{G}$. Hence $\mathcal{G}$ is a p-locally finite collection in (X, τ).

ii) Let $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ be a locally finite collection of subsets of A. Then there exists a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of (X, τ) such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$. Now consider $\mathcal{G}^* = \{G_\alpha \cap B : \alpha \in \Delta\}$, by Lemma 1.1, $\mathcal{G}^*$ is a p-locally finite collection of preopen subset of (B, τ_B) such that $F_\alpha \subseteq G_\alpha \cap B$ for all $\alpha \in \Delta$. Thus A is α P-expandable in (B, τ_B). □

Corollary 3.5. *Let A be a subset of a space (X, τ).*

i. If A is pre-clopen in (X, τ) and β P-expandable, then A is α P-expandable.

ii. If A is semi-open in (X,τ) and $\alpha\,$P-expandable, then A is $\beta\,$P-expandable.

Note that Example 3.2 shows that the assumption A is pre-clopen in Corollary 3.5 can not be replaced by the statement A is preclosed.

Lemma 3.6. *If A is a closed subset of a space (X,τ), then any locally finite collection of subsets of A is a locally finite collection in X.*

Proposition 3.7. *Let (X,τ) be a P-expandable space, then:*

i. Every regular closed subset of (X,τ) is $\beta\,$P-expandable.

ii. Every closed subsets of (X,τ) is $\alpha\,$P-expandable.

Proof. i) Let A be a regular closed subset of a P-expandable space (X,τ). Let $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ be a locally finite collection of subset of A. Since A is closed by Lemma 3.6, $\mathcal{F}$ is locally finite in (X,τ), so there exists a p-locally finite collection of preopen subset of (X,τ), say $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ such that $F_\alpha \subseteq G_\alpha$, for each $\alpha \in \Delta$. Consider $\mathcal{G}^* = \{G_\alpha \cap A : \alpha \in \Delta\}$. Then, by Lemma 1.1 and the fact that $RC(X,\tau) \subseteq SO(X,\tau)$, $\mathcal{G}^*$ is a p-locally finite collection of preopen subsets of A such that $F_\alpha \subseteq G_\alpha \cap A$ for each $\alpha \in \Delta$. Thus A is $\beta\,$P-expandable.

ii) It is follow from Lemma 3.6. □

Let $\{(X_\alpha,\tau_\alpha) : \alpha \in \Delta\}$ be a collection of topological spaces such that $X_\alpha \cap X_\beta = \phi$ for each $\alpha \neq \beta$. Let $X = \underset{\alpha\in\Delta}{\cup} X_\alpha$ be topologized by $\tau = \{G \subseteq X : G \cap X_\alpha \in \tau_\alpha$ for each $\alpha \in \Delta\}$. Then (X,τ) is called the sum of the spaces $\{(X_\alpha,\tau_\alpha) : \alpha \in \Delta\}$ and we write $X = \underset{\alpha\in\Delta}{\oplus} X_\alpha$.

Theorem 3.8. *The topological sum $\underset{\alpha\in\Delta}{\oplus} X_\alpha$ is P-expandable if and only if (X_α,τ_α) is P-expandable, for each $\alpha \in \Delta$.*

Proof. Necessity follows from Proposition 3.7. To prove sufficiency, let $\mathcal{U}$ be a locally finite collection of $\underset{\alpha\in\Delta}{\oplus} X_\alpha$. For each $\alpha \in \Delta$ the family $\mathcal{U}_\alpha = \{U \cap X_\alpha : U \in \mathcal{U}\}$ is a locally finite collection of the P-expandable space (X_α,τ_α). Therefore there exists a p-locally finite collection $\mathcal{G}_\alpha = \{G_{U_\alpha} : U \in \mathcal{U}\}$ of a preopen subsets of (X_α,τ_α) such that for all $\alpha \in \Delta$, $U \cap X_\alpha \subseteq G_{U_\alpha}$ for all $U \in \mathcal{U}$. Put $G_U = \underset{\alpha\in\Delta}{\cup} G_{U_\alpha}$ and $\mathcal{G}^* = \{G_U : U \in \mathcal{U}\}$. We note that (i) G_U is preopen in X for each $U \in \mathcal{U}$(by Lemma 1.1) (ii) G^* is p-locally finite in X. Let $x \in X$. Then there exists $\alpha_\circ \in \Delta$ such that $x \in X_{\alpha_\circ}$. So there exists a preopen subset $W_{\alpha_\circ}$ of $X_{\alpha_\circ}$ such that $W_{\alpha_\circ}$ intersects at most finitely many member of $\mathcal{G}_{\alpha_\circ}$, say $G_{U_{1_{(\alpha_\circ)}}}, G_{U_{2_{(\alpha_\circ)}}}, ...G_{U_{n_{(\alpha_\circ)}}}$. Note that $G_{U_\beta} \cap W_{\alpha_\circ} = \phi$ for each $U \in \mathcal{U}$ and so for every $U \in \mathcal{U} - \{U_1, ..., U_n\}$, $W_{\alpha_\circ} \cap G_U = \phi$. Thus $\mathcal{G}^*$ is p-locally finite in X such that for each $U \in \mathcal{U}$, $U = U \cap X = U \cap (\underset{\beta\in\Delta}{\cup} X_\beta) \subseteq G_U$. □

Theorem 3.9. *Let $f : (X,\tau) \to (Y,\sigma)$ be a M-preopen and strongly preclosed surjective continuous function such that $f^{-1}(y)$ is strongly compact relative to (X,τ) for every $y \in Y$. If (X,τ) is* P*-expandable then (Y,σ) is* P*-expandable.*

Proof. Assume that (X,τ) is P-expandable and $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ is a locally finite collection of subsets of Y. Then $f^{-1}(\mathcal{F}) = \{f^{-1}(F_\alpha) : \alpha \in \Delta\}$ is a locally finite collection of subsets of the P-expandable (X,τ) and so there is a p-locally finite $\{G_\alpha : \alpha \in \Delta\}$ of preopen subsets of X such that $f^{-1}(F_\alpha) \subseteq G_\alpha$ for each $\alpha \in \Delta$. Since f is M-preopen and by Theorem 1.6, the collection $f(G_\alpha)$ is p-locally finite collection of preopen subsets of Y such that $F_\alpha \subseteq f(G_\alpha)$. □

Theorem 3.10. *Let $f : (X,\tau) \to (Y,\sigma)$ be a countably perfect preirresolute continuous function. If (Y,σ) is* P*-expandable, then so is (X,τ).*

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ be a locally finite collection of subsets of X, by Lemma 1.5, $\{f(F_\alpha) : \alpha \in \Delta\}$ is a locally finite collection in Y. Hence there is a p-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subsets of Y such that $f(F_\alpha) \subseteq G_\alpha$ for each $\alpha \in \Delta$. Then, by Theorem 1.6, $F_\alpha \subseteq f^{-1}f(F_\alpha) \subseteq f^{-1}(G_\alpha)$ and $\{f^{-1}(G_\alpha) : \alpha \in \Delta\}$ is a p-locally finite collection of preopen subsets of X. □

It clear that every continuous open function is preirresolute and M-preopen [8].

Corollary 3.11. *Let (X,τ) be compact and (Y,σ) be* P*-expandable. Then the product space $(X,\tau) \times (Y,\sigma)$ is P-expandable.*

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