ON BI-Γ-IDEALS IN Γ-SEMIGROUPS WITH INVOLUTION

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Abstract. The concept of involution in semigroups was given by Nordahl et al [13]. In this paper, we introduce involution in Γ-semigroups. Also, we define bi-Γ-ideals in Γ-semigroups with involution and prove many interesting results characterizing Γ-semigroups with involution by using bi-Γ-ideals.
Keywords: Γ-semigroups, involution, quasi-Γ-ideal, bi-Γ-ideal.

1. Introduction and preliminaries

An involution semigroup S will mean a bijection $x \to x^\star$ of S onto itself, satisfying $(a^\star)^\star = a$, $(ab)^\star = b^\star a^\star$. If we consider involutions on various algebraic structures, it is generally needed that the defined involution is also an antiautomorphism of the underlying algebraic structures of period two. In that sense, involutions depict a fixed kind of internal symmetry of such systems. The natural example of an algebraic involution is the transposition of matrices in the algebra of matrices over a ring. Furthermore, an involution can be taken as a fundamental operation, and consequently, a part of the algebra on which it acts. For example, an involution semigroup is a triple $(S, \cdot, \star)$ such that $(S, \cdot)$ is a semigroup, while $\star$ is an involution on S such that $(xy)^\star = y^\star x^\star$ holds for all $x, y \in S$. In a similar fashion, if $(S, +, \cdot)$ is a semiring, then $(S, +, \cdot, \star)$ is

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called an involution semiring, provided that $\star$ is an involution of S satisfying the identities $(x+y)^\star = x^\star + y^\star, (xy)^\star = y^\star x^\star$. If it requires that semirings may be equipped with a zero 0, and/or an identity 1, then the involution satisfies $0^\star = 0$ and $1^\star = 1$. The motivation for investigation of involution semigroups arises from a wide range of involution rings, involution algebras, by leaving the additive structure. Γ-semigroups with involution is in a quite fascinating way strongly related to various classes of ideals like Γ-ideals, quasi-Γ-ideals and bi-Γ-ideals. Abbasi et al. [8] defined the involution in po-Γ-semigroups and studied many results on prime, semiprime and weakly prime ideals in involution po-Γ-semigroups. Furthermore, they characterized intra-regular involution po-Γ-semigroups. Various varieties of semigroups and algebras have unary operations imposed on them, including the classes of groups, inverse semigroups [6], cellular algebras [5], algebras [2], [6], [9], [15], primitive involution rings [7, 10] and regular $\star$-semigroups [13]. Scheiblich [4] constructed examples of bands for which two involutions lead to non-isomorphic regular $\star$-semigroups. For other results and examples, we refer [2],[3], [11], [12] and [14].
For Γ-semigroups, we refer [1]. In order to prove our main results, we introduce the following definitions and examples:

Definition 1.1. *Let A and Γ be any two nonempty sets. If there exists a mapping $A \times \Gamma \times A \to A$ such that $a\gamma b \in A \forall a, b \in A$ and $\gamma \in \Gamma$ and $A^\star \subseteq A$. Then A is called Γ-semigroup with involution.*

Example 1.1. Let $A = \left\{ x : x = \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in R \right\}$ and $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\begin{gathered} * : A^* \to A \\ \text{s.t. } x^* \to x^T \end{gathered}$$

Then A is a Γ-semigroup with involution as $A \times \Gamma \times A \longrightarrow A$ and $A^T \subset A$.

Example 1.2. Let us consider $A = (0, 2]$ and $* : A \to A$ such that
$a^* \to 1/a \forall a \in A$.
Then it is not a semigroup. If we define $\Gamma = \{1/4n : n \in N\}$. Then A is a Γ-semigroup with involution $\star$.

Definition 1.2. *Let S be a Γ-semigroup with involution $\star$. A sub-Γ-semigroup B of a Γ-semigroup S with involution $\star$ is called a bi-Γ-ideal of S with involution if $B\Gamma S\Gamma B \subseteq B$ and $B^\star \subseteq B$.*

Example 1.3. Let $S = \left\{ x : x = \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in R \right\}$ and $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Then S is a Γ-semigroup with involution, where

$$\begin{gathered} * : A^* \to A \\ \text{s.t. } x^* \to x^T \end{gathered}$$

Let $A = \left\{ x : x = \begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix} : \alpha \in R \right\}$. Then A is a bi-Γ-ideal of S.

Definition 1.3. *A non-empty subset Q of a Γ-semigroup S with involution $\star$ is called a quasi-Γ-ideal of S if*
$Q\Gamma S \cap S\Gamma Q \subseteq Q$ *and* $Q^\star \subseteq Q$.

2. Γ-semigroups with involution

In this section, we prove a number of important results concerning characterizations of Γ-semigroups with involution using bi-Γ-ideals.

Theorem 2.1. *Every quasi-Γ-ideal of a Γ-semigroup S with involution $\star$ is a bi-Γ-ideal of a Γ-semigroup S with involution $\star$.*

Proof. Let Q be a quasi-Γ-ideal of a Γ-semigroup S with involution $\star$. Then $Q\Gamma S \cap S\Gamma Q \subseteq Q$ and $Q^\star \subseteq Q$. Now, $Q\Gamma S\Gamma Q \subseteq Q\Gamma S\Gamma S \subseteq Q\Gamma S$.
Also, $Q\Gamma S\Gamma Q \subseteq S\Gamma S\Gamma Q \subseteq S\Gamma Q$. This implies that $Q\Gamma S\Gamma Q \subseteq Q\Gamma S \cap S\Gamma Q$. This further implies that $Q\Gamma S\Gamma Q \subseteq Q$ and $Q^\star \subseteq Q$. Hence, Q is a bi-Γ-ideal of a Γ-semigroup S with involution. □

Theorem 2.2. *Let S be a Γ-semigroup with order preserving involution $\star$. Then:*
(1) $(x\Gamma s\Gamma y)^\star = y^\star\Gamma S\Gamma x^\star$, *for any* $x, y \in S$.
(2) $(S\Gamma x\Gamma S)^\star = S\Gamma x^\star\Gamma S$, *for any* $x \in S$.

Proof. Let $t \in (x\Gamma s\Gamma y)^\star$. By definition, $t^\star \in x\Gamma s\Gamma y$, $t^\star \in x\beta s\gamma y$ for some $s \in S$ and $\beta,\gamma \in \Gamma$. This implies that
$t \in (x\beta s\gamma y)^\star \subseteq y^\star\beta s^\star\gamma x^\star \subseteq y^\star\Gamma S\Gamma x^\star$, because $\star$ is an order preserving involution. Thus $(x\Gamma s\Gamma y)^\star \subseteq y^\star\Gamma S\Gamma x^\star$.
On the other hand if $t \in y^\star\Gamma S\Gamma x^\star$, then for some $s \in S$, and $\beta, \gamma \in \Gamma$, we have $t \in y^\star\beta s^\star\gamma x^\star$. This implies that $t^\star \in x\beta s^\star \subseteq x\Gamma s\Gamma y$. As $t \in (x\Gamma s\Gamma y)^\star$. Therefore, $y^\star\Gamma S\Gamma x^\star \subseteq (x\Gamma s\Gamma y)^\star$.
Consequently, $(x\Gamma s\Gamma y)^\star = y^\star\Gamma S\Gamma x^\star$.
(2) The proof is similar to (1). □

Theorem 2.3. (1) *Suppose that A is a sub-Γ-semigroup of a Γ-semigroup S with involution $\star$, $s \in S$ and $(s\Gamma A\Gamma s) \cap A \neq \emptyset$, then $(s\Gamma A\Gamma s) \cap A$ is bi-Γ-ideal of A.*
(2) *Let S be a Γ-semigroup with involution and T a non-empty subset of S. Then, $T \cup T\Gamma S\Gamma T$ is the bi-Γ-ideal of S with involution $\star$.*

Proof.

$$
\begin{aligned}
(1)(s\Gamma A\Gamma s \cap A)\Gamma A\Gamma(s\Gamma A\Gamma s \cap A) &\subseteq [(s\Gamma A\Gamma s)\Gamma A \cap A\Gamma A]\Gamma(s\Gamma A\Gamma s \cap A)\\
&\subseteq [(s\Gamma A\Gamma s)\Gamma A \cap A]\Gamma(s\Gamma A\Gamma s \cap A)\\
&\subseteq [[(s\Gamma A\Gamma s\Gamma A)\Gamma(s\Gamma A\Gamma s)] \cap [A\Gamma(s\Gamma A\Gamma s)\Gamma A \cap A)]]\\
&\subseteq [(s\Gamma A\Gamma s) \cap (A\Gamma s\Gamma A\Gamma s)] \cap A\\
&\subseteq (s\Gamma A\Gamma s \cap A).
\end{aligned}
$$

Hence $(s\Gamma A\Gamma s)\cap A$ is a bi-Γ-ideal of A.
Now

$$\begin{aligned}[(s\Gamma A\Gamma s\cap A)]^{\star} &\subseteq (s\Gamma A\Gamma s)^{\star}\cap A^{\star}\\ &\subseteq [(s\Gamma A)\Gamma s]^{\star}\cap A^{\star}\\ &\subseteq s^{\star}\Gamma(s\Gamma A)^{\star}\cap A^{\star}\\ &\subseteq s^{\star}\Gamma(A^{\star}\Gamma s^{\star})\cap A^{\star}\\ &\subseteq (s\Gamma A\Gamma s)\cap A.\end{aligned}$$

Hence $(s\Gamma A\Gamma s)\cap A$ is bi-Γ-ideal of A with involution.

(2) Let $B = T\cup T\Gamma S\Gamma T$. Then $T\subseteq B$. So

$$\begin{aligned}&B\Gamma S\Gamma B = (T\cup T\Gamma S\Gamma T)\Gamma S\Gamma(T\cup T\Gamma S\Gamma T)\\ &\subseteq [T(\Gamma S\Gamma)(T\cup T\Gamma S\Gamma T)]\cup[T\Gamma S\Gamma T(\Gamma S\Gamma)(T\cup T\Gamma S\Gamma T)]\\ &\subseteq [T(\Gamma S\Gamma)T\cup T(\Gamma S\Gamma)T\Gamma S\Gamma T]\cup[T\Gamma S\Gamma T(\Gamma S\Gamma)T\cup T\Gamma S\Gamma T(\Gamma S\Gamma)T\Gamma S\Gamma T]\\ &\subseteq [T\Gamma S\Gamma T\cup T\Gamma S\Gamma T]\cup[T\Gamma S\Gamma T\cup T\Gamma S\Gamma T]\\ &= T\Gamma S\Gamma T\cup T\cup T\Gamma S\Gamma T = B.\end{aligned}$$

Hence $B = T\cup T\Gamma S\Gamma T$ is bi-Γ-ideal. Consider:

$$\begin{aligned}(T\cup T\Gamma S\Gamma T)^{\star} &\subseteq T^{\star}\cup(T\Gamma S\Gamma T)^{\star}\\ &\subseteq T\cup[(T\Gamma S)\Gamma T]^{\star}\\ &\subseteq T\cup[T^{\star}\Gamma(T\Gamma S)^{\star}]\\ &\subseteq T\cup[T^{\star}\Gamma s^{\star}\Gamma T^{\star}]\\ &\subseteq T\cup T\Gamma S\Gamma T.\end{aligned}$$

Hence $T\cup T\Gamma S\Gamma T$ is a bi-Γ-ideal with involution. □

Theorem 2.4. *Let S be a Γ-semigroup with order preserving involution $\star$. Then the following statements hold:*
(1) If $\{A_i^{\star} : i\in I\}$ is a family of left(resp., right) Γ-ideals of S, then the intersection $\cap A_i^{\star}\neq\emptyset$ is a left (resp.,right)Γ-ideals of S.
(2) If $\{A_i^{\star} : i\in I\}$ is a family of bi-Γ-ideals of S, then the intersection $\cap A_i^{\star}\neq\emptyset$ is a bi-Γ-ideal of S.
(3) If $\{A_i^{\star} : i\in I\}$ is a family of quasi-Γ-ideal of S, then the intersection $\cap A_i^{\star}\neq\emptyset$ is a quasi-Γ-ideal of S.

Proof. (1) Let $\{A_i^{\star} : i\in I\}$ be a family of left-Γ-ideals of S. Then $S\Gamma A_i^{\star}\subseteq A_i^{\star}$.
Consider:

$$\begin{aligned}S\Gamma\cap A_i^{\star} &\subseteq S\Gamma A_i^{\star}\\ &\subseteq A_i^{\star}\text{ for all i}\in\text{I}\\ &\subseteq \cap A_i^{\star}\end{aligned}$$

(2) Let $\{A_i^{\star} : i\in I\}$ be a family of bi-Γ-ideals of S. Then $A_i^{\star}\Gamma S\Gamma A_i^{\star}\subseteq A_i^{\star}$. Now consider:

$$\begin{aligned}\cap A_i^{\star}\Gamma S\Gamma\cap A_i^{\star} &\subseteq A_i^{\star}\Gamma S\Gamma A_i^{\star}\\ &\subseteq A_i^{\star}\text{ for all i}\in\text{I}\\ &\subseteq \cap A_i^{\star}.\end{aligned}$$

(3) Let $\{A_i^\star : i \in I\}$ be a family of quasi-Γ-ideals of S. Then $A_i^\star \Gamma S \cap S\Gamma A_i^\star \subseteq A_i^\star$. Consider $\cap A_i^\star \Gamma S \cap S\Gamma \cap A_i^\star \subseteq A_i^\star \Gamma S \cap S\Gamma A_i^\star \subseteq A_i^\star$ for all $\mathrm{i} \in \mathrm{I} \subseteq \cap \mathrm{A}_{\mathrm{i}}^\star$. □

Theorem 2.5. *Let S be a Γ-semigroup with order preserving involution $\star$. Then:*

(1) *$A^\star$ is a left (resp.,right) Γ-ideal for any right (resp.,left) Γ-ideal A of S.*
(2) *$B^\star$ is a bi-Γ-ideal for any bi-Γ-ideal B of S.*
(3) *$Q^\star$ is a quasi-Γ-ideal for any quasi-Γ-ideal Q of S.*

Proof. (1) Let A be right-Γ-ideal of S. Then $A\Gamma S \subseteq A$ and $S^\star = S$. Consider:

$$S\Gamma A^\star = S^\star \Gamma A^\star = (A\Gamma S)^\star \subseteq A^\star.$$

Thus $A^\star$ is a left-Γ-ideal of S.

(2) Let B be a bi-Γ-ideal of S. This implies $B\Gamma S\Gamma B \subseteq B$ and $S^\star = S$.

Now consider $B^\star \Gamma S \Gamma B^\star = B^\star \Gamma S^\star \Gamma B^\star = (B\Gamma S\Gamma B)^\star \subseteq B^\star$. Hence $B^\star$ is a bi-Γ-ideal of S.

(3) Let Q be quasi-Γ-ideal of S. Then $Q\Gamma S \cap S\Gamma Q \subseteq Q$ and $S^\star = S$. Now consider

$$\begin{aligned} Q^\star \Gamma S \cap S\Gamma Q^\star &\subseteq Q^\star \Gamma S^\star \cap S^\star \Gamma Q^\star \\ &\subseteq S^\star \Gamma Q^\star \cap Q^\star \Gamma S^\star \\ &\subseteq (Q\Gamma S)^\star \cap (S\Gamma Q)^\star \\ &\subseteq (Q\Gamma S \cap S\Gamma Q)^\star \\ &\subseteq Q^\star. \end{aligned}$$

Hence $Q^\star$ is a quasi-Γ-ideal of S. □

Theorem 2.6. *Let S be a Γ-semigroup with involution *. If $A = A^\star \Gamma A^\star$, then $A^\star \cap B^\star = A\Gamma B$ for any Γ-ideals A and B of S.*

Proof. Let A and B be two Γ-ideals of S. By Theorem 2.5, $A^\star$ and $B^\star$ are Γ-ideals of S. Now we have $A\Gamma B \subseteq A\Gamma S \subseteq A = A^\star \Gamma A^\star \subseteq A^\star$. Similarly, $A\Gamma B \subseteq S\Gamma B \subseteq B = B^\star \Gamma B^\star \subseteq B^\star$. Thus $A\Gamma B \subseteq A^\star \cap B^\star$.
On the other hand, $A^\star \cap B^\star$ is a Γ-ideal of S. This implies that $A^\star \cap B^\star = (A^\star \cap B^\star)^\star \Gamma (A^\star \cap B^\star)^\star = (A \cap B)\Gamma(A \cap B) \subseteq A\Gamma B$. So $A^\star \cap B^\star = A\Gamma B$. □

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