CHARACTERIZATIONS OF MV-ALGEBRAS IN TERMS OF CUBIC SETS

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Abstract. The operations of cubic sum, cubic product, cubic intersection, cubic union are given in MV-algebras, and the concepts of cubic MV-ideals and cubic prime MV-ideals in MV-algebras are introduced. Then some characterizations of cubic MV-ideals and cubic prime MV-ideals are obtained. The image set of cubic prime MV-ideals is proved to be a chain under the order relation $\preceq$ by discussing the properties of cubic prime ideals, and the cubic prime MV-ideal theory and extension theorem of MV-algebras are presented. Finally, the quotient structure of cubic MV-ideals is constructed by cubic cosets, and three isomorphism theorems concerning the quotient of cubic MV-ideals are presented by using the notion of invariant cubic sets.
Keywords: MV-algebra, cubic MV-ideal, cubic prime MV-ideal, quotient structure.

1. Introduction

Non-classical logic systems which lay logical foundation for dealing with uncertain information processing and fuzzy information in computer science, have become one of the most active research directions in artificial intelligence field. The study of logic algebraic systems not only promotes the development of non-classical mathematical logics, but also enriches the content and methods of algebras. MV-algebras were introduced by Chang [1] as the algebraic counterpart of Łukasiewicz infinite-valued calculus, and MV-algebras entered deeply in many areas of mathematics and logics. The notions of pseudo MV-algebras [2]

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and generalized MV-algebras [3] as two non-commutative but equivalent generalizations of MV-algebras have independently appeared, and they are used for algebraic foundations of non-commutative fuzzy logic.

Ideal theory is a very effectively tool to study logical algebras and the completeness of the corresponding nonclassical logics. On the one hand, ideals are closely related to congruence relations with which one can associate quotient algebras; on the other hand, the sets of provable formulas in the corresponding inference systems from the point of view of uncertain information can be described by fuzzy ideals of those algebraic semantics. A number of researches have motivated to develop nonclassical logics, and also to enrich the ideal theory of algebras [4, 5, 6]. In addition, based on the fuzzy set theory introduced by Zadeh, the related fuzzy structures (i.e., the fuzzification) of ideals in MV-algebras were further studied [7, 8]. Hedayati [9] extended the notions of fuzzy ideals to $(\in, \in \vee q)$-fuzzy (implicative) ideals in pseudo MV-algebras by using the concept of quasicoincidence of a fuzzy value with a fuzzy set. Using falling shadows theory, [10] proposed the concept of falling fuzzy (implicative) ideals which as a generalization of a $T_{\wedge}$-fuzzy (implicative) ideal in MV-algebras. Moreover, based on the concept of the soft set, [11] established the int-soft ideal theory in pseudo MV -algebras.

Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [12] introduced a new notion, called a cubic set, and investigated several properties, then they applied the cubic theory to BCK/BCI-algebras, and proposed cubic P-ideals and cubic α-ideals [13, 14]. Continue the Jun's work in [15], Khan et al. [16] introduced the concepts of cubic h-ideals, cubic h-bi-ideals and cubic h-quasi-ideals in hemirings, and provided some basic properties. Combining cubic sets and soft sets, [17] introduce the notions of cubic soft o-subalgebras and (closed) cubic soft ideals in BCK/BCI-algebras, and investigate related properties.

The paper aims to investigate ideals of MV-algebras based on the cubic theory. The concepts of cubic MV-ideals and cubic prime MV-ideals in MV-algebras are given, and some characterizations of them are present by the introduced cubic operations. Inspired by the fuzzy prime filter theorem in [18], the cubic prime MV-ideal theorem is provided in MV-algebras. A congruence relation on an MV-algebra is constructed via a cubic MV-ideal. Furthermore, a quotient structure of MV-algebras is constructed by cubic cosets, and some certain isomorphism theorems are proved by using the notion of invariant cubic sets.

2. Preliminaries

In this section, we will provide basic terminologies and notations of MV-algebras which are necessary for the understanding of subsequent results.

An algebra $(M, \oplus, \neg, 0)$ of type $(2, 1, 0)$ is called an MV-algebra if it satisfies the following axioms: for any $x, y, z \in M$,

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, (MV2) $x \oplus y = y \oplus x$,

(MV3) $x \oplus 0 = x$,

(MV4) $\neg\neg x = x$,

(MV5) $x \oplus \neg 0 = \neg 0$,

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Let $(M, \oplus, \neg, 0)$ be an MV-algebra, for any $x, y \in M$, we put $1 = \neg 0$, $x \otimes y = \neg(\neg x \oplus \neg y)$, $x \to y = \neg x \oplus y$, $x \ominus y = x \otimes \neg y$, $x \vee y = \neg(\neg x \oplus y) \oplus y = (x \ominus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y) = (x \oplus \neg y) \otimes y$. In what follows, unless mentioned otherwise, $(M, \oplus, \neg, 0)$ is an MV-algebra and will often be referred to by its support set M.

Proposition 2.1 ([19, 20])**.** *Let $(M, \oplus, \neg, 0)$ be an MV-algebra. Then the following assertions are valid: for any $x, y, z, s, t \in M$,*

(1) *$x \leq y$ if and only if $\neg x \oplus y = 1$ if and only if $x \ominus y = 0$;*

(2) *$x \ominus y \leq z$ if and only if $x \leq y \oplus z$;*

(3) *$x \otimes \neg x = 0$, $x \oplus \neg x = 1$, $(x \ominus y) \wedge (y \ominus x) = 0$;*

(4) *$x \otimes y \leq x \wedge y \leq x \vee y \leq x \oplus y$;*

(5) *$x \ominus y = \neg y \ominus \neg x$, $x \ominus z \leq (x \ominus y) \oplus (y \ominus z)$;*

(6) *$(x \oplus s) \ominus (y \oplus t) \leq (x \ominus y) \oplus (s \ominus t)$;*

(7) *if $x \leq y$, then $\neg y \leq \neg x$, $x \otimes z \leq y \otimes z$ and $x \oplus z \leq y \oplus z$;*

(8) *$(x \vee y) \ominus y = x \ominus y$, $x \ominus (x \wedge y) = x \ominus y$.*

Let $(M, \oplus, \neg, 0)$ be an MV-algebra and I a nonempty set of M. Then I is called an ideal of M if it satisfies: for any $x, y \in M$, (1) $x, y \in I$ implies $x \oplus y \in I$; (2) $x \leq y$ and $y \in I$ imply $x \in I$. An ideal I is proper iff $I \neq M$. We say that an ideal P is prime iff it is proper and satisfies for any $x, y \in M$, either $x \ominus y \in P$ or $y \ominus x \in P$.

Filters, the order duals of lattice ideals, have a variety of applications in logic and topology. Since MV-algebra M is a lattice, we can give the notion of lattice filters. A nonempty subset F of M is called a lattice filer if it satisfies: for any $x, y \in M$, (1) $x, y \in F$ implies $x \oplus y \in F$; (2) $x \leq y$ and $x \in F$ imply $y \in F$[21].

Let M_1 and M_2 be MV-algebras. A function $f : M_1 \to M_2$ is a homomorphism iff it satisfies the following conditions: for any $x, y \in M_1$, (1) $f(0) = 0$, (2) $f(x \oplus y) = f(x) \oplus f(y)$, (3) $f(\neg x) = \neg f(x)$.

Now we will recall the concept of interval-valued fuzzy sets. A closed subinterval $\tilde{a} = [a^-, a^+]$ of a closed unit interval $[0, 1]$ is called an interval number, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $D[0,1]$ the set of all interval numbers. We define the operations $\wedge$, $\vee$, $\geq$, $\leq$ and $=$ in case of two elements in $D[0,1]$. Consider two elements $\tilde{a}_1 = [a_1^-, a_1^+], \tilde{a}_2 = [a_2^-, a_2^+]$ in $D[0,1]$, then

(1) $\tilde{a}_1 \geq \tilde{a}_2$ if and only if $a_1^- \geq a_2^-$ and $a_1^+ \geq a_2^+$;

(2) $\tilde{a}_1 \leq \tilde{a}_2$ if and only if $a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$;

(3) $\tilde{a}_1 = \tilde{a}_2$ if and only if $a_1^- = a_2^-$ and $a_1^+ = a_2^+$;

(4) $\tilde{a}_1 \wedge \tilde{a}_2 = [\min\{a_1^-, a_2^-\}, \min\{a_1^+, a_2^+\}]$;

(5) $\tilde{a}_1 \vee \tilde{a}_2 = [\max\{a_1^-, a_2^-\}, \max\{a_1^+, a_2^+\}]$;

(6) $\mathrm{rinf}_{i\in\Lambda}\tilde{a}_i = [\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+]$, where $\tilde{a}_i \in D[0,1]$, $i \in \Lambda$;

(7) $\mathrm{rsup}_{i\in\Lambda}\tilde{a}_i = [\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+]$, where $\tilde{a}_i \in D[0,1]$, $i \in \Lambda$;

other operations $>$ and $<$ can be defined analogously.

An interval-valued fuzzy set (briefly, IVF-set) $\tilde{\mu}_A$ defined on a nonempty set X is given by

$$\tilde{\mu}_A = \{(x, [\mu_A^-(x), \mu_A^+(x)])|x \in X\},$$

where $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in X$. Then the ordinary fuzzy sets $\mu_A^- : X \to [0,1]$ and $\mu_A^+ : X \to [0,1]$ are called a lower fuzzy set and an upper fuzzy set of $\tilde{\mu}_A$, respectively.

3. Cubic MV-ideals of MV-algebras

In this section, we define some cubic operations on MV-algebras, then introduce a new notion called cubic MV-ideal of MV-algebras and study several properties of it.

Definition 3.1 ([12, 13]). *Let X be a nonempty set. A cubic set A in X as an object having the following form:*

$$A = \{(x, \tilde{\mu}_A(x), \lambda_A(x))|x \in X\},$$

which is briefly denoted by $A = (\tilde{\mu}_A, \lambda_A)$, where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an IVF set in X and λ_A is a fuzzy set in X. In order to facilitate our subsequent discussion, for any $x \in X$, the number $A(x) = (\tilde{\mu}_A(x), \lambda_A(x))$ is called a cubic element, where the numbers $\tilde{\mu}_A(x)$ and $\lambda_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A, and $\mu_A^+(x) + \lambda_A(x) \leq 1$.

For two cubic elements $A(x)$ and $A(y)$ of the cubic set A, we give the following operations:

(1) $A(x) \preceq A(y)$ if and only if $\tilde{\mu}_A(x) \leq \tilde{\mu}_A(y)$, $\lambda_A(x) \geq \lambda_A(y)$;

(2) $A(x) \prec A(y)$ if and only if $\tilde{\mu}_A(x) < \tilde{\mu}_A(y)$, $\lambda_A(x) > \lambda_A(y)$;

(3) $A(x) \succeq A(y)$ if and only if $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$, $\lambda_A(x) \leq \lambda_A(y)$;

(4) $A(x) \succ A(y)$ if and only if $\tilde{\mu}_A(x) > \tilde{\mu}_A(y)$, $\lambda_A(x) < \lambda_A(y)$;

(5) $A(x) = A(y)$ if and only if $\tilde{\mu}_A(x) = \tilde{\mu}_A(y)$, $\lambda_A(x) = \lambda_A(y)$;

(6) $A(x) \veebar A(y) = (\tilde{\mu}_A(x) \vee \tilde{\mu}_A(y), \lambda_A(x) \wedge \lambda_A(y))$;

(7) $A(x) \barwedge A(y) = (\tilde{\mu}_A(x) \wedge \tilde{\mu}_A(y), \lambda_A(x) \vee \lambda_A(y))$.

If $A_i = (\tilde{\mu}_{A_i}, \lambda_{A_i})$ $(i \in \Lambda)$ are cubic elements, where Λ is an index set, then we define:

$$\overline{\sup}_{i\in\Lambda} A_i = \left(\mathrm{rsup}_{i\in\Lambda}\tilde{\mu}_{A_i}, \inf_{i\in\Lambda} \lambda_{A_i}\right).$$

Let X be a nonempty set and A a nonempty subset of X. The cubic characteristic function of A is defined as $\chi_A = \{(x, \tilde{\mu}_{\chi_A}(x), \lambda_{\chi_A}(x))|x \in X\}$, where

$$\tilde{\mu}_{\chi_A}(x) = \begin{cases} [1,1], & x \in A, \\ [0,0], & \text{otherwise}, \end{cases} \quad \lambda_{\chi_A}(x) = \begin{cases} 0, & x \in A, \\ 1, & \text{otherwise}. \end{cases}$$

Let $A = (\tilde{\mu}_A, \lambda_A)$ and $B = (\tilde{\mu}_B, \lambda_B)$ be two cubic sets of X, we put $A \sqsubseteq B$ if and only if $A(x) \preceq B(x)$ for any $x \in X$; $A \sqsubset B$ if and only if $A(x) \prec B(x)$ for any $x \in X$.

In what follows, we introduce the operations $\circledast$ and $\circledcirc$ which provide interesting further characterizations of cubic MV-ideals in the subsequent discussions.

Definition 3.2. *Let $A = (\tilde{\mu}_A, \lambda_A)$ and $B = (\tilde{\mu}_B, \lambda_B)$ be two cubic sets of M. Then:*

(1) *the cubic sum $\circledast$ of A and B is defined as*

$$A \circledast B = \{(x, (A \circledast B)(x))|x \in M\} := \{(x, (\tilde{\mu}_A + \tilde{\mu}_B)(x), (\lambda_A + \lambda_B)(x))|x \in M\},$$

where $(A \circledast B)(x) = \overline{\sup}\{A(y) \barwedge A(z)|x = y \oplus z, y, z \in M\}$.

(2) *the cubic product $\circledast$ of A and B is defined as*

$$A \circledcirc B = \{(x, (A \circledcirc B)(x))|x \in M\} := \{(x, (\tilde{\mu}_A \circ \tilde{\mu}_B)(x), (\lambda_A \circ \lambda_B)(x))|x \in M\},$$

where $(A \circledcirc B)(x) = \overline{\sup}\{A(y) \barwedge A(z)|x = y \otimes z, y, z \in M\}$.

Inspired by [16], we can also give the intersection and union of two cubic sets as follows. Let A and B be cubic sets of an MV-algebra M. The intersection and union of A and B, denote by $A \sqcap B$ and $A \sqcup B$ respectively, are cubic sets:

$$A \sqcap B = \{(x, A(x) \barwedge B(x))|x \in M\} := \{(x, (\tilde{\mu}_A \Cup \tilde{\mu}_B)(x), (\lambda_A \Cap \lambda_B)(x))|x \in M\},$$
$$A \sqcup B = \{(x, A(x) \veebar B(x))|x \in M\} := \{(x, (\tilde{\mu}_A \Cap \tilde{\mu}_B)(x), (\lambda_A \Cup \lambda_B)(x))|x \in M\}.$$

Lemma 3.3. *For any cubic sets A, B, C, D and E in an MV-algebra M, then we have:*

(1) $A \circledast (B \sqcup C) = (A \circledast B) \sqcup (A \circledast C)$;

(2) $A \circledcirc (B \sqcup C) = (A \circledcirc B) \sqcup (A \circledcirc C)$;

(3) $A \circledast (B \sqcap C) = (A \circledast B) \sqcap (A \circledast C)$;

(4) $A \circledcirc (B \sqcap C) = (A \circledcirc B) \sqcap (A \circledcirc C)$;

(5) $A \sqcup (B \sqcap C) = (A \sqcup B) \sqcap (A \sqcup C)$;

(6) $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C)$;

(7) $A \circledast (B \circledast C) = (A \circledast B) \circledast C$;

(8) $A \circledcirc (B \circledcirc C) = (A \circledcirc B) \circledcirc C$;

(9) *if* $D \sqsubseteq A$, $E \sqsubseteq B$, *then* $D \circledast E \sqsubseteq A \circledast B$;

(10) *if* $D \sqsubseteq A$, $E \sqsubseteq B$, *then* $D \circledcirc E \sqsubseteq A \circledcirc B$.

Proof. (5), (6), (9) and (10) are straightforward. The proofs of (2), (3) and (4) are similar to that of (1), and the proof of (8) is similar to that of (7), therefore, we only give the proofs of (1) and (7).

(1) For any $x \in M$, assume that there exist $y, z \in X$ such that $x = y \oplus z$, then

$$\begin{aligned}
&(\tilde{\mu}_A + (\tilde{\mu}_B \Cup \tilde{\mu}_C))(x) = \text{rsup}\,\{\tilde{\mu}_A(y) \wedge (\tilde{\mu}_B \Cup \tilde{\mu}_C)(z) | x = y \oplus z\}\\
&= \text{rsup}\,\{\tilde{\mu}_A(y) \wedge (\tilde{\mu}_B(z) \vee \tilde{\mu}_C(z)) | x = y \oplus z\}\\
&= \text{rsup}\,\{(\tilde{\mu}_A(y) \wedge \tilde{\mu}_B(z)) \vee (\tilde{\mu}_A(y) \wedge \tilde{\mu}_C(z)) | x = y \oplus z\}\\
&= \text{rsup}\,\{(\tilde{\mu}_A(y) \wedge \tilde{\mu}_B(z)\} \vee \text{rsup}\,\{(\tilde{\mu}_A(y) \wedge \tilde{\mu}_C(z) | x = y \oplus z\}\\
&= (\tilde{\mu}_A + \tilde{\mu}_B)(x) \vee (\tilde{\mu}_A + \tilde{\mu}_C)(x)\\
&= ((\tilde{\mu}_A + \tilde{\mu}_B) \Cup (\tilde{\mu}_A + \tilde{\mu}_C))(x),\\
&(\lambda_A + (\lambda_B \Cap \lambda_C))(x) = \inf\,\{\max\,\{\lambda_A(y), (\lambda_B \Cap \lambda_C)(z)\}\,| x = y \oplus z\}\\
&= \inf\,\{\max\,\{\lambda_A(y), \min\,\{\lambda_B(z), \lambda_C(z)\}\}\,| x = y \oplus z\}\\
&= \inf\,\{\min\,\{\max\,\{\lambda_A(y), \lambda_B(z)\}\,, \max\,\{\lambda_A(y), \lambda_C(z)\}\}\,| x = y \oplus z\}\\
&= \min\{\inf\{\max\{\lambda_A(y), \lambda_B(z)\}| x{=}y \oplus z\}, \inf\{\max\{\lambda_A(y), \lambda_C(z)\}| x = y \oplus z\}\}\\
&= \min\,\{(\lambda_A + \lambda_B)(x), (\lambda_A + \lambda_C)(x)\}\\
&= ((\lambda_A + \lambda_B) \Cap (\lambda_A + \lambda_C))(x).
\end{aligned}$$

Hence, $A \circledast (B \sqcup C) = (A \circledast B) \sqcup (A \circledast C)$

(7) For any $x \in M$, assume that there exist $y, z \in M$ such that $x = y \oplus z$, then

$$
\begin{aligned}
&(\tilde{\mu}_A + (\tilde{\mu}_B + \tilde{\mu}_C))(x) = \mathrm{rsup}_{x=y\oplus z}\{\tilde{\mu}_A(y) \wedge (\tilde{\mu}_B + \tilde{\mu}_C)(z)\} \\
&= \mathrm{rsup}_{x=y\oplus z}\left\{\tilde{\mu}_A(y) \wedge \mathrm{rsup}_{z=z_1\oplus z_2}\{\tilde{\mu}_B(z_1) \wedge \tilde{\mu}_C(z_2)\}\right\} \\
&= \mathrm{rsup}_{x=y\oplus z}\left\{\mathrm{rsup}_{z=z_1\oplus z_2}\{\tilde{\mu}_A(y) \wedge \tilde{\mu}_B(z_1) \wedge \tilde{\mu}_C(z_2)\}\right\} \\
&= \mathrm{rsup}_{x=y\oplus (z_1\oplus z_2)}\{\tilde{\mu}_A(y) \wedge \tilde{\mu}_B(z_1) \wedge \tilde{\mu}_C(z_2)\}, \\
&((\tilde{\mu}_A + \tilde{\mu}_B) + \tilde{\mu}_C)(x) = \mathrm{rsup}_{x=y\oplus z}\{(\tilde{\mu}_A + \tilde{\mu}_B)(y) \wedge \tilde{\mu}_C(z)\} \\
&= \mathrm{rsup}_{x=y\oplus z}\left\{\mathrm{rsup}_{y=y_1\oplus y_2}\{\tilde{\mu}_A(y_1) \wedge \tilde{\mu}_B(y_1)\} \wedge \tilde{\mu}_C(z)\right\} \\
&= \mathrm{rsup}_{x=y\oplus z}\left\{\mathrm{rsup}_{y=y_1\oplus y_2}\{\tilde{\mu}_A(y_1) \wedge \tilde{\mu}_B(y_2) \wedge \tilde{\mu}_C(z)\}\right\} \\
&= \mathrm{rsup}_{x=(y_1\oplus y_2)\oplus z}\{\tilde{\mu}_A(y_1) \wedge \tilde{\mu}_B(y_2) \wedge \tilde{\mu}_C(z) \in M\},
\end{aligned}
$$

since $x = y \oplus (z_1 \oplus z_2) = (y_1 \oplus y_2) \oplus z$, then $(\tilde{\mu}_A + (\tilde{\mu}_B + \tilde{\mu}_C))(x) = ((\tilde{\mu}_A + \tilde{\mu}_B) + \tilde{\mu}_C)(x)$.

$$
\begin{aligned}
&(\lambda_A + (\lambda_B + \lambda_C))(x) = \inf_{x=y\oplus z}\{\max\{\lambda_A(y), (\lambda_B + \lambda_C)(z)\}\} \\
&= \inf_{x=y\oplus z}\left\{\max\left\{\lambda_A(y), \inf_{z=z_1\oplus z_2}\{\max\{\lambda_B(z_1), \lambda_C(z_2)\}\}\right\}\right\} \\
&= \inf_{x=y\oplus z}\left\{\inf_{z=z_1\oplus z_2}\{\max\{\lambda_A(y), \lambda_B(z_1), \lambda_C(z_2)\}\}\right\} \\
&= \inf_{x=y\oplus (z_1\oplus z_2)}\{\max\{\lambda_A(y), \lambda_B(z_1), \lambda_C(z_2)\}\}, \\
&((\lambda_A + \lambda_B) + \lambda_C)(x) \\
&= \inf_{x=y\oplus z}\{\max\{(\lambda_A + \lambda_B)(y), \lambda_C(z)\}\} \\
&= \inf_{x=y\oplus z}\left\{\max\left\{\inf_{y=y_1\oplus y_2}\{\max\{\lambda_A(y_1), \lambda_B(y_2)\}\}, \lambda_C(z)\right\}\right\} \\
&= \inf_{x=y\oplus z}\left\{\inf_{y=y_1\oplus y_2}\{\max\{\lambda_A(y_1), \lambda_B(y_2), \lambda_C(z)\}\}\right\} \\
&= \inf_{x=(y_1\oplus y_2)\oplus z}\{\max\{\lambda_A(y_1), \lambda_B(y_2), \lambda_C(z)\}\},
\end{aligned}
$$

since $x = y \oplus (z_1 \oplus z_2) = (y_1 \oplus y_2) \oplus z$, hence

$$
\begin{aligned}
&\inf_{x=y\oplus (z_1\oplus z_2)}\{\max\{\lambda_A(y), \lambda_B(z_1), \lambda_C(z_2)\}\} \\
&= \inf_{x=y\oplus (z_1\oplus z_2)}\{\max\{\lambda_A(y), \lambda_B(z_1), \lambda_C(z_2)\}\},
\end{aligned}
$$

and so $(\lambda_A + (\lambda_B + \lambda_C))(x) = ((\lambda_A + \lambda_B) + \lambda_C)(x)$.

Therefore $A \circledast (B \circledast C) = (A \circledast B) \circledast C$. □

Lemma 3.4. *Let A and B be nonempty subsets of an MV-algebra M. Then the followings hold:*

(1) *$A \subseteq B$ if and only if $\chi_A \sqsubseteq \chi_B$;*

(2) *$\chi_A \sqcup \chi_B = \chi_{A\cup B}$, $\chi_A \sqcap \chi_B = \chi_{A\cap B}$;*

(3) *$\chi_A \circledast \chi_B = \chi_{A\oplus B}$, where $A \oplus B = \{x \oplus y | x \in A, y \in B\}$;*

(4) *$\chi_A \circledcirc \chi_B = \chi_{A\otimes B}$, where $A \otimes B = \{x \otimes y | x \in A, y \in B\}$.*

Proof. (1) and (2) is obviously, the proof of (4) is similar to that of (3), here we only need to prove (3). For any $x \in M$, we consider two cases.

(i) if $x \in A \oplus B$, then there exist $y_1 \in A$ and $z_1 \in B$ such that $x = y_1 \oplus z_1$. Then $\tilde{\mu}_{\chi_{A\oplus B}}(x) = [1,1]$. Since $(\tilde{\mu}_{\chi_A} + \tilde{\mu}_{\chi_B})(x) = \text{rsup}\,\{\tilde{\mu}_{\chi_A}(y) \wedge \tilde{\mu}_{\chi_B}(z) | x = y \oplus z\}$ $\geq \tilde{\mu}_{\chi_A}(y_1) \wedge \tilde{\mu}_{\chi_B}(z_1) = [1,1]$, then $(\tilde{\mu}_{\chi_A} + \tilde{\mu}_{\chi_B})(x) = [1,1] = (\tilde{\mu}_{\chi_{A\oplus B}}(x)$, and so, $\tilde{\mu}_{\chi_{A\oplus B}}(x) = [1,1]$.

$(\lambda_{\chi_A} + \lambda_{\chi_B})(x) = \inf\limits_{x=y\oplus z} \{\max\{\lambda_{\chi_A}(y), \lambda_{\chi_B}(z)\}\} \leq \max\{\lambda_{\chi_A}(y_1), \lambda_{\chi_B}(z_1)\} = 0$, therefore $(\lambda_{\chi_A} + \lambda_{\chi_B})(x) = 0 = \lambda_{\chi_{A\oplus B}}(x)$.

(ii) if $x \notin A \oplus B$, then there exist $y_1 \in M\backslash A$ or $z_1 \in M\backslash A$ such that $x = y_1 \oplus z_1$, then $(\tilde{\mu}_{\chi_A} + \tilde{\mu}_{\chi_B})(x) = \text{rsup}\,\{\tilde{\mu}_{\chi_A}(y) \wedge \tilde{\mu}_{\chi_B}(z) | x = y \oplus z\} = [0,0] = \tilde{\mu}_{\chi_{A\oplus B}}(x)$, and $(\lambda_{\chi_A} + \lambda_{\chi_B})(x) = \inf\limits_{x=y\oplus z} \{\max\{\lambda_{\chi_A}(y) \wedge \lambda_{\chi_B}(z)\}\} = 1 = \lambda_{\chi_{A\oplus B}}(x)$.

Therefore, we have $\chi_A \circledast \chi_B = \chi_{A\oplus B}$. □

Definition 3.5. *Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set of an MV-algebra M. Then A is called a cubic MV-ideal of M if it satisfies the following conditions: for any $x, y \in M$, if (1) $A(x) \barwedge A(y) \preceq A(x \oplus y)$; (2) $x \leq y$ implies $A(y) \preceq A(x)$.*

For better understanding the notion of cubic MV-ideals, we illustrate it by the following example.

Example 3.6. Let $M = \{0, a, b, 1\}$ be a set such that $0 < a < 1$ and $0 < b < 1$. The operations $\oplus$ and $\neg$ are defined as follows:

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$\neg$	0	a	b	1
	1	b	a	0

then $(M, \oplus, \neg, 0)$ is an MV-algebra. Define a cubic set $A = (\tilde{\mu}_A, \lambda_A)$ in M as follows:

$$\tilde{\mu}_A(x) = \begin{cases} [0.8, 0.9], & x = 0, \\ [0.3, 0.4], & x = a, \\ [0.2, 0.5], & x = b, \\ [0.1, 0.3], & x = 1; \end{cases} \qquad \lambda_A(x) = \begin{cases} 0.1, & x = 0, \\ 0.5, & x = a, \\ 0.4, & x = b, \\ 0.6, & x = 1, \end{cases}$$

it is to check that A is a cubic MV-ideal of M.

(2) Let I be an ideal of MV-algebra M, $\Gamma_1 = (\tilde{\alpha}_1, \beta_1)$ and $\Gamma_2 = (\tilde{\alpha}_2, \beta_2)$ be cubic elements such that $(\tilde{\alpha}_1, \beta_1) \prec (\tilde{\alpha}_2, \beta_2)$. Define a function as follows:

$$I_{\Gamma_1}^{\Gamma_2}(x) = \begin{cases} \Gamma_2, & x \in I, \\ \Gamma_1, & x \notin I. \end{cases}$$

Routine calculation shows that $I_{\Gamma_1}^{\Gamma_2}$ is a cubic MV-ideal of M. Here the cubic set $I_{\Gamma_1}^{\Gamma_2}$ is called the generalized cubic characteristic function of I.

Proposition 3.7. *Let A be a cubic set of an MV-algebra M. Then A is a cubic MV-ideal of M if and only if for any $x, y \in M$,*

(1) $A(x) \preceq A(0)$;

(2) $A(y) \barwedge A(x \ominus y) \preceq A(x)$.

Proof. The sufficiency is very clear, we now give the proof of the necessity. For any $x, y \in M$, if $x \leq y$, then $A(x) \succeq A(y) \barwedge A(x \ominus y) = A(y) \barwedge A(0) = A(y)$. Notice that $(x \oplus y) \ominus y \leq x$, we have $A((x \oplus y) \ominus y) \succeq A(x)$, and $A(x \oplus y) \succeq A(y) \barwedge A((x \oplus y) \ominus y) \succeq A(x) \barwedge A(y)$, therefore A is a cubic MV-ideal of M. □

Note that the concept of level sets in the fuzzy set theory, Khan et al. [16] give the notion of cubic level sets which serves as a bridge between of cubic sets and crisp sets.

Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set of a nonempty set X, $r \in [0, 1]$ and $[s, t] \in D[0, 1]$ such that $r + t \leq 1$. The set

$$L(A; ([s, t], r)) = \{x \in X | \tilde{\mu}_A(x) \geq [s, t], \lambda_A(x) \leq r\}$$

is called a $([s, t], r)$-cubic level set of A. The proof of the next proposition is obviously, and will be omitted.

Proposition 3.8. *Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set of M. Then the following statements are equivalent:*

(1) *A is a cubic MV-ideal;*

(2) *for any $r \in [0, 1]$, $[s, t] \in D[0, 1]$ and $r + t \leq 1$, the nonempty cubic level set $L(A; ([s, t], r))$ is an ideal of M.*

Analogues to the notion of cubic MV-ideals, we can present the notion of cubic lattice filters as follows.

Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set of an MV-algebra M. Then A is called a cubic lattice filter of M if it satisfies that for any $x, y \in M$, $A(x) \barwedge A(y) = A(x \wedge y)$. It is easy verify that if A is a cubic lattice filter of M, then for any $r \in [0, 1]$, $[s, t] \in D[0, 1]$ and $r + t \leq 1$, the nonempty cubic level set $L(A; ([s, t], r))$ is a lattice filter of M.

Theorem 3.9. *Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic set of an MV-algebra M. Then A is a cubic MV-ideal of M if and only if $z \ominus x \leq y$ implies $A(x) \barwedge A(y) \preceq A(z)$ for any $x, y, z \in M$.*

Proof. Assume that A is a cubic MV-ideal of M and there exist $x, y, z \in M$ such that $z \ominus x \leq y$, then $A(z \ominus x) \succeq A(y)$. In view of Proposition 3.7, we have $A(z) \succeq A(x) \barwedge A(z \ominus x) \succeq A(x) \barwedge A(y)$.

Conversely, it follows immediately from $0 \ominus x = 0 \leq x$ that $A(0) \succeq A(x) \barwedge A(x) = A(x)$. Notice that $x \ominus (x \ominus y) \leq y$, we obtain that $A(x) \succeq A(y) \barwedge A(x \ominus y)$. Thus A is a cubic MV-ideal of M. □

Proposition 3.10. *Let A be a cubic set of an MV-algebra M. Then A is a cubic MV-ideal of M if and only if for any $x, y \in M$,*

(1) $A(x) \barwedge A(y) \preceq A(x \oplus y)$;

(2) $A(y) \preceq A(x \otimes y)$.

Proof. The proof will be complete if we show that (2) is equivalent to the condition (2) of Definition 3.5. Assume that A is a cubic MV-ideal of M, since $x \otimes y \leq y$ for any $x, y \in M$, then we get the condition (2) of Proposition 3.10.

Conversely, suppose that the condition (2) of Proposition 3.10 holds. For any $x, y \in M$, if $x \leq y$, then $(\neg y \oplus x) \otimes y = x \wedge y = x$, and hence $A(x) = A((\neg y \oplus x) \otimes y) \succeq A(y)$, therefore (2) of Definition 3.5 is valid. □

Proposition 3.11. *Let A be a cubic MV-ideal of M. Then the following results hold: for any $x, y, z \in M$,*

(1) *if* $A(x \ominus y) = A(0)$, *then* $A(y) \preceq A(x)$;

(2) $A(x \vee y) = A(x) \barwedge A(y)$;

(3) $A(x \oplus y) = A(x) \barwedge A(y)$;

(4) $A(x) \barwedge A(\neg x) = A(0)$;

(5) $A(x \ominus y) \barwedge A(y \ominus z) \preceq A(x \ominus z)$.

Proof. (1) Since A is a cubic MV-ideal of M, then we have $A(y) \barwedge A(x \ominus y) = A(y) \preceq A(x)$ by Proposition 3.7.

(2) Using $(x \vee y) \leq x \oplus y$ and together with Theorem 3.9, we obtain that $A(x) \barwedge A(y) \preceq A(x \vee y)$. As for the reverse inequality, from $x, y \leq x \vee y$, we have $A(x \vee y) \preceq A(x)$ and $A(x \vee y) \preceq A(y)$, and so $A(x \vee y) \preceq A(x) \barwedge A(y)$. Hence (2) is valid.

(3) Since $(x \vee y) \leq x \oplus y$, one more application of Definition 3.5 yields $A(x \oplus y) \preceq A(x \vee y) = A(x) \barwedge A(y)$. The reverse inequality follows from Proposition 3.10, which completes the proof of (3).

(4) is a consequence of (3).

(5) is immediately from Proposition 2.1 (5) and Theorem 3.9. □

Theorem 3.12. *Let A be a cubic set of M. Then A is a cubic MV-ideal of M if and only if the following conditions are valid:*

(1) $A \circledast A \sqsubseteq A$;

(2) $\chi_M \circledcirc A \sqsubseteq A$.

Proof. Assume that A is a cubic MV-ideal of M and x is an element of M. We will first show that (1) holds. Let x be expressed as $x = y \oplus z$ for some $y, z \in M$. Then we get that $\tilde{\mu}_A(x) = \tilde{\mu}_A(y \oplus z) \geq \tilde{\mu}_A(y) \wedge \tilde{\mu}_A(z)$, $\lambda_A(x) = \lambda_A(y \oplus z) \leq \max\{\lambda_A(y), \lambda_A(z)\}$. And so $\tilde{\mu}_A(x) \geq \text{rsup}\{\tilde{\mu}_A(y) \wedge \tilde{\mu}_A(z) | x = y \oplus z\} = (\tilde{\mu}_A + \tilde{\mu}_A)(x)$, $\lambda_A(x) \leq \inf\{\max\{\lambda_A(y), \lambda_A(z)\} | x = y \oplus z\} = (\lambda_A + \lambda_A)(x)$, hence $A \circledast A \sqsubseteq A$.

For any $x \in M$, we have $(\tilde{\mu}_{\chi_M} \circ \tilde{\mu}_A)(x) = \text{rsup}\{\tilde{\mu}_{\chi_M}(y) \wedge \tilde{\mu}_A(z) | x = y \otimes z\} = \text{rsup}\{[1,1] \wedge \tilde{\mu}_A(z) | x = y \otimes z\} = \text{rsup}\{\tilde{\mu}_A(z) | x = y \otimes z\} \leq \text{rsup}\{\tilde{\mu}_A(y \otimes z) | x = y \otimes z\} = \tilde{\mu}_A(x)$, and $(\lambda_{\chi_M} \circ \lambda_A)(x) = \inf\{\max\{\lambda_{\chi_M}(y), \lambda_A(z)\} | x = y \otimes z\} = \inf\{\max\{0, \lambda_A(z)\} | x = y \otimes z\} = \inf\{\lambda_A(z) | x = y \otimes z\} \geq \inf\{\lambda_A(y \otimes z) | x = y \otimes z\} = \lambda_A(x)$, therefore $\chi_M \circledcirc A \sqsubseteq A$.

Conversely, suppose that $A \circledast A \sqsubseteq A$ and $\chi_M \circledcirc A \sqsubseteq A$. For any $x, y \in M$, we get that $\tilde{\mu}_A(x \oplus y) \geq (\tilde{\mu}_A + \tilde{\mu}_A)(x \oplus y) \geq \tilde{\mu}_A(x) \wedge \tilde{\mu}_A(y)$ and $\lambda_A(x \oplus y) \leq (\lambda_A + \lambda_A)(x \oplus y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$, that is $A(x) \barwedge A(y) \preceq A(x \oplus y)$. Due to the fact that $\tilde{\mu}_A(x \otimes y) \geq (\tilde{\mu}_{\chi_M} \circ \tilde{\mu}_A)(x \otimes y) \geq \tilde{\mu}_{\chi_M}(x) \wedge \tilde{\mu}_A(y) = \tilde{\mu}_A(y)$ and $\lambda_A(x \otimes y) \leq (\lambda_{\chi_M} \circ \lambda_A)(x \otimes y) \leq \max\{\lambda_{\chi_M}(x), \lambda_A(y)\} = \lambda_A(y)$, that is, $A(y) \preceq A(x \otimes y)$, we get that A is a cubic MV-ideal of M by Proposition 3.10. □

Proposition 3.13. *Let A and B be cubic MV-ideals of M. Then the following results are valid:*

(1) *$A \sqcap B$ is a cubic MV-ideal of M;*

(2) *if $A \circledast B$ is an inverse isotone mapping, then $A \circledast B$ is a cubic MV-ideal of M.*

Proof. (1) It is obviously.

(2) For any $x, y, z \in M$, if $z \ominus x \leq y$, that is, $z \leq x \oplus y$, according to Proposition 3.11 (3), we have $(A \circledast B)(x) \barwedge (A \circledast B)(y) = \overline{\sup}\{A(x_1) \barwedge B(x_2) | x = x_1 \oplus x_2\} \barwedge \overline{\sup}\{A(y_1) \barwedge B(y_2) | y = y_1 \oplus y_2\} = \overline{\sup}\{A(x_1) \barwedge B(x_2) \barwedge A(y_1) \barwedge B(y_2) | x = x_1 \oplus x_2, y = y_1 \oplus y_2\} \preceq \overline{\sup}\{(A(x_1) \barwedge A(y_1)) \barwedge (B(x_2) \barwedge B(y_2)) | x \oplus y = (x_1 \oplus y_1) \oplus (x_2 \oplus y_2)\} = \overline{\sup}\{(A(x_1 \oplus y_1)) \barwedge (B(x_2 \oplus y_2)) | x \oplus y = (x_1 \oplus y_1) \oplus (x_2 \oplus y_2)\} = (A \circledast B)(x \oplus y) \preceq (A \circledast B)(z)$. Thus $A \circledast B$ is a cubic MV-ideal of M. □

Let X be a non-empty set and $(\tilde{\alpha}, \beta)$ a cubic element. For any $x \in X$, if $\mathcal{C}_{(\tilde{\alpha},\beta)}(x) = (\tilde{\alpha}, \beta)$, then $\mathcal{C}_{(\tilde{\alpha},\beta)}$ is called a constant cubic set of X. For the sake of convenience, a nonconstant cubic MV-ideal is called a proper cubic MV-ideal. We define the image set $Im(A)$ of the cubic set A of X as: $Im(A) = \{A(x) | x \in X\}$.

Definition 3.14. *Let A be a proper cubic MV-ideal of M. For any $x, y \in M$, if $A(x \ominus y) = A(0)$ or $A(y \ominus x) = A(0)$, then A is called a cubic prime MV-ideal.*

In what follows we will show some characterizations of cubic prime MV-ideals.

Proposition 3.15. *Let A be a proper cubic MV-ideal of M. Then A is a cubic prime MV-ideal of M if and only if for any $r \in [0,1]$, $[s,t] \in D[0,1]$ and $r + t \leq 1$, the nonempty cubic level set $L(A; [s,t], r)$ is a prime ideal of M.*

Proposition 3.16. *Let A be a proper cubic MV-ideal of M. Then the following assertions are equivalent:*

(1) *A is a cubic prime MV-ideal of M;*

(2) *$A(x \wedge y) = A(0)$ implies $A(x) = A(0)$ or $A(y) = A(0)$ for any $x, y \in M$.*

Proof. Assume that A is a cubic prime MV-ideal of M. Let $A(x \wedge y) = A(0)$. Notice that $x \ominus (x \wedge y) = x \ominus y$ and A is a cubic MV-ideal of M, we have $A(x) \succeq A(x \wedge y) \barwedge A(x \ominus (x \wedge y)) = A(x \ominus (x \wedge y)) = A(x \ominus y)$ by Proposition 3.7, and so $A(0) \succeq A(x) \succeq A(x \ominus y)$. Similarly $A(0) \succeq A(y) \succeq A(y \ominus x)$. From A is a cubic prime MV-ideal, it follows that $A(x \ominus y) = A(0)$ or $A(y \ominus x) = A(0)$, hence $A(x) = A(0)$ or $A(y) = A(0)$.

Conversely, suppose that (2) is valid. For any $x, y \in M$, $(x \ominus y) \wedge (y \ominus x) = 0$, then $A((x \ominus y) \wedge (y \ominus x)) = A(0)$. Then $A(x \ominus y) = A(0)$ or $A(y \ominus x) = A(0)$ by hypothesis, therefore A is a cubic prime MV-ideal. □

Theorem 3.17. *Let A be a proper cubic MV-ideal of M. Then A is a cubic prime MV-ideal of M if and only if $Im(A)$ is a chain under the order relation $\preceq$, and $A(x \wedge y) = A(x) \veebar A(y)$ for any $x, y \in M$.*

Proof. Suppose that A is a cubic prime MV-ideal of M, then $A(x \ominus y) = A(0)$ or $A(y \ominus x) = A(0)$ for any $x, y \in M$. If $A(x \ominus y) = A(0)$, consider that $x \ominus (x \ominus y) \leq x \wedge y$, we get that $A(x) \succeq A(x \ominus y) \barwedge A(x \wedge y) = A(x \wedge y)$ by Theorem 3.9. Combining with $A(x \wedge y) \succeq A(x)$ and $A(x \wedge y) \succeq A(y)$, we have $A(x \wedge y) = A(x)$ and $A(x) \succeq A(y)$. Similarly, if $A(y \ominus x) = A(0)$, we can prove that $A(x \wedge y) = A(y)$ and $A(y) \succeq A(x)$. Hence, $Im(A)$ is a chain under the order relation $\preceq$, and $A(x \wedge y) = A(x) \veebar A(y)$.

Conversely, due to the fact that $(x \ominus y) \wedge (y \ominus x) = 0$, we obtain that $A(0) = A((x \ominus y) \wedge (y \ominus x)) = A(x \ominus y) \veebar A(y \ominus x)$. Since $Im(A)$ is a chain, then $A(0) = A(x \ominus y)$ or $A(0) = A(y \ominus x)$, thus A is a cubic prime MV-ideal. □

Proposition 3.18. *Let A, B be cubic sets of M. If A is a cubic prime MV-ideal of M, and B is a proper cubic MV-ideal of M such that $A \sqsubseteq B$ and $A(0) = B(0)$, then B is a cubic prime MV-ideal of M.*

Proof. Since A is a cubic prime MV-ideal of M, then $A(x \ominus y) = A(0)$ or $A(y \ominus x) = A(0)$ for any $x, y \in M$. If $A(x \ominus y) = A(0)$, from $A \sqsubseteq B$ and $A(0) = B(0)$ it follows that $B(x \ominus y) = B(0)$. Similarly, $B(y \ominus x) = B(0)$ if $A(y \ominus x) = A(0)$. Thus, B is a cubic prime MV-ideal. □

Proposition 3.19. *Let A and $\mathcal{C}_{(\tilde{\alpha},\beta)}$ be a cubic set and a constant cubic set of M, respectively. If A is a cubic prime MV-ideal of M and $(\tilde{\alpha}, \beta) \prec A(0)$, then $A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$ is a cubic prime MV-ideal of M.*

Proof. For any $x, y, z \in M$, if $z \ominus x \leq y$, then $A(z) \succeq A(x) \barwedge A(y)$ by Theorem 3.9, and so $(A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)})(z) = A(z) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(z) \succeq (A(x) \barwedge A(y)) \vee \mathcal{C}_{(\tilde{\alpha},\beta)}(z) = (A(x) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(z)) \barwedge (A(y) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(z)) = (A(x) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(x)) \barwedge (A(y) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(y)) = (A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)})(x) \barwedge (A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)})(y)$. Therefore, $A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$ is a cubic MV-ideal of M.

Nextly, we will show that $A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$ is prime. In view that A is a cubic prime MV-ideal of M, and $(\tilde{\alpha}, \beta) \prec A(0)$, we get that $(A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)})(0) = A(0) \veebar \mathcal{C}_{(\tilde{\alpha},\beta)}(0) = A(0) \veebar (\tilde{\alpha}, \beta) = A(0) \neq A(1)$, thus $A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$ is a proper cubic MV-ideal of M. Since $(A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)})(0) = A(0)$ and $A \sqsubseteq A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$, then $A \sqcup \mathcal{C}_{(\tilde{\alpha},\beta)}$ is a cubic prime MV-ideal of M by Proposition 3.18 . □

Definition 3.20. *Let $(\tilde{\alpha}, \beta)$ be a cubic element, and X be a set of some cubic elements. The cubic element $(\tilde{\alpha}, \beta)$ satisfies the chain property on X if it is comparable with all elements of X.*

Lemma 3.21. [19] *Let I be an ideal of MV-algebra M and $F(\neq \emptyset)$ be a lattice filter of M with $I \cap F = \emptyset$. There is a prime ideal P of M such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Inspired by the fuzzy prime filter theorem in [18], we give the cubic prime ideal theorem in MV-algebras as follows.

Theorem 3.22. (*Cubic Prime Ideal Theory*) *Let A be a proper cubic MV-ideal of M with $A(0) \prec ([1, 1], 0)$. Suppose that there is a cubic lattice filter B of M such that $A \sqcap B \sqsubset \mathcal{C}_{(\tilde{\alpha},\beta)}$, and $(\tilde{\alpha}, \beta)$ satisfies the chain property on $Im(A) \cup Im(B)$, then there is a cubic prime MV-ideal D such that $A \sqsubseteq D$ and $D \sqcap B \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$.*

Proof. Since A is a proper cubic MV-ideal, then $L(A, A(0))$ is a proper ideal of M. Nextly, we consider the following three cases relative to the $(\tilde{\alpha}, \beta)$-cubic level sets of A and B:

Case (1): $L(B; (\tilde{\alpha}, \beta)) = \emptyset$. It follows that $B \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$. Taking $F = \{0\}$, then there is a prime ideal P of M such that $L(A, A(0)) \subseteq P$ by Lemma 3.21. Here we put $D = P_{A(0)}^{([1,1],0)}$, that is, D is the generalized cubic characteristic function of the ideal P. And so D is a cubic prime MV-ideal of M, with $A \sqsubseteq D$ and $D \sqcap B \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$.

Case (2): $L(A; (\tilde{\alpha}, \beta)) = \emptyset$. We obtain that $A \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$ and $A(0) \prec (\tilde{\alpha}, \beta)$. If P is the prime ideal given in case (1), and $D = P_{A(0)}^{(\tilde{\alpha},\beta)}$, then D is a cubic prime MV-ideal of M, with $A \sqsubseteq D \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$ and $D \sqcap B \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$.

Case (3): $L(A;(\tilde{\alpha},\beta)) \neq \emptyset$ and $L(B;(\tilde{\alpha},\beta)) \neq \emptyset$. Then $L(B;(\tilde{\alpha},\beta))$ is a lattice filter of M, and $(\tilde{\alpha},\beta) \preceq A(0)$. From $A \sqcap B \sqsubset \mathcal{C}_{(\tilde{\alpha},\beta)}$, we obtain that $L(A;(\tilde{\alpha},\beta)) \cap L(B;(\tilde{\alpha},\beta)) = \emptyset$, thus $L(A;(\tilde{\alpha},\beta)) \neq M$, and so $L(A;(\tilde{\alpha},\beta))$ is a proper ideal of M. By Lemma 3.21, it follows that there is a prime lattice ideal P of M such that $L(A;(\tilde{\alpha},\beta)) \subseteq P$ and $L(B;(\tilde{\alpha},\beta)) \cap P = \emptyset$. We put $D = P^{([1,1],0)}_{(\tilde{\alpha},\beta)}$, it is easy to see that D is a cubic prime MV-ideal of M, and we will show that $A \sqsubseteq D$ and $D \sqcap B \sqsubseteq \mathcal{C}_{(\tilde{\alpha},\beta)}$. For any $x \in M$, if $x \in P$, then $x \notin L(B;(\tilde{\alpha},\beta))$, and so $A(x) \preceq ([1,1],0) = D(x)$ and $B(x) \prec (\tilde{\alpha},\beta)$, hence $(D \sqcap B)(x) \prec (\tilde{\alpha},\beta) = \mathcal{C}_{(\tilde{\alpha},\beta)}(x)$; if $x \notin P$, then $x \notin L(A;(\tilde{\alpha},\beta))$, thus $A(x) \prec (\tilde{\alpha},\beta) = \mathcal{C}_{(\tilde{\alpha},\beta)}(x)$ and $(D \sqcap B)(x) \preceq (\tilde{\alpha},\beta) = \mathcal{C}_{(\tilde{\alpha},\beta)}(x)$. Therefore, in any case we have $A(x) \preceq \mathcal{C}_{(\tilde{\alpha},\beta)}(x)$ and $(D \sqcap B)(x) \preceq \mathcal{C}_{(\tilde{\alpha},\beta)}(x)$. □

4. Quotient structures of MV-algebras based on cubic MV-ideals

In the section, we defined the quotient structure of cubic MV-ideals, then present three isomorphism theorems concerning the quotient of cubic MV-ideals.

Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic MV-ideal of an MV-algebra M and $x \in M$. The cubic set A^x is called the cubic coset of A which is defined as: for any $y \in M$,

$$A^x(y) = A(x \ominus y) \barwedge A(y \ominus x).$$

We denote M/A the set of all cubic cosets with respect to A.

Lemma 4.1. *Let A be a cubic MV-ideal of an MV-algebra M. Then the following assertions hold: for any $x, y, z, s, t \in M$,*

(1) $A^x = A^y$ *if and only if* $A(x \ominus y) = A(y \ominus x) = A(0)$*;*

(2) *the set* $A_* = \{x \in M | A(x) = A(0)\}$ *is an ideal of* M*;*

(3) $A^x(y) = A^y(x)$*;*

(4) $A^x(y) = A^{\neg x}(\neg y)$*;*

(5) $A^x(y) \barwedge A^y(z) \preceq A^x(z)$*;*

(6) $A^x(y) \barwedge A^s(t) \preceq A^{x \oplus s}(y \oplus t)$*.*

Proof. (1) Suppose that $A^x = A^y$, then $A^x(x) = A^y(x)$, it follows that $A(x \ominus x) = A(0) = A(y \ominus x) \barwedge A(x \ominus y)$. Thus $A(x \ominus y) = A(y \ominus x) = A(0)$.

Conversely, assume that $A(x \ominus y) = A(y \ominus x) = A(0)$. According to Proposition 3.11 (5), we get that $A(x \ominus z) \succeq A(x \ominus y) \barwedge A(y \ominus z) = A(y \ominus z)$ and $A(z \ominus x) \succeq A(z \ominus y) \barwedge A(y \ominus x) = A(z \ominus y)$. Hence $A^x(z) = A(x \ominus z) \barwedge A(z \ominus x) \succeq A(y \ominus z) \barwedge A(z \ominus y) = A^y(z)$. Similarly, $A^y(z) \succeq A^x(z)$, therefore $A^x = A^y$.

(2) and (3) are obviously.

(4) is immediately from Proposition 2.1 (5).

(5) Since $x \ominus z \leq (x \ominus y) \oplus (y \ominus z)$ and $z \ominus x \leq (y \ominus x) \oplus (z \ominus y)$, then $A(x \ominus y) \barwedge A(y \ominus z) \preceq A(x \ominus z)$ and $A(y \ominus x) \barwedge A(z \ominus y) \preceq A(z \ominus x)$ by 3.11 (3), whence (5) follows form the monotonicity of $\barwedge$.

(6) The proof of (6) is similar to that of (5). □

As an immediate consequence of Lemma 4.1, we have

Proposition 4.2. *Let A be a cubic MV-ideal of M. A relation $\equiv_A$ on M is defined as follows: for any $x, y \in M$,*

$$x \equiv_A y \textit{ if and only if } A(x \ominus y) = A(y \ominus x) = A(0),$$

Then $\equiv_A$ is a congruence relation on M.

Given $x \in M$, the equivalence class of x with respect to $\equiv_A$ will be denoted by $[x]_A$ and the quotient set $M/\equiv_A$. Since $\equiv_A$ is a congruence, defining the operations on the set $M/\equiv_A$ as: $\neg[x]_A = [\neg x]_A$ and $[x]_A \oplus [y]_A = [x \oplus y]_A$ for any $x, y \in M$. Then the system $(M/\equiv_A, \oplus, \neg, [0]_A)$ becomes an MV-algebra.

The next corollary is an easy consequence of Lemma 4.1 and Proposition 4.2.

Corollary 4.3. *Let A be a cubic MV-ideal of M. Then*

(1) *$A^x = A^y$ if and only if $x \equiv_A y$, for any $x, y \in M$;*

(2) *$A_* = [0]_A$.*

Let A be a cubic MV-ideal of an MV-algebra M. For any $A^x, A^y \in M/A$, we define $A^x \vee A^y = A^{x \vee y}$, $A^x \wedge A^y = A^{x \wedge y}$, $A^x \oplus A^y = A^{x \oplus y}$, $\neg A^x = A^{\neg x}$, and the order $\leq$ on M/A by $A^x \leq A^y$ if and only if $A^x \vee A^y = A^y$.

Lemma 4.4. *Let A be a cubic MV-ideal of M. Then $A^x \leq A^y$ if and only if $A(x \ominus y) = A(0)$.*

Proof. For any $x, y \in M$, $A^x \leq A^y$ if and only if $A^x \vee A^y = A^{x \vee y} = A^y$ if and only if $A(y \ominus (x \vee y)) = A((x \vee y) \ominus y) = A(0)$, that is, $A(x \ominus y) = A(0)$. □

Corollary 4.5. *Let A be a cubic MV-ideal of M. Then $A^x \leq A^y$ if and only if $A^x \wedge A^y = A^x$.*

Theorem 4.6. *Let A be a cubic MV-ideal of M. Then $(M/A, \oplus, \neg, A^0)$ is an MV-algebra, which is called a cubic quotient MV-algebra.*

Proof. We can claim that the operations on M/A are well-defined. In fact, if $A^x = A^y$ and $A^s = A^t$, according to Corollary 4.3, we have that $x \equiv_A y$ and $s \equiv_A t$, and so $x \oplus s \equiv_A y \oplus t$, it follows that $A^{x \oplus s} = A^{y \oplus t}$. Similarly, we can prove $A^{x \vee s} = A^{y \vee t}$ and $A^{x \wedge s} = A^{y \wedge t}$. Then we can easily check that M/A is an MV-algebra. □

Theorem 4.7. *Let A be a cubic MV-ideal of M. Then $M/A \cong M/\equiv_A$.*

Proof. Define a map $\varphi : M/A \to M/\equiv_A$ by $\varphi(A^x) = [x]_A$ for any $x \in M$. Assume that $A^x, A^y \in M/A$, then $A^x = A^y$ if and only if $x \equiv_A y$, which implies that φ is an one-to-one function. Obviously, φ is surjective. Moreover, $\varphi(A^x \oplus A^y) = \varphi(A^{x\oplus y}) = [x \oplus y]_A = [x]_A \oplus [y]_A = \varphi(A^x) \oplus \varphi(A^y)$, $\varphi(\neg A^x) = \varphi(A^{\neg x}) = [\neg x]_A = \neg[x]_A = \neg\varphi(A^x)$, thus φ is an isomorphism and the proof is complete. □

From the above theorem we immediately obtain:

Corollary 4.8. *Let $f : M_1 \to M_2$ be a homomorphism of MV-algebras and A be a cubic MV-ideal of M with $ker f = A_*$. Then $M/A \cong f(M_1)$.*

Definition 4.9. *Let A be a cubic MV-ideal of M and B a cubic set of M. A cubic set B/A of the MV-algebra M/A is defined as follows: for any $A^x \in M/A$,*

$$(B/A)(A^x) = \overline{\sup}\{B(y)|A^x = A^y, y \in M\},$$

and B/A is called a quotient cubic set of M/A.

Proposition 4.10. *Let A, B be cubic MV-ideals of M. Then B/A is a cubic MV-ideal of M/A.*

Proof. For any $x_1, x_2 \in M$, $(B/A)(A^{x_1} \oplus A^{x_2}) = (B/A)(A^{x_1\oplus x_2}) = \overline{\sup}\{B(y_1 \oplus y_2)|A^{x_1\oplus x_2} = A^{y_1\oplus y_2}\} \geq \overline{\sup}\{B(y_1)\overline{\wedge}B(y_2)|A^{x_1} = A^{y_1}, A^{x_2} = A^{y_2}\} = B/A(A^{x_1})\overline{\wedge} B/A(A^{x_2})$.

For any $x_1, x_2 \in M$ such that $A^{x_1} \leq A^{x_2}$, then $A^{x_1} = A^{x_1} \wedge A^{x_2}$, and $(B/A)(A^{x_1}) = \overline{\sup}\{B(y_1)|A^{x_1} = A^{y_1}\} = \overline{\sup}\{B(y_1)|A^{x_1} = A^{y_1}, A^{x_2} = A^{y_2}\} = \overline{\sup}\{B(y_1\wedge y_2)|A^{x_1} = A^{y_1\wedge y_2}, A^{x_2} = A^{y_2}\} \geq \overline{\sup}\{B(y_2)|A^{x_2} = A^{y_2}\} = B/A(A^{x_2})$.

Thus B/A is a cubic MV-ideal of M/A. □

Definition 4.11. *Let f be a mapping from an MV-algebra M_1 into an MV-algebra M_2, and A, B be cubic sets of M_1 and M_2, respectively. Then*

(1) *the inverse image $f^{-1}(B)$ of B under f is defined as $f^{-1}(B)(x) = B(f(x))$, for any $x \in M_1$;*

(2) *the image $f(A)$ of A under f is defined as*

$$f(A)(y) = \begin{cases} \overline{\sup}\{A(x)|f(x) = y\}, & f^{-1}(y) \neq \emptyset, \\ ([0, 0], 1), & \textit{otherwise}. \end{cases}$$

The following result can be easy proved together with Definition 3.5, and so we omit the proof.

Proposition 4.12. *Let $f : M_1 \to M_2$ be a homomorphism of MV-algebras and A, B be cubic MV-ideals of M_1 and M_2, respectively. Then*

(1) *the inverse image $f^{-1}(B)$ is a cubic MV-ideal of M_1;*

(2) *the image* $f(A)$ *is a cubic MV-ideal of* M_2.

Proposition 4.13. *Let* A, B *be cubic MV-ideals of an MV-algebra* M *and* $\varphi : M \to M/A$ *a natural homomorphism, that is,* $\varphi(x) = A^x$ *for any* $x \in M$. *Then*

(1) $\varphi(B) = B/A$ *for any cubic set* B *of* M;

(2) $\varphi^{-1}(B)/A = B$ *for any cubic set* B *of* M/A.

Proof. (1) For any $A^x \in M/A$, we get that $\varphi(B)(A^x) = \overline{\sup}\{B^t|\varphi(t) = A^x, t \in M\} = \overline{\sup}\{B^t|A^t = A^x, t \in M\} = B/A(A^x)$. And therefore $\varphi(B) = B/A$.

(2) For any $A^x \in M/A$, we obtain that $(\varphi^{-1}(B)/A)(A^x) = \overline{\sup}\{\varphi^{-1}(B)(t)|A^t = A^x, t \in M\} = \overline{\sup}\{B(\varphi(t))|A^t = A^x, t \in M\} = \overline{\sup}\{B(A^t)|A^t = A^x, t \in M\} = B(A^x)$. Thus $\varphi^{-1}(B)/A = B$. □

For the purpose of investigating homomorphism theorems of MV-algebras based on cubic MV-ideals, we introduce the following notions.

Let $f : M_1 \to M_2$ be a homomorphism of MV-algebras and A, B be cubic MV-ideals of M_1 and M_2, respectively. If $f(A) \sqsubseteq B$, we say that A is weakly homomorphic to B, and we write $A \sim B$. If $f(A) = B$, we say that A is homomorphic to B, and we write $A \approx B$. If f is bijective and $f(A) = B$, we say that A is isomorphic to B, and we write $A \cong B$.

As an immediate consequence of the above Proposition 4.13, we record here the following result.

Corollary 4.14. *Let* A, B *be cubic MV-ideals of* M. *Then* $B \approx B/A$.

Theorem 4.15. *Let* A, B *be cubic MV-ideals of an MV-algebra* M_1, *and* $f : M_1 \to M_2$ *be an epimorphism of MV-algebras such that* $kerf = A_*$. *Then* $B/A \cong f(B)$.

Proof. Define a map $\varphi : M_1/A \to M_2$ by $\varphi(A^x) = f(x)$ for any $x \in M_1$. Then for any $x_1, x_2 \in M_1$, we have that $A^{x_1} = A^{x_2}$ if and only if $x_1 \ominus x_2, x_2 \ominus x_1 \in A_* = kerf$ if and only if $f(x_1 \ominus x_2) = f(x_2 \ominus x_1) = f(0) = 0$ if and only if $f(x_1) = f(x_2)$. Therefore φ is an one-to-one function. It follows that φ is surjective due to the fact that f is a surjective function. For any $x_1, x_2, x \in M_1$, $\varphi(A^{x_1} \oplus A^{x_2}) = \varphi(A^{x_1 \oplus x_2}) = f(x_1 \oplus x_2) = f(x_1) \oplus f(x_2) = \varphi(A^{x_1}) \oplus \varphi(A^{x_2})$, $\varphi(\neg A^x) = \varphi(A^{\neg x}) = f(\neg x) = \neg f(x) = \neg\varphi(A^x)$, thus φ is a homomorphism.

Moreover, for any $y \in M_2$, $\varphi(B/A)(y) = \overline{\sup}\{(B/A)(A^x)|\varphi(A^x) = y\} = \overline{\sup}\{\overline{\sup}\{B(z)|A^z = A^x\}|f(x) = y\} = \overline{\sup}\{B(z)|A^z = A^x, f(x) = y\} = \overline{\sup}\{B(z)|f(z) = f(x), f(x) = y\} = f(B)(y)$, that is $\varphi(B/A) = f(B)$. And so $B/A \cong f(B)$. □

Definition 4.16. *Let* $f : M_1 \to M_2$ *be a homomorphism of MV-algebras and* A *be a cubic MV-ideal of* M_1. A *is called an invariant cubic set with respect to* f *if* $f(x_1) = f(x_2)$ *implies* $A(x_1) = A(x_2)$, *for any* $x_1, x_2 \in M_1$.

Proposition 4.17. *Let $f : M_1 \to M_2$ be a homomorphism of MV-algebras, and A be a cubic MV-ideal of M_1. Then A is invariant with respect to f if and only if $kerf \subseteq A_*$.*

Proof. Suppose that A is invariant with respect to f. For any $x \in kerf$, we have $f(x) = 0 = f(0)$, then $A(x) = A(0)$, hence $x \in A_*$, and so $kerf \subseteq A_*$.

Conversely, assume that $kerf \subseteq A_*$. For any $x_1, x_2 \in M_1$, if $f(x_1) = f(x_2)$, according to the proof of Theorem 4.15 we get that $x_1 \ominus x_2, x_2 \ominus x_1 \in kerf \subseteq A_*$, and so $A(x_1 \ominus x_2) = A(x_2 \ominus x_1) = A(0)$. Noting that A is a cubic MV-ideal of M_1, then $A(x_2) \preceq A(x_1)$ and $A(x_1) \preceq A(x_2)$ by Proposition 3.11 (1). Therefore $A(x_1) = A(x_2)$, and thus A is invariant with respect to f. □

Lemma 4.18. *Let $f : M_1 \to M_2$ be an epimorphism of MV-algebras, and A be a cubic MV-ideal of M_1 such that A is invariant with respect to f. Then for any $x_1, x_2 \in M_1$, $A^{x_1} = A^{x_2}$ if and only if $f(A)^{f(x_1)} = f(A)^{f(x_2)}$.*

Proof. Suppose that $A^{x_1} = A^{x_2}$, using Lemma 4.1 we obtain that $A(x_1 \ominus x_2) = A(x_2 \ominus x_1) = A(0)$. Since A is a cubic MV-ideal of M_1 and f is an epimorphism, it follows that $f(A)$ is a cubic MV-ideal of M_2 by Proposition 4.12. Then $f(A)(f(x_1) \ominus f(x_2)) = f(A)(f(x_1 \ominus x_2)) = f^{-1}(f(A))(x_1 \ominus x_2) = A(x_1 \ominus x_2) = A(0) = f(A)(0)$. Similarly, we can show that $f(A)(f(x_2) \ominus f(x_1)) = f(A)(0)$, thus $f(A)^{f(x_1)} = f(A)^{f(x_2)}$.

Conversely, let $f(A)^{f(x_1)} = f(A)^{f(x_2)}$. It follows that $f(A)(f(x_1) \ominus f(x_2)) = f(A)(f(x_1 \ominus x_2)) = f(A)(0) = A(0)$. And $A(x_1 \ominus x_2) = f^{-1}(f(A))(x_1 \ominus x_2) = f(A)(f(x_1 \ominus x_2)) = A(0)$, analogously, $A(x_2 \ominus x_1) = A(0)$. Hence $A^{x_1} = A^{x_2}$. □

Theorem 4.19. *Let $f : M_1 \to M_2$ be an epimorphism of MV-algebras, and A, B be cubic MV-ideals of M_1. Then $B/A \cong f(B)/f(A)$.*

Proof. Define $h : M_1/A \to M_2/f(A)$ by $h(A^x) = f(A)^{f(x)}$ for any $x \in M_1$, it follows from Lemma 4.18, we get that h is an one-to-one function. Note that f is a surjective function, hence h is surjective. For any $x_1, x_2, x \in M_1$,

$h(A^{x_1} \oplus A^{x_2}) = h(A^{x_1 \oplus x_2}) = f(A)^{f(x_1 \oplus x_2)} = f(A)^{f(x_1) \oplus f(x_2)} = f(A)^{f(x_1)} \oplus f(A)^{f(x_2)} = h(A^{x_1}) \oplus h(A^{x_2})$, $h(\neg A^x) = h(A^{\neg x}) = f(A)^{f(\neg x)} = f(A)^{\neg f(x)} = \neg h(A^x)$, thus h is a homomorphism.

Moreover, according to Definition 4.9, Definition 4.11 and Lemma 4.18, for any $x \in M_1$, we obtain that

$$
\begin{aligned}
& h^{-1}(f(B)/f(A))(A^x) = (f(B)/f(A))(h(A^x)) = (f(B)/f(A))(f(A)^{f(x)}) \\
& = \overline{\sup}\{f(B)(y) | f(A)^{f(x)} = f(A)^y, y \in M_2\} \\
& = \overline{\sup}\{\overline{\sup}\{B(z) | f(z) = y, z \in M_1\} | f(A)^{f(x)} = f(A)^y, y \in M_2\} \\
& = \overline{\sup}\{B(z) | f(A)^{f(z)} = f(A)^x, z \in M_1\} \\
& = \overline{\sup}\{B(z) | A^z = A^x, z \in M_1\} = (B/A)(A^x),
\end{aligned}
$$

that is, $h^{-1}(f(B)/f(A)) = B/A$. Due to the fact that h is isomorphic, we get that $h(B/A) = f(B)/f(A)$, hence $B/A \cong f(B)/f(A)$. □

Corollary 4.20. *Let $f : M_1 \to M_2$ be an epimorphism of MV-algebras, and A, B be cubic MV-ideals of M_2. Then $f^{-1}(B)/f^{-1}(A) \cong B/A$.*

Lemma 4.21. *Let A, B be cubic MV-ideals of M and $A \sqsubseteq B$. If $A(0) = B(0)$, then $B^x = B^y$ if and only if $(B/A)(A^x) = (B/A)(A^y)$, for any $x, y \in M$.*

Proof. For any $x, y \in M$, if $B^x = B^y$, then $B(x \ominus y) = B(y \ominus x) = B(0)$. And we have $(B/A)(A^{x\ominus y}) = \overline{\sup}\{B(z)|A^z = A^{x\ominus y}, z \in M\} \succeq B(x \ominus y) = B(0) = (B/A)(A^0)$, thus $(B/A)(A^{x\ominus y}) = (B/A)(A^x \ominus A^y) = (B/A)(A^0)$, which means $(B/A)(A^y) \preceq (B/A)(A^x)$ by Proposition 3.11 (1). Similarly, we can prove $(B/A)(A^x) \preceq (B/A)(A^y)$, hence $(B/A)(A^x) = (B/A)(A^y)$.

Conversely, if $(B/A)(A^x) = (B/A)(A^y)$, it follows that $(B/A)(A^{x\ominus y}) = (B/A)(A^x \ominus A^y) = \overline{\sup}\{B(z)|A^z = A^{x\ominus y}, z \in M\} = (B/A)(A^0) = B(0)$ by Proposition 3.11 (1). To prove $B^x = B^y$, we only need to show that $B(x \ominus y) = B(y \ominus x) = B(0)$. For any $z \in M$, if $A^z = A^{x\ominus y}$, then $A(z \ominus (x \ominus y)) = A(0)$ by Lemma 4.1 (1). Note that B is a cubic MV-ideal of M and $A \sqsubseteq B$, from Proposition 3.7 (2) we get that $B(x\ominus y) \succeq B(z) \barwedge B(z\ominus(x\ominus y)) = B(z) \barwedge A(z\ominus(x\ominus y)) = B(z) \barwedge A(0) = B(z) \barwedge B(0) = B(z)$. And so $B(x\ominus y) \succeq \overline{\sup}\{B(z)|A^z = A^{x\ominus y}, z \in M\} = B(0)$, thus $B(x \ominus y) = B(0)$. Similarly, we can show $B(y \ominus x) = B(0)$, hence $B^x = B^y$. □

Theorem 4.22. *Let A, B, C be cubic MV-ideals of M and $A \sqsubseteq B$. If $A(0) = B(0)$, then $(C/A)/(B/A) \cong C/B$.*

Proof. It is easy to prove that $(C/A)/(B/A)$, C/B are cubic MV-ideals of MV-algebras $(M/A)/(B/A)$ and M/B, respectively.

Define $h : (M/A)/(B/A) \to M/B$ by $h((B/A)^{A^x}) = B^x$ for any $x \in M$, from Lemma 4.21 we get that h is an one-to-one function. Obviously, h is surjective. For any $x_1, x_2, x \in M$,

$$\begin{aligned} &h((B/A)^{A^{x_1}} \oplus (B/A)^{A^{x_2}}) = h((B/A)^{A^{x_1} \oplus A^{x_2}}) \\ &= h((B/A)^{A^{x_1 \oplus x_2}}) = B^{x_1 \oplus x_2} \\ &= B^{x_1} \oplus B^{x_2} = h((B/A)^{A^{x_1}}) \oplus h((B/A)^{A^{x_2}}), \end{aligned}$$

and $h(\neg(B/A)^{A^x}) = h((B/A)^{\neg A^x}) = h((B/A)^{A^{\neg x}}) = B^{\neg x} = \neg B^x = \neg h((B/A)^{A^x})$, thus h is a homomorphism. Moreover, using Lemma 4.21 we have that: for any $x \in M$, $(C/A)/(B/A)((B/A)^{A^x}) = \overline{\sup}\{(C/A)(A^w)|(B/A)^{A^w} = (B/A)^{A^x}, w \in M\} = \overline{\sup}\{\overline{\sup}\{C(v)|A^v = A^w, v \in M\}|(B/A)^{A^w} = (B/A)^{A^x}, w \in M\} = \overline{\sup}\{C(v)|(B/A)^{A^v} = (B/A)^{A^x}, v \in M\} = \overline{\sup}\{C(v)|B^v = B^x, v \in M\} = (C/B)(B^x) = (C/B)(h((B/A)^{A^x})) = h^{-1}(C/B)((B/A)^{A^x})$, that is, $(C/A)/(B/A) = h^{-1}(C/B)$. Since h is isomorphic, we get that $(C/A)/(B/A) \cong C/B$. □

Theorem 4.23. *Let A, B be cubic MV-ideals of M and $A(0) = B(0)$. If $A \circledast B$ is an inverse isotone mapping, then $A/(A \sqcap B) \sim (A \circledast B)/B$ and $A/(A \sqcap B) \sim (A \circledast B)/A$.*

Proof. Since $A \circledast B$ is an inverse isotone mapping, it follows that $A \circledast B$ is a cubic MV-ideal by Proposition 3.13. It is easy to obtain that $A/(A \sqcap B)$, $(A \circledast B)/B$ are cubic MV-ideals of MV-algebras $M/(A \sqcap B)$ and M/B, respectively.

Define the map $h : M/(A \sqcap B) \to M/B$ by $h((A \sqcap B)^x) = B^x$ for any $x \in M$. For any $x_1, x_2 \in M$, if $(A \sqcap B)^{x_1} = (A \sqcap B)^{x_2}$, from Lemma 4.1 it follows that $(A \sqcap B)(x_1 \ominus x_2) = (A \sqcap B)(x_2 \ominus x_1) = (A \sqcap B)(0) = B(0)$, so we get that $B(x_1 \ominus x_2) = B(x_2 \ominus x_1) = B(0)$, thus $B^{x_1} = B^{x_2}$. hence h is an one-to-one function. Obviously, h is surjective.

Moreover, for any $x_1, x_2, x \in M$, we have $h((A \sqcap B)^{x_1} \oplus (A \sqcap B)^{x_2}) = h((A \sqcap B)^{x_1 \oplus x_2}) = B^{x_1 \oplus x_2} = B^{x_1} \oplus B^{x_2} = h((A \sqcap B)^{x_1}) \oplus h((A \sqcap B)^{x_2})$, and $h(\neg(A \sqcap B)^x) = h((A \sqcap B)^{\neg x}) = B^{\neg x} = \neg B^x = \neg h((A \sqcap B)^x)$, thus h is a homomorphism.

For any $x \in M$,

$$
\begin{aligned}
&((A \circledast B)/B)(B^x) = \overline{\sup}\{A \circledast B)(t) | B^t = B^x\} \\
&= \overline{\sup}\{A(t_1) \barwedge B(t_2) | t = t_1 \oplus t_2, B^t = B^x\} \succeq \overline{\sup}\{A(t) \barwedge B(0) | B^t = B^x\} \\
&= \overline{\sup}\{A(t) | B^t = B^x\}, \\
&h(A/(A \sqcap B))(B^x) = \overline{\sup}\{(A/(A \sqcap B))((A \sqcap B)^z) | h((A \sqcap B)^z) = B^x\} \\
&= \overline{\sup}\{(A/(A \sqcap B))((A \sqcap B)^z) | B^z{=}B^x\}{=}\overline{\sup}\{A(t) | (A \sqcap B)^t{=}(A \sqcap B)^z, B^z{=}B^x\} \\
&\preceq \overline{\sup}\{A(t) | B^t = B^z, B^z = B^x\} = \overline{\sup}\{A(t) | B^t = B^x\}.
\end{aligned}
$$

It follows that $h(A/(A \sqcap B))(B^x) \preceq ((A \circledast B)/B)(B^x)$, hence $h(A/(A \sqcap B)) \sqsubseteq ((A \circledast B)/B)$, and so $A/(A \sqcap B) \sim (A \circledast B)/B$. Similarly, we can prove $A/(A \sqcap B) \sim (A \circledast B)/A$. □

Definition 4.24. *The cubic MV-ideal A of an MV-algebra M has cubic sup-property if for any nonempty subset H of M, there exists $x_0 \in H$ such that $A(x_0) = \sup\{A(x) | x \in H\}$.*

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