t-PROPERTY OF METRIC SPACES AND FIXED POINT THEOREMS

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Abstract. The prime goal of this article is to prove some fixed point results in partially ordered metric spaces that are not necessarily complete. This is achieved by introducing the concept of t-property. Moreover, we have established the existence of fixed points

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for variant contractive mappings in the framework of incomplete ordered metric spaces. Few examples have been given to illustrate the new concepts and results.
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1. Introduction and preliminaries

Uniqueness of fixed points for contraction mappings in complete metric spaces was proved long ago in 1922 by Banach [10]. It was popular by the name of Banach Contraction Principle. This has played a pivotal role in the evolution of fixed point theory. This principle has been generalized in framework of different spaces, see [4, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 28]. In 2004, Ran-Reuring [26], Nieto and Rodríguez-López [24] and many others have generalized Banach Contraction Principle in the setting of ordered metric spaces, see [2, 3, 5, 6, 8, 9, 25, 27, 29, 30]. In this article, t-property of partially ordered metric spaces has been introduced.Using this concept, we present some fixed point results for variant contraction mappings.

Definition 1.1 ([7])**.** *An ordered metric space $(X, d, \preceq)$ is said to be $\overline{O}$-complete if every increasing Cauchy sequence in X converges in X. In an ordered metric space, completeness implies $\overline{O}$-completeness, but the converse is not true in general.*

Now, we introduce some definitions.

Definition 1.2. *Let $(X, \preceq)$ be any ordered set and $x, y \in X$. Such x is called a strict upper bound of y, if $y \preceq x$ and $y \neq x$. We denote it by $y \prec x$.*

Definition 1.3. *Let $(X, d, \preceq)$ be any ordered metric space. X has the t-property if every strictly increasing Cauchy sequence $\{x_n\}$ in X has a strict upper bound in X, i.e., there exists $u \in X$ such that $x_n \prec u$.*

We present the following examples illustrating Definition 1.3.

Example 1.1. Let $X = \mathbb{R}, \mathbb{Q}, (a, b], a, b \in \mathbb{R}$ be equipped with the natural ordering $\leq$ and the usual metric. Then X has t-property.

Example 1.2. Let $X = \{(x, y) : x, y \in \mathbb{Q}\}$. We define $\preceq$ in X by $(x_1, x_2) \preceq (y_1, y_2)$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$. Let d be the Euclidean metric on X. Then $(X, d, \preceq)$ has the t-property.

Example 1.3. Let $X = C[a, b]$ be equipped with the metric d defined as $d(f, g) = \int_a^b \mid f - g \mid dx$. Then (X, d) is not a complete metric space. Now, we define $\preceq$ in X as: $f \preceq g$ iff $f(x) \leq g(x)$, for each $x \in [a, b]$. Obviously, $(C[a, b], d, \preceq)$ has t-property.

In the following example, the increasing Cauchy sequence does not have any strict upper bound.

Example 1.4. Let us consider $X = \{(x, y, z) : x, y, z \in \mathbb{Q}\ with\ max\{x, y, z\} < \sqrt{2}\}$. Endow X with the Euclidean metric on $\mathbb{R}^3$. Define $\preceq$ in X by $(x_1, y_1, z_1) \preceq (x_2, y_2, z_2)$ if $x_1 \leq x_2$, $y_1 \leq y_2$ and $z_1 \leq z_2$. Consider $x_n = (q_n, q_n, q_n)$ in X such that $q_0 = 1$ and $\{q_n\}$ is strictly increasing in $\mathbb{Q}$. We have that $q_n < \sqrt{2}$ for all $n \geq 0$. Also, $\{x_n\}$ is a strictly increasing Cauchy sequence in X, but it does not have any strict upper bound in X.

Remark 1.1. Mention that every totally ordered complete metric space has t-property provided that there exists a strictly increasing Cauchy sequence. But, every metric space having t-property is not complete. This fact is described In Example 1.1 (except the case $\mathbb{R}$), Example 1.2 and Example 1.3.

2. Main Result

In all our given results, the completeness of the metric space is omitted. To overcome this lack, we require that the space has the t-property. Our first fixed point result is

Theorem 2.1. *Let $(X, d, \preceq)$ be an ordered metric space satisfying the t-property. Let $f : X \to X$ be a self-mapping. Assume that f is monotonic non-decreasing. Further, if*
(1) *there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$;*
(2) *for all $x, y \in X$ with $x \prec y$,*

$$d(y, f(y)) \leq \alpha d(x, f(x)), \tag{1}$$

where $\alpha \in (0, 1)$. Then f has at least one fixed point in X. Moreover, every strict upper bound of a fixed point is also a fixed point.

Proof. By assumption (1), we have $x_0 \preceq f(x_0)$. If $x_0 = f(x_0)$, the proof is completed. Otherwise, choose $x_1 = f(x_0)$ such that $x_0 \prec x_1$. By monotonicity of f, we have $f(x_0) \preceq f(x_1)$, that is, $x_1 \preceq f(x_1)$. If $x_1 = f(x_1)$, the proof is completed. Otherwise, choose $x_2 = f(x_1)$ such that $x_1 \prec x_2$. Again, by monotonicity of f, we have $f(x_1) \preceq f(x_2)$. Continuing in this process, we get a strictly increasing sequence $\{x_n\}$ in X such that

$$x_{n+1} = f(x_n). \tag{2}$$

As $x_0 \prec x_1$, by (1), we have

$$d(x_1, f(x_1)) \leq \alpha d(x_0, f(x_0)). \tag{3}$$

Again as $x_1 \prec x_2$, by (1), we have

$$d(x_2, f(x_2)) \leq \alpha d(x_1, f(x_1)). \tag{4}$$

Using (3) in (4), we get

$$d(x_2, f(x_2)) \leq \alpha^2 d(x_0, f(x_0)).$$

Continuing in this way, we get

$$d(x_n, f(x_n)) \leq \alpha^n d(x_0, f(x_0)). \tag{5}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. For $n < m$, by using triangular inequality, (2) and (5), we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m), \\ &= d(x_n, f(x_n)) + d(x_{n+1}, f(x_{n+1})) + ... + d(x_{m-1}, f(x_{m-1})), \\ &\leq \alpha^n d(x_0, f(x_0)) + \alpha^{n+1} d(x_0, f(x_0)) + ... + \alpha^{m-1} d(x_0, f(x_0)), \\ &\leq (\alpha^n + \alpha^{n+1} + ... + \alpha^{m-n-1}) d(x_0, f(x_0)), \\ &\leq \frac{\alpha^n}{1-\alpha} d(x_0, f(x_0)). \end{aligned}$$

This shows that $\{x_n\}$ is an increasing Cauchy sequence in X, which has the t-property, so there exists $u \in X$ such that $x_n \prec u$ for all n. Thus, from (1) and (5), we have

$$d(u, f(u)) \leq \alpha d(x_n, f(x_n)) \leq \alpha^{n+1} d(x_0, f(x_0)) \to 0, \quad as \quad n \to \infty.$$

Thus $f(u) = u$. Hence u is a fixed point of f. Now, let k be any strict upper bound of u in X, that is, $u \prec k$. By (1), we have

$$d(k, f(k)) \leq \alpha d(u, f(u)) = 0,$$

so $k = f(k)$, that is, k is also a fixed point of f in X. □

Example 2.1. Let $X = \{a_n : \ a_{n+1} = 3a_n + 1 \text{ for } n \geq 0 \text{ and } a_0 = -1\} \cup (-1, 0]$. Then $X = \{..., -41, -14, -5, -2, -1\} \cup (-1, 0]$. Endow X with the usual metric on $\mathbb{R}$ and the natural ordering $\leq$. Clearly, $(X, d, \preceq)$ has the t-property. Define $f : X \to X$ by

$$f(x) = \begin{cases} 3x + 1, & if x < -1, \\ x, & if x \geq -1. \end{cases}$$

Obviously, f is non-decreasing. Now, it remains to prove that f satisfies (1). Let $x, y \in X$ with $x < y$. If $y \geq -1$, then $f(y) = y$, so $d(y, f(y)) = 0$ and the proof is completed. Assume now that $x < y \leq -2$. Then $d(y, f(y)) = -(2y + 1)$ and $d(x, f(x)) = -(2x + 1)$. It should be noted that for $x, y \in X$ with $x < y \leq -2$, we have $y \geq \frac{5}{12}x$. Then

$$\begin{aligned} d(y, f(y)) &= -(2y + 1) \\ &\leq -\frac{5}{6}x - 1 \\ &= -\frac{1}{2}[\frac{5x + 6}{3}] \\ &\leq -\frac{1}{2}(2x + 1) \\ &= \alpha d(x, f(x)), \end{aligned}$$

where $\alpha = \frac{1}{2} \in (0,1)$. Hence all the conditions of Theorem 2.1 are satisfied. Therefore f has at least one fixed point in X. In fact, any element in the set $[0,1]$ is a fixed point of f.

On the other hand, neither Banach Contraction, nor Kannan-type and nor Chatterjea-type contraction holds. Indeed, by taking $x = -5$ and $y = -2$, we have

$$d(f(x), f(y)) > kd(x,y) \quad \text{for all } k \in (0,1),$$

$$d(d(x), f(y)) > k[d(x, f(x)) + d(y, f(y)] \quad \text{for all } k \in (0, \frac{1}{2}),$$

and

$$d(d(x), f(y)) > k[d(x, f(y)) + d(y, f(x)] \quad \text{for all } k \in (0, \frac{1}{2}).$$

Theorem 2.2. *Let $(X, d, \preceq)$ be an $\overline{O}$-complete ordered metric space. Let $f: X \to X$ be a self-mapping such that f is continuous and monotonic non-decreasing. Further if*
(1) there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$.
(2) for all $x, y \in X$ with $x \prec y$, $x \neq f(x)$ and for any $\alpha \in (0, \frac{1}{2})$,

$$d(y, f(y)) \leq \alpha[d(x,y) + d(f(x), f(y)]. \tag{6}$$

Then f has at least one fixed point in X.

Proof. As Theorem 2.1, we construct a strictly increasing sequence $\{x_n\}$ in X such that

$$x_{n+1} = f(x_n). \tag{7}$$

As $x_0 \prec x_1$, by using (6) and (7), we have

$$\begin{aligned} d(x_1, f(x_1)) &\leq \alpha[d(x_0, x_1) + d(f(x_0), f(x_1)] \\ &= \alpha d(x_0, f(x_0)) + \alpha d(x_1, f(x_1)). \end{aligned} \tag{8}$$

Then

$$d(x_1, f(x_1)) \leq \frac{\alpha}{1-\alpha} d(x_0, f(x_0)). \tag{9}$$

Again as $x_1 \prec x_2$, by using (6) and (7), we have

$$\begin{aligned} d(x_2, f(x_2)) &\leq \alpha[d(x_1, x_2) + d(f(x_1), f(x_2)], \\ &= \alpha d(x_1, f(x_1)) + \alpha d(x_2, f(x_2)). \end{aligned} \tag{10}$$

Then

$$d(x_2, f(x_2)) \leq \frac{\alpha}{1-\alpha} d(x_1, f(x_1)).$$

By using (9),

$$d(x_2, f(x_2)) \leq (\frac{\alpha}{1-\alpha})^2 d(x_0, f(x_0)).$$

Continuing this process, we get

$$d(x_n, f(x_n)) \leq (\frac{\alpha}{1-\alpha})^n d(x_0, f(x_0)). \tag{11}$$

As $0 < \alpha < \frac{1}{2}$, we get $0 < k = \frac{\alpha}{1-\alpha} < 1$. (11) becomes

$$d(x_n, f(x_n)) \leq k^n d(x_0, f(x_0)). \tag{12}$$

As Theorem 2.1, $\{x_n\}$ is an increasing Cauchy sequence in X. Since (X, d) is $\overline{O}$-complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{13}$$

Since f is continuous,

$$\lim_{n\to\infty} f(x_n) = f(u). \tag{14}$$

Taking $n \to \infty$ in (12) and making use of (13) and (14), we obtain $d(u, f(u)) = 0$. Hence u is a fixed point of f in X. □

Now, we are going to prove Theorem 2.2 when f is not continuous.

Definition 2.1 (SICU-property)**.** *An ordered metric space $(X, d, \preceq)$ is said to have SICU-property, if every strictly increasing convergent sequence has the limit as it's strict upper bound, i.e., if $\{x_n\}$ is strictly increasing convergent sequence with $x_n \to x \Rightarrow x_n \prec x$, for all n.*

Example 2.2. Let $X = \mathbb{R}^n$ endowed with Euclidean metric and $\preceq$ is defined as $(x_1, x_2, ..., x_n) \preceq (y_1, y_2, ..., y_n)$, if $x_i \leq y_i$, for all $i = 1, 2, ..., n$. Then $\mathbb{R}^n$ has SICU-property.

Theorem 2.3. *In Theorem 2.2, if we leave the continuity of f, but assume that $(X, d, \preceq)$ has SICU-property, then f has at least one fixed point in X.*

Proof. Going through same lines of proof in Theorem 2.2, we get $x_n \to u$ such that $x_n \prec u$. Thus, by using triangular inequality and (6), we have

$$\begin{aligned} d(u, f(u)) &\leq \alpha[d(x_n, u) + d(f(x_n), f(u)], \\ &\leq \alpha[d(x_n, u) + d(f(x_n), x_n) + d(x_n, u) + d(u, f(u))]. \end{aligned}$$

Then

$$d(u, f(u)) \leq \frac{\alpha}{1-\alpha}[2d(x_n, u) + d(x_n, f(x_n))].$$

By using (12) and taking $n \to \infty$, we have $d(u, f(u)) \leq 0$. Thus $f(u) = u$. □

Example 2.3. Let $X = \mathbb{R}^2$ be endowed with the Euclidean metric. Consider $(x, y) \preceq (u, v)$ iff $x \leq u$ and $y \leq v$. Then $(X, d, \preceq)$ is $\overline{O}$-complete and has SICU-property.Take $A = \{1, 3, 5, 7, ...\}$ as the set of all positive odd numbers. Let $E \subset X$ defined by $E = \{(a, b) : a \in A \ \ and \ \ 0 < b < 1\}$. Clearly, for all $(a, b), (c, d) \in E$ such that $(a, b) \prec (c, d)$, we have

$$a + \frac{3}{2} \leq c.$$

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = \begin{cases} (x, 1), & \text{if } (x, y) \in E \\ (x, y), & \text{if } (x, y) \in E^c. \end{cases}$$

It is easy to verify that f is non-decreasing. We show that f satisfies (6). For any $x = (x_1, y_1), y = (x_2, y_2) \in X$ with $x \prec y$ and $x \neq f(x)$, there exist only two cases:
(i) $x \in E$ and $y \in E^c$,
(ii) $x, y \in E$.

Case (i). $y \in E^c$, so $y = f(y)$, that is, $d(y, f(y)) = 0$. Hence (6) holds.

Case (ii) $x = (x_1, y_1), y = (x_2, y_2) \in E$. Then $0 < y_1, y_2 < 1$. Also, by definition of E, we have $x_2 - x_1 \geq \frac{3}{2}$. Also,

$$d(y, f(y)) = d((x_2, y_2), (x_2, 1)) = (1 - y_2) < 1,$$

and

$$d(x, y) = d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \geq (x_2 - x_1).$$

Similarly, $d(f(x), f(y)) = (x_2 - x_1)$. Taking $\alpha = \frac{1}{3} \in (0, \frac{1}{2})$, we have

$$\alpha[d(x, y) + d(f(x), f(y)] \geq \frac{1}{3} 2(x_2 - x_1) \geq \frac{2}{3}\frac{3}{2} = 1 > d(y, f(y)).$$

All the conditions of Theorem 2.3 are satisfied. Thus f has a fixed point in X. Any elemnt in E^c is a fixed point of f.

Example 2.4. Let $X = \{a_n : \ a_{n+1} = 5a_n + 1 \text{ for } n \geq 0 \text{ and } a_0 = -1\} \cup (-1, 0]$. Then $X = \{..., -94, -19, -4, -1\} \cup (-1, 0]$. Endow X with the usual metric on $\mathbb{R}$ and the natural ordering $\leq$. Then $(X, d, \preceq)$ is an $\overline{O}$-complete ordered metric space and satisfies the SICU-property. Define $f : X \to X$ by

$$f(x) = \begin{cases} 5x + 1, & \text{if } x \leq -1, \\ x, & \text{if } x > -1. \end{cases}$$

Then f is non-decreasing. We shall prove that f satisfies (6). Let $x, y \in X$ with $x < y$. If $y > -1$, then $d(y, f(y)) = 0$ and so (6) holds. Assume that $x < y \leq -1$.

Then $d(y, f(y)) = -(4y+1)$, $d(x, y) = (y-x)$ and $d(f(x), f(y)) = 5(y-x)$. It should be noted that for $x, y \in X$ with $x < y \leq -1$, we have $y \geq \frac{x}{3}$ or $-2x \geq -6y$. Taking $\alpha = \frac{1}{3} \in (0, \frac{1}{2})$, we get

$$\alpha[d(x, y) + d(f(x), f(y))] = 2y - 2x \geq 2y - 6y = -4y \geq -(4y+1) = d(y, f(y)).$$

Thus all the conditions of Theorem 2.3 are satisfied, and hence there exists a fixed point of f in X. Any $x \in [-1, 0]$ is a fixed point of f.

Example 2.5. Let $X = \{a_1, a_2, a_3, a_4\}$ be any ordered set where $\preceq$ is defined as: $a_i \preceq a_j$ iff $i \leq j$. If we define a metric $d : X \times X \to [0, \infty)$ by

$$d(a_i, a_i) = 0, \forall i = 1, 2, 3, 4.$$

$$d(a_i, a_{i+1}) = d(a_{i+1}, a_i) = 1, \text{ for } i = 1, 2, 3$$

$$d(a_i, a_{i+2}) = d(a_{i+2}, a_i) = 2, \text{ for } i = 1, 2$$

$$d(a_1, a_4) = d(a_4, a_1) = 3.$$

Note that $(X, d, \preceq)$ is a finite ordered metric space. We define $f : X \to X$ by $f(a_1) = a_2, f(a_2) = a_2, f(a_3) = a_4, f(a_4) = a_4$. If $a_i \preceq a_j$ for all $i \leq j$, then f is a monotonic non-decreasing mapping. Let $x, y \in X$ such that $x \prec y$ with $x \neq f(x)$. Take $x = a_1$ and $y = a_3$. Otherwise, $y = f(y)$, so $d(y, f(y)) = 0$ and the proof is completed in this case. For $x = a_1$ and $y = a_3$, We have $d(x, y) = d(a_1, a_3) = 2$, $d(f(x), f(y)) = d(f(a_1), f(a_3)) = d(a_2, a_4) = 2$ and $d(y, f(y)) = d(a_3, a_4) = 1$. Taking $\alpha = \frac{1}{3} \in (0, \frac{1}{2})$, then

$$d(y, f(y)) \leq \alpha[d(x, y) + d(f(x), f(y))].$$

For $x \prec y = a_3, a_4$, we have $d(y, f(y)) = 0$. Thus all conditions of Theorem 2.2 are satisfied. The elements a_2 and a_4 are fixed points of f.

Here, neither Banach Contraction, nor Kannan-type contraction and nor Chatterjea-type contraction is satisfied. This can be proved by taking $x = a_1$ and $y = a_3$.

Now, let Φ be set of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying:
(i) ϕ is non-decreasing;
(ii) $\phi(t) < t, \quad \forall t > 0$;
(iii) $\lim_{r \to t^+} \phi(r) < t, \quad \forall t > 0$.

We state the following known lemma.

Lemma 2.1 ([1]). *Let $\phi \in \Phi$ and $\{u_n\}$ be a given sequence such that $u_n \to 0^+$ as $n \to \infty$. Then $\phi(u_n) \to 0^+$ as $n \to \infty$. Also $\phi(0) = 0$.*

Theorem 2.4. *Let $(X, d, \preceq)$ be any ordered metric space having the t-property and $f : X \to X$ be a monotonic non-decreasing self-mapping. Assume for all $x, y \in X$ with $x \prec y$, we have*

$$d(y, f(y)) \leq \phi(d(x, f(x))), \tag{15}$$

where $\phi \in \Phi$. Suppose that the series $\sum_{n\geq 1} \phi^n(t)$ converges for all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$, then f has at least one fixed point in X. Moreover, every strict upper bound of fixed point of f is again a fixed point of f.

Proof. As Theorem 2.1 and without loss of generality, we construct a strictly increasing sequence $\{x_n\}$ in X such that

$$x_{n+1} = f(x_n). \tag{16}$$

We take $D_n = d(x_n, f(x_n))$. Since $x_n \neq f(x_n)$ $\forall n$, we have $D_n > 0$ for all n. As $x_n \prec x_{n+1}$ for all n, using (15), we get

$$D_{n+1} = d(x_{n+1}, f(x_{n+1})) \leq \phi(d(x_n, f(x_n))) = \phi(D_n) < D_n. \tag{17}$$

This shows that $\{D_n\}$ is a monotonic decreasing sequence in $\mathbb{R}^+$, so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} D_n = r. \tag{18}$$

From (17), we have

$$\lim_{n\to\infty} \phi(D_n) = r. \tag{19}$$

Suppose that $r > 0$. By (19) and $\lim_{r\to t^+} \phi(r) < t$ for $t > 0$, we get

$$r = \lim_{n\to\infty} \phi(D_n) = \lim_{D_n\to r^+} \phi(D_n) < r,$$

which is a contradiction, that is, $r = 0$, i.e.,

$$\lim_{n\to\infty} D_n = 0. \tag{20}$$

From (15)

$$d(x_1, f(x_1)) \leq \phi(d(x_0, f(x_0))).$$

Repeating this process n times, we get

$$D_n = d(x_n, f(x_n)) \leq \phi^n(d(x_0, f(x_0))), \text{ for all } n \geq 1.$$

Since $\sum_{n\geq 1} \phi^n(t)$ converges for all $t > 0$, we have that $\sum_{n\geq 1} D_n$ converges. We shall show that $\{x_n\}$ is a Cauchy sequence in X. As $\{x_n\}$ is strictly decreasing sequence, for $n, m \in \mathbb{N}$ with $n < m$, we have by using (15), (16) and (20)

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &= d(x_n, f(x_n)) + d(x_{n+1}, f(x_{n+1})) + ... + d(x_{m-1}, f(x_{m-1})) \\ &= D_n + D_{n+1} + ... + D_{m-1} \leq \sum_{k=n}^{\infty} D_k \to 0, \text{ as } n \to \infty. \end{aligned}$$

Thus $\{x_n\}$ is a monotonic increasing Cauchy sequence in X, which has the t-property, so there exists $u \in X$ such that $x_n \prec u$ for all n. By using (15) and (19), we have

$$d(u, f(u)) \leq \phi(d(x_n, f(x_n))) = \phi(D_n) \to 0, \text{ as } n \to \infty.$$

This shows that u is a fixed point of f in X. Let $z \in X$ be any strict upper bound of u, *i.e.*, $u \prec z$. By using (15) and Lemma 2.1, we have

$$d(z, f(z)) \leq \phi(d(u, f(u))) = \phi(0) = 0.$$

Hence z is also a fixed point of f in X. □

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