THE 1-PLANARITY OF INTERSECTION GRAPH OF IDEALS OF A RING

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Abstract. A graph $G = (V(G), E(G))$ is called 1-planar if it can be drawn in the plane such that every edge of the graph is cut by at most one other edge of the graph. For any ring R, the ideal intersection graph of R, denoted by $G(R)$, is the graph whose vertices are the nontrivial proper ideals of R and two distinct vertices are adjacent if they have nontrivial intersection. In this paper, we characterize when the intersection graph $G(R)$ of a ring R, is 1-planar.
Keywords: intersection graph of ideals of a ring, 1-planar graph, artinian ring.

1. Introduction

Through out this paper a graph means a finite simple graph, i.e. a graph without loops and multiple edges. Recall that a graph is called planar, if it can be drawn in the plane with nonintersecting edges except of the ends. Always it is interesting to characterize when a graph is planar. One generalization of planar graphs is called 1-planar. A graph is called 1-planar if it can be drawn in the plane with every edge is cut in at most one point except of the ends. This class of graphs is interesting in computer sciences, especially in networks. It was studied extensively in literature, see [3], [7], and [8]. The following lemma, summarizes the 1-planarity of a graph G with seven vertices or less, see [7].

Lemma 1. *The graph $K_7 - K_3$ is the unique 7-vertex minimal non 1-planar graph.*

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All rings, R, in this paper are commutative with unity $1 \neq 0$. A local ring R, is a ring with only one maximal ideal, say M. An Artinian ring R is the direct product of a finite number of Artinian local rings, $R = \prod_{i=1}^{m} R_i$. Moreover, every ideal of R_i is finitely generated. For an Artinian local ring R with unique maximal ideal M, there exists a least positive integer k such that $M^k = 0$. In this case we say that the local ring R has nilpotency index k, see [2].

Let $G(R)$ be the ideal intersection graph of a ring R, with vertices are the proper nontrivial ideals of the ring R, and two vertices are adjacent if they intersect nontrivially. Akbari et. al. [1] proved that, R is Artinian if $G(R)$ is finite. Moreover, if $G(R)$ is finite and connected, then $\operatorname{diam}(G(R)) \leq 2$. From now on, it is enough to consider Artinian rings, to characterize when the intersection graph $G(R)$, is 1-planar.

The idea behind introducing the intersection graph, $G(R)$, of a ring R is to study the interrelationship between algebraic properties of the ring R and the graph theoretic properties of the graph $G(R)$. So, $G(R)$ was investigated by many authors in literature, see [1], [4] and [9]. Jafari and Rad in [6] characterized when the intersection graph, $G(R)$, of ideals of a ring R is planar. Here in the same spirit we characterize when the intersection graph $G(R)$, is 1-planar.

For undefined notions and terminology, the reader is referred to [2] and [5].

2. The 1-planarity of intersection graph $G(R)$

As the first result, we characterize when an Artinian ring, which is the direct product of at least three local rings, is 1-planar.

Theorem 1. *Let $R = \prod_{i=1}^{m} R_i$, for $m \geq 3$. The graph $G(R)$ is 1-planar if and only if R is the product of three fields.*

Proof. Firstly, assume that $m = 3$ and that R_1, R_2, and R_3 are fields. Then $G(R)$ has exactly six vertices. So, $G(R)$ is 1-planar. But, if at least one of R_1, R_2, and R_3 is not a field, say R_1, with nontrivial maximal ideal M_1. Then $G(R)$ has at least the following vertices $R_1 \times R_2 \times \{0\}$, $R_1 \times \{0\} \times \{0\}$, $M_1 \times \{0\} \times \{0\}$, $M_1 \times R_2 \times \{0\}$, $R_1 \times \{0\} \times R_3$, $M_1 \times \{0\} \times R_3$, $M_1 \times R_2 \times R_3$. These vertices induce the complete subgraph K_7. Hence, by Lemma 1, the result follows.

Secondly, assume that $m > 3$, then $|G(R)| \geq 7$ and K_7 is an induced subgraph of $G(R)$. Thus, by lemma 1, $G(R)$ is non 1-planar graph. □

Lemma 2. *Let R be a direct product of a field, say F_1, and a local ring, say R_2. Then $G(R)$ is 1-planar if and only if $G(R_2)$ is isomorphic to K_1, K_2, or P_3.*

Proof. Assume that $G(R_2)$ is isomorphic to K_2, then there is an ideal I_1 of R_2 such that $\{0\} \subsetneq I_1 \subsetneq M \subsetneq R_2$, and $V(G(R)) = V(G(F_1 \times R_2)) = \{F_1 \times \{0\}, F_1 \times I_1, F_1 \times M, \{0\} \times I_1, \{0\} \times M, \{0\} \times R_2\}$. Then $|G(R)| = 6$, hence $G(R)$ is 1-planar. While, if $G(R_2)$ is isomorphic to K_1 then $|G(R)| < 6$. Moreover, if $G(R_2)$ is isomorphic to P_3, then there are two ideals I_1 and I_2 of R_2 such that $\{0\} \subsetneq I_1 \subsetneq M \subsetneq R_2$, $\{0\} \subsetneq I_2 \subsetneq M \subsetneq R_2$, and $I_1 \cap I_2 = \emptyset$. Hence, $G(R) = G(F_1 \times R_2)$ is 1-planar, see Figure 1.

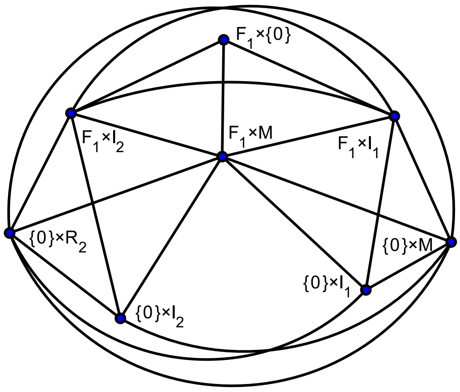

Figure 1: $G(R) = G(F_1 \times R_2)$, where $G(R_2)$ is isomorphic to P_3

On the other hand, assume that $|G(R_2)| = 3$ but not isomorphic to P_3, then $G(R_2)$ is isomorphic to K_3, where $\{0\} \subsetneq I_1 \subsetneq I_2 \subsetneq M \subsetneq R_2$. Then the set of vertices $F_1 \times M$, $F_1 \times I_1$, $F_1 \times I_2$, $\{0\} \times I_1$, $\{0\} \times I_2$, $\{0\} \times M$, $\{0\} \times R_2$, induces a subgraph isomorphic to K_7 in the graph $G(R)$. Hence, by Lemma 1, $G(R)$ is non 1-planar.

Moreover, assume that $|G(R_2)| \geq 4$, then the simplest connected intersection graph of $G(R_2)$, is a star graph with three edges, R_2 is local ring. Though, we have three mutually disjoint, say I_1, I_2, I_3, non trivial and non maximal ideals, that are subsets of maximal ideal, say M. Then the set of vertices $\{0\} \times R_2$, $\{0\} \times M$, $\{0\} \times I_1$, $F_1 \times M$, $F_1 \times I_1$, $F_1 \times I_2$, $F_1 \times I_3$, induce a subgraph isomorphic to $K_7 - P_2$ in the graph of $G(R)$. Hence, by Lemma 1, $G(R)$ is non 1-planar. □

Lemma 3. *Let R be a direct product of two local rings, $R = R_1 \times R_2$, that are not fields. Then $G(R)$ is 1-planar if and only if each R_i has only one proper nontrivial ideal.*

Proof. If both rings R_1 and R_2 has at most one nontrivial proper ideal, say M_1 and M_2 respectively. Then $G(R)$ has at most the following vertices $\{0\} \times R_2$, $\{0\} \times M_2$, $M_1 \times \{0\}$, $M_1 \times R_2$, $R_1 \times \{0\}$, $M_1 \times M_2$, and $R_1 \times M_2$. The graph of $G(R)$ is isomorphic to a subgraph of $K_7 - C_4$. Hence by Lemma 1, $G(R)$ is 1-planar.

On the other hand, if both R_1 and R_2 are not fields with one of them has at least two nontrivial proper ideals, say $I \subset M_2 \subset R_2$, and R_1 has $M_1 \neq 0$, then $G(R)$ has at least seven mutually adjacent vertices, namely $\{0\} \times M_2$, $R_1 \times$

M_2, $R_1 \times I$, $\{0\} \times I$, $M_1 \times R_2$, $M_1 \times M_2$, and $M_1 \times I$. These induce the complete subgraph K_7. Hence, by Lemma 1, $G(R)$ is non 1-planar. □

Now, we summarize the case when R is the direct product of two local rings.

Theorem 2. *For $R = R_1 \times R_2$, $G(R)$ is 1-planar if and only if one of the following holds:*

1. *R_1 is a field and $G(R_2)$ is isomorphic to a path of length at most three.*
2. *Each R_i has at most one proper non trivial ideal.*

Finally, we consider the local Artinian rings.

Theorem 3. *If a ring R is a local ring whose maximal ideal is principal, then $G(R)$ is 1-planar if and only if $|G(R)| \leq 6$*

Proof. Let R be a local ring whose maximal ideal is principal. Then R is a chained principal ideal ring. So, $G(R)$ is a complete graph. Hence, by Lemma 1, the result follows. □

Theorem 4. *Let R be a local non principal ideal ring, whose maximal ideal has exactly two generators x and y. Then $G(R)$ is 1-planar if and only if either $M^2 = 0$, or $M^4 = 0$ with $M =< x, y >$ and $x^2 = 0 = y^2$.*

Proof. If either $M^2 = 0$, or $M^4 = 0$ with $M =< x, y >$ and $x^2 = 0 = y^2$. Then $|G(R)| \leq 5$, hence by Lemma 1, $G(R)$ is 1-planar.

Conversely, assume that M has nilpotency index 4 with $M =< x, y >$ and $x^4 = 0 = y^2$. Then $G(R)$ has a subgraph, that is induced by the vertices $< x, y >$, $< x^2, y >$, $< x^2 >$, $< x^3, y >$, $< x^3 >$, $< x >$, and $< xy >$, which is isomorphic to the complete K_7. Moreover, assume that M has nilpotency index equals 3 with $M =< x, y >$ and $x^3 = 0 = y^3$. Then $G(R)$ has the subgraph, say G_1, that is induced by the vertices $< x >, < y >, < x^2 >, < x, y >, < x, y^2 >$, $< x^2, y >$, and $< x^2, y^2 >$, that is isomorphic to the graph $K_7 - P_2$. Now, assume that M has nilpotency index equals r with $r \geq 5$ and $M =< x, y >$, then $G(R)$ has a subgraph that is isomorphic to K_7. Hence, for all these cases, $G(R)$ is non 1-planar by Lemma 1. □

Theorem 5. *Let R be a local ring with maximal ideal M, that can't be generated by less than three generators. Then $G(R)$ is 1-planar if and only if $M^2 = 0$ and $M =< x, y, z >$.*

Proof. If $M =< x, y, z >$ and $M^2 = 0$, then $G(R)$ has at most the following vertices $< x >$, $< y >$, $< z >$, $< x, y >$, $< x, z >$, $< y, z >$, $< x, y, z >$. Then the graph of $G(R)$ is isomorphic to a proper subgraph of $K_7 - k_3$, hence by Lemma 1, $G(R)$ is 1-planar.

Assume that R is a local ring with maximal ideal M with three generators or more. If the nilpotency index of M equals r with $r \geq 3$. Then $|G(R)| \geq 8$ and $G(R)$ has a subgraph isomorphic to G_1, that is defined in the proof of the previous theorem. Hence the result follows. □

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References

[1] S. Akbari, R. Nikandish, M.J. Nikmehr, *Some results on the intersection graphs of ideals of rings*, Journal of Algebra and its Applications, 12 (2013), 1250200-1 - 1250200-13.

[2] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra,* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[3] O. V. Borodin, *Solution of Ringel's problems on the vertex-face coloring of plane graphs and on the coloring of 1-planar graphs*, Diskret. Anal. Novosibirsk, 41 (1984), 12 - 26(in Russian).

[4] I. Chakrabarty, S. Ghosh, T.K. Mukherjee, M.K. Sen, *Intersection graphs of ideals of rings*, Discrete Mathemaics, 309 (2009), 5381 - 5392.

[5] J. Gross and J. Yellen, *Hand Book of Graph Theory (1st ed.),* CRC PRESS, Inc 2003.

[6] S. H. Jafari and N. J. Rad, *Planarity of intersection graphs of ideals of rings*, International Electronic Journal of Algebra, 8 (2010), 161-166.

[7] V. P. Korzhik, *Minimal non-1-planar graphs*, Discrete Mathemaics, 308 (2008), 1319-1327.

[8] G. Ringel, *Ein Sechsfarbenproblem auf der kugel*, Abh. Math. Sem. Univ. Hamburg, 29 (1965), 107-117.

[9] E. Szpilrajn-Marczewski, *Sur deux properties des classes d'ensembles*, Fund. Math., 33 (1945), 303-307.

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