# NON-LINEAR STABILITY OF $L_4$ IN THE R3BP WHEN THE SMALLER PRIMARY IS A HETEROGENEOUS TRIAXIAL RIGID BODY WITH $N$ LAYERS

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**Abstract.** In the present paper, the non-linear stability of the triangular libration point ($L_4$) in the restricted three-body problem(R3BP) when less massive primary is a heterogeneous triaxial rigid body has been studied with the assumption that the primary has N layers having different densities. Following the procedure of Birkhoff's normalization, we normalized the Hamiltonian up to second order and the co-ordinates $(x, y)$ are expanded in double D'Alembert series. The non-linear stability of the triangular libration point is discussed by applying Moser's modified version of Arnold's theorem (1961) as well as following the procedure as adopted by Bhatnagar and Hallan (1983). It is observed that Moser's theorem is applicable in the range of linear stability, except for three mass ratios depending upon heterogeneous triaxial rigid body.

**Keywords:** R3BP, libration point $L_4$, non-linear stability, heterogeneous triaxial rigid body, densities.

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## 1. Introduction

The restricted three-body problem (R3BP) concerned the motion of the infinitesimal mass under the gravitational influence of two finite bodies: There are two masses, called primaries, moving in circular orbits around their common center of mass. There is another mass, which is infinitesimal, moving in the plane of motion of the primaries such that it does not influence the motion of the primaries but is influenced by them. To describe the motion of the third mass is called the circular restricted problem. In case the primaries move in elliptical orbits instead of circular orbits, the problem is known as pseud restricted problem. R3BP has been stated by many authors. In the restricted three-body problem, distinct particular solutions exist. These particular solutions are known as Lagrangian points and in the bi-dimensional system, five Lagrangian points exist. Three of these are the collinear points and the other two are the triangular points. Several authors have investigated the stability of these points in linear sense and found that the collinear points are unstable where as non collinear points are stable in the some range.

Many mathematicians and astronomers have discussed non-linear stability by taking different aspects of the restricted three-body problem and made valuable contributions. Deprit and Deprit (1967) discussed the stability of the triangular Lagrangian points. Bhatnagar and Hallan(1983) studied the effect of perturbations in Coriolis and Centrifugal force on the nonlinear stability of the equilibrium points in the restricted problem of three bodies . Gyorgyey (1985) investigated on the non-linear stability of motions around $L_5$ in the elliptic restricted problem of three bodies. Krzysztof et al. (1991) studied about the libration points in the restricted photo-gravitational three-body problem. Sharma et al. (1997) discussed on the effect of oblateness on the non-linear stability of $L_4$ in the restricted three-body problem. Esteban et al. (2001) analyzed the rotating stratified heterogeneous oblate spheroid in Newtonian physics. Jain et al. (2001) studied on the non-linear stability of $L_4$ in the restricted three-body problem when the primaries are triaxial rigid bodies. Andres et al.(2001) studied the non-linear stability of the equilibria in the gravity field of a finite straight segment. Chandra et al. (2004) discussed the effect of oblateness on the non-linear stability of the triangular liberation points of the restricted three-body problem in the presence of resonances. Aggarwal et al. (2006) investigated the Non-linear stability of $L_4$ restricted three-body problem for radiated axes symmetric primaries with resonances. Kushvah et al.(2007) studied the non-linear stability in the generalized photo-gravitational restricted three-body problem with Poynting-Robertson drag. Singh (2011) examined the non-linear stability in the restricted three-body problem with oblate variable mass. Ishwar et al. (2012) investigated the non-linear stability in photo-gravitational non-planer R3BP with oblate smaller primary. Jain et al. (2014) studied the non-linear stability of $L_4$ in the restricted problem when the primaries are finite straight segment under resonances. Ansari (2017 a, b) investigated the dynamical behavior in the restricted three-body problem with perturbations. Shalini et al. (2016, 2017) studied the stability of $L_4$ in the R3BP by taking the smaller primary is a heterogeneous spheroid with layers.

In the present work, we propose to discuss the R3BT with the assumption that the massive primary is a point mass and less massive primary is a heterogeneous triaxial rigid body with N layers, having different densities $\rho_i$ and axes $(a_i, b_i, c_i)$, $(i = 1, 2, 3, \ldots, N)$ respectively. The main objective of this paper is to study of the stability of libration point $L_4$ in non-linear sense. For this we will apply Moser's modified version of Arnold's theorem (1961) and procedure as adopted by Bhatnagar and Hallan (1983).

This paper should be read in conjunction with the papers by Bhatnagar and Hallan (1983) and Shalini (2017). As to save space, we are not mentioning the values of various variables given in those papers, although they are used in this paper.

This paper is organized as follows: In section Introduction we have reviewed the literature related to R3BP under different perturbations. In section Equations of motion we have derived the potential of heterogeneous triaxial rigid body with N layers and mean motion of the primaries and further, formulated the equations of motion of the proposed system. Section Location of Triangular Points,we have obtained the coordinates of non-collinear libration points. Section First order normalization, deals with the first order normalization. In section Second order normalization, we have determined the second order normalization. In section Second order coefficients in the frequencies, we have found the second order frequencies. In section Stability, we have checked the non-linear stability of triangular libration points. Section Conclusion, contains the conclusion of the obtained results.

## 2. Equations of motion

Let $m_1$ and $m_2$ $(m_1 > m_2)$ be the two masses of the primaries at $P_1$ and $P_2$ respectively as shown in Fig. (1) are moving in circular orbits around their common center of mass $O$ which is taken as the origin, $OP_1 = 1-\mu$ and $OP_2 = \mu$, as the distances of primaries from the center of mass. Let $m_1$ be point mass and $m_2$ a heterogeneous triaxial rigid body with $N$ layers having axes ($a_i$, $b_i$, $c_i$) $(a_i > b_i > c_i)$ and densities $\rho_i$. An infinitesimal mass $m_3$ which is much less than masses of the primaries is moving in the plane of motion of $m_1$ and $m_2$. Let $\overrightarrow{r}$, $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ as the distances of infinitesimal mass, first primary and second primary from center of mass respectively. $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ are the gravitational forces acting on $m_3$ due to $m_1$ and $m_2$ respectively. Also let us consider that the principal axes of heterogeneous triaxial rigid body remain parallel to the synodic axes $Oxyz$ throughout the motion and the equatorial plane of $m_2$ is coincide with the plane of motion of $m_1$ and $m_2$.$(Fig.1(a),(b))$ The equation of motion of $m_3$ in the vector form is

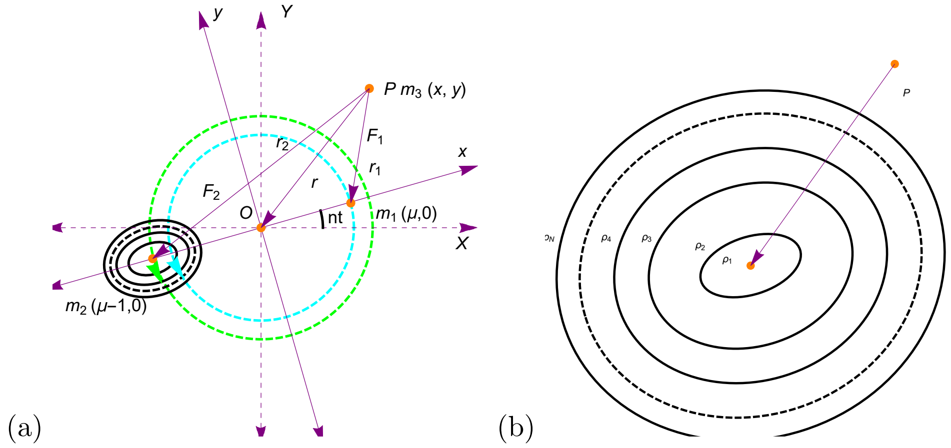


Figure 1: (a): Configuration of the restricted three-body problem with $m_2$ as heterogeneous triaxial rigid body, (b): Heterogeneous triaxial rigid body with $N$ Layers

$$m_3\left(\frac{\partial^2\overrightarrow{r}}{\partial t^2}+2\overrightarrow{\omega}\times\frac{\partial\overrightarrow{r}}{\partial t}+\frac{\partial\overrightarrow{\omega}}{\partial t}\times\overrightarrow{r}+\overrightarrow{\omega}\times(\overrightarrow{\omega}\times\overrightarrow{r})\right)=\overrightarrow{F}, \tag{1}$$

where $\overrightarrow{r}=\overrightarrow{OP}$, $\overrightarrow{\omega}=\mathrm{n}\,\widehat{k}$ = angular velocity = constant, $\overrightarrow{F}$ = total force acting on $m_3$.

The gravitational potential of the heterogeneous triaxial rigid body of mass $m_2$, with $N$ layers of densities $\rho_i$ and axes $(a_i, b_i, c_i)$, $\rho_i < \rho_{i+1}$, $a_i < a_{i+1}$, $b_i < b_{i+1}$, $c_i < c_{i+1}$ at the point P is

$$V_2 = V_{NN} + V_{(N-1)N} + \ldots + V_{2N} + V_{1N}(say), \tag{2}$$

where $V_{NN}, V_{(N-1)N}, \ldots, V_{2N}, V_{1N}$, are the potential of the triaxial rigid body of densities $\rho_{NN}, \rho_{(N-1)N}, ......, \rho_{2N}, \rho_{1N}$, for the regions $N, N-1, \ldots, 1$, respectively. Here, $V_{NN} = V'_{NN} - V'_{N(N-1)}(say)$, where $V'_{NN}$ = potential of the triaxial rigid body of axes $(a_N, b_N, c_N)$ with homogeneous density $\rho_N$ throughout at $P, = \frac{-4\pi\rho_N G}{3r_2}a_N b_N c_N[1+\frac{1}{10r_2^2}(2a_N^2-b_N^2-c_N^2-\frac{3}{r_2^2}(a_N^2-b_N^2)y^2)]$, and $V'_{N(N-1)}$ = potential of the triaxial rigid body of axes $(a_{N-1}, b_{N-1}, c_{N-1})$ with homogeneous density $\rho_N$ throughout at $P = \frac{-4\pi\rho_N G}{3r_2}a_{N-1}b_{N-1}c_{N-1}[1+\frac{1}{10r_2^2}(2a_{N-1}^2-b_{N-1}^2-c_{N-1}^2-\frac{3}{r_2^2}(a_{N-1}^2-b_{N-1}^2)y^2)]$.

Thus,

$$\begin{cases} V_{NN}=\frac{-4\pi\rho_N G}{3r_2}[a_N b_N c_N\{1+\frac{1}{10r_2^2}(2a_N^2-b_N^2-c_N^2-\frac{3}{r_2^2}(a_N^2-b_N^2)y^2)\} \\ -a_{N-1}b_{N-1}c_{N-1}\{1+\frac{1}{10r_2^2}(2a_{N-1}^2-b_{N-1}^2-c_{N-1}^2-\frac{3}{r_2^2}(a_{N-1}^2-b_{N-1}^2)y^2)\}]. \end{cases} \tag{3}$$

Similarly, we can find $V_{(N-1)N}, \ldots, V_{2N}, V_{1N}$, and substituting in Eq.(2), we have

$$V_2 = -\frac{m_2 G}{r_2} - \frac{k_1' G}{2r_2^3} + \frac{k_2' G y^2}{2r_2^5}. \tag{4}$$

where

$$k_1' = \frac{4\pi}{3}\sum_{i=1}^{N}((\rho_i - \rho_{i+1})a_i b_i c_i \sigma_{i,1}), \quad k_2' = \frac{4\pi}{3}\sum_{i=1}^{N}((\rho_i - \rho_{i+1})a_i b_i c_i \sigma_{i,2}),$$

$\sigma_{i,1} = \frac{(2a_i^2 - b_i^2 - c_i^2)}{5}$, $\sigma_{i,2} = \frac{(a_i^2 - b_i^2)}{5}$, $\sigma_{N+1} \neq 0$.

Hence, the total potential at P due to $m_1$ and $m_2$ is given by $V = -\frac{m_1 G}{r_1} - \frac{m_2 G}{r_2} - \frac{k_1' G}{2r_2^3} + \frac{k_2' G y^2}{2r_2^5}$.

Let us fix the units of mass,length and time with the assumption that the gravitational constant G, the sum of the masses and the distance between both primaries be equal to unity.

Then the equations of motion in Synodic co-ordinates system and dimensionless variables are

$$\begin{cases} \ddot{x} - 2n\dot{y} = \frac{\partial \Omega}{\partial x} \\ \ddot{y} + 2n\dot{x} = \frac{\partial \Omega}{\partial y} \end{cases} \tag{5}$$

where $\Omega = n(\frac{x^2+y^2}{2}) + \frac{1-\mu}{r_1} - \frac{\mu}{r_2} + \frac{k_1}{2r_2^3} - \frac{3k_2 y^2}{r_2^5}$, $r_1^2 = (x-\mu)^2 + y^2$, $r_2^2 = (x - \mu + 1)^2 + y^2$,

$$k_1 = \frac{4\pi}{3}\sum_{i=1}^{N}((\rho_i' - \rho_{i+1}')a_i' b_i' c_i' \sigma_{i,1}'), \quad k_2 = \frac{4\pi}{3}\sum_{i=1}^{N}((\rho_i' - \rho_{i+1}')a_i' b_i' c_i' \sigma_{i,2}'),$$

$\sigma_{i,1}' = \frac{(2a_i'^2 - b_i'^2 - c_i'^2)}{5R^2}$, $\sigma_{i,2}' = \frac{(a_i'^2 - b_i'^2)}{5R^2}$, $a_i' = \frac{a_i}{R}$, $b_i' = \frac{b_i}{R}$, $c_i' = \frac{c_i}{R}$, $\rho_i' = \frac{\rho_i}{M}$, $\rho_{N+1}' \neq 0$, $M = m_1 + m_2$, $k_1, k_2 << 1$, R = dimensional distance between the primaries.

### 2.1 Mean motion

The potential of the triaxial rigid body is $-(\frac{m_2 G}{R} + \frac{k_1 G}{2R^3})$. Let the distances of $m_1$ and $m_2$ from the center of mass $O$ be $a'$ and $b'$ respectively. Since $m_1$ and $m_2$ are moving in circular orbits about $O$, we have $m_1 a' n^2 = (\frac{m_2 G}{R^2} + \frac{3k_1 G}{2R^4})m_1$ and $m_2 b' n^2 = (\frac{m_2 G}{R^2} + \frac{3k_1 G}{2R^4})m_1$.

Adding these equations, we have $n^2 = (\frac{m_2 G}{(a'+b')^3} + \frac{3k_1 G}{2(a'+b')^5})(\frac{m_1+m_2}{m_2})$. Using the dimensionless variables, we get the mean motion as

$$n = 1 + ck_1, \tag{6}$$

where $c = \frac{3}{4\mu}$.

## 3. Locations of triangular points

The locations of triangular libration points are solutions of the Eq. (5) obtained by making all the derivatives equals to zero (i.e.$\Omega_x = 0$ and $\Omega_y = 0$). i.e.:

$$
\begin{aligned}
& nx - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x-\mu+1)}{r_2^3} - \frac{(3k_1)(x+1-\mu)}{2r_2^5} \\
& + \frac{(15k_2)(x+1-\mu)y^2}{2r_2^7} = 0, \qquad (7)
\end{aligned}
$$

And

$$(n - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_1}{2r_2^5} - \frac{3k_2}{r_2^5} + \frac{15k_2y^2}{2r_2^7})y = 0. \qquad (8)$$

From equation (8), we have two cases, either $y = 0$, or

$$(n - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_1}{2r_2^5} - \frac{3k_2}{r_2^5} + \frac{15k_2y^2}{2r_2^7}) = 0. \qquad (9)$$

The collinear libration points are the solution of the equation (7), when $y = 0$. The non-collinear libration points are the solutions of the equations (7) and (9).

If we put $k_1 = k_2 = 0$ in equations (7) and (8), we get the classical case of the R3BP and consequently. $r_1 = r_2 = 1$ is the required solution. Now, we discuss only the location of libration point $L_4$. For this, we suppose that the solution of the above equations when $y \neq 0$ are $r_1 = 1 + \pi$, $r_2 = 1 + \pi'$ , $\pi, \pi' \ll 1$. Putting the values of $r_1$ and $r_2$ in the equations $r_1^2 = (x-\mu)^2 + y^2$, $r_2^2 = (x-\mu+1)^2 + y^2$ and solving, we get $x = \mu - \frac{1}{2} - (\pi - \pi')$, $y = \sqrt{\frac{3}{2} - \frac{1}{\sqrt{3}}(\pi + \pi')}$.

Now, we substitute the values of $r_1, x, y$ in equations (7) and (9) and rejecting the second and higher order terms of $\pi$ and $\pi'$, we get the co-ordinates of the stationary points $L_4(x, y)$ and $L_5(x, -y)$ as $x = \mu - \frac{1}{2} + p_1k_1 + p_2k_2$, $y = \frac{\sqrt{3}}{2} + p_3k_1 + p_4k_2$, respectively, where $p_1 = \frac{-1}{2\mu}$, $p_2 = \frac{7\mu-11}{8\mu(\mu-1)}$, $p_3 = \frac{3-4c\mu}{6\sqrt{3}\mu}$, $p_4 = \frac{11-5\mu}{8\sqrt{3}\mu(\mu-1)}$.

## 4. First order normalization

Following the procedure as adopted by Bhatnagar and Hallan (1983) to derived the first order normalization. The Lagrangian function of the equation(5) is given by

$$\Gamma = \frac{1}{2}\{\dot{x}^2 + \dot{y}^2 + n^2(x^2+y^2) + 2n(x\dot{y} - y\dot{x})\} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{k_1}{2r_2^3} - \frac{3k_2y^2}{2r_2^5}. \qquad (10)$$

Now shifting the origin to $L_4(x,y)$, and expanding $\Gamma$ in power series of $x$ and $y$, it can be expressed as $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + ...$, where

$$\begin{aligned}
\Gamma_0 &= \frac{1}{8}(11+\gamma^2+t_1k_1-9k_2),\\
\Gamma_1 &= \frac{-\dot{x}}{2}(\sqrt{3}+t_2k_1+2p_4k_2)-\frac{\dot{y}}{2}(\gamma+t_3k_1-2p_2k_2)\\
&\quad + x(t_4k_1+t_5k_2)+y(t_6k_1+t_7k_2),\\
\Gamma_2 &= \frac{1}{2}(\dot{x^2}+\dot{y^2})+n(x\dot{y}-y\dot{x})-\frac{3}{32}(8\sqrt{3}\gamma+t_8k_1+t_9k_2)xy\\
&\quad + \frac{1}{64}(72+t_{10}k_1+t_{11}k_2)y^2+\frac{1}{64}(24+t_{12}k_1+t_{13}k_2)x^2,\\
\Gamma_3 &= \frac{3}{32}(-2\sqrt{3}+t_{14}k_1+t_{15}k_2)y^3+\frac{x^3}{32}(-14\gamma+t_{16}k_1+t_{17}k_2)y^2\\
&\quad - \frac{3xy^2}{32}(-22\gamma+t_{18}k_1+t_{19}k_2)+\frac{3x^2y}{64}(2\sqrt{3}+t_{20}k_1+t_{21}k_2)x^2,\\
\Gamma_4 &= \frac{x^4}{512}(-148+t_{22}k_1+t_{23}k_2)+\frac{3y^4}{256}(2+t_{24}k_1+t_{25}k_2)y^2\\
&\quad - \frac{15xy^3}{64}(6\sqrt{3}\gamma+t_{26}k_1+t_{27}k_2)+\frac{5x^3y}{64}(10\sqrt{3}\gamma+t_{28}k_1+t_{29}k_2)\\
&\quad + \frac{1}{64}(1+t_{30}k_1+t_{31}k_2)x^2y^2,
\end{aligned}$$

all the values of $t_i$, $(i=1,2,...31)$ are refer in Appendix.

Corresponding to the Lagrangian function $\Gamma$ given by equation (5), the Hamiltonian function is given by $H = -\Gamma + p_x\dot{x} + p_y\dot{y}$, where $p_x$ and $p_y$ are the momenta coordinates and given by $p_x = \frac{\partial\Gamma}{\partial\dot{x}} = \dot{x} - ny$, $p_y = \frac{\partial\Gamma}{\partial\dot{y}} = \dot{y} + nx$. Finally, the Hamiltonian function becomes $H(x,y,p_x,p_y) = \frac{1}{2}(p_x^2+p_y^2) + n(yp_x - xp_y) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{k_1}{2r_2^3} + \frac{k_2y^2}{2r_2^5}$. Applying the following translation $x \to (x - \frac{\gamma}{2}p_1k_1 + p_2k_2)$, $y \to (y + \frac{\sqrt{3}}{2}p_3k_1 + p_4k_2)$, $p_x \to p_x - n(\frac{\sqrt{3}}{2} + p_3k_1 + p_4k_2)$, $p_y \to p_y + n(\frac{-\gamma}{2} + p_1k_1 + p_2k_2)$, One can find the Hamiltonian $H$ as $H = \sum_{k=0}^{\infty} H_k$, where $H_k =$ the sum of the terms of $k^{th}$ degree homogenous in variables $x, y, p_x, p_y$.

Now

$$\begin{aligned}
H_0 &= -\Gamma_0,\\
H_1 &= \frac{1}{64}(-40-12(11+11\gamma p_1-3\sqrt{3}p_3)k_1\\
&\quad + (141-132\gamma p_2+36\sqrt{3}p_4)k_2),\\
H_2 &= \frac{1}{2}(p_x^2+p_y^2)+n(yp_x-xp_y)+Ex^2+Fy^2+2Gxy, \qquad (11)\\
H_3 &= -\Gamma_3,\\
H_4 &= -\Gamma_4,
\end{aligned}$$

where $E = \frac{1}{64}(8+c_1k_1+c_2k_2)$, $F = \frac{1}{64}(-40+c_3k_1+c_4k_2)$, $G = \frac{3}{16}(8\sqrt{3}\gamma + c_5k_1+c_6k_2)$, $c_1 = 12(-1+7\gamma p_1)$, $c_2 = 3(45+28\gamma p_2+4\sqrt{3}p_4)$, $c_3 = -12(11+$

$11\gamma p_1 - 3\sqrt{3}p_3)$, $c_4 = 141 - 132\gamma p_2 + 36\sqrt{3}p_4$, $c_5 = 4(-5\sqrt{3} + \sqrt{3}p_1 - 11\gamma p_3)$, $c_6 = 65\sqrt{3} + 4\sqrt{3}p_2 - 44\gamma p_4)$.

To investigate the stability of motion as in Whittaker (1965), we consider the following set of linear equations in the variables $x$ and $y$

$$\begin{aligned} -\lambda p_x &= \frac{\partial H_2}{\partial x} = 2Ex + Gy - np_y, \\ -\lambda p_y &= \frac{\partial H_2}{\partial y} = 2Fy + Gx - np_x, \\ \lambda x &= \frac{\partial H_2}{\partial p_x} = p_x + ny, \end{aligned} \tag{12}$$

$$\lambda y = \frac{\partial H_2}{\partial p_y} = p_y - nx, \quad i.e. AX = 0,$$

where

$$A = \begin{pmatrix} 2E & G & \lambda & -n \\ G & 2F & n & \lambda \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}.$$

The equation (12) will have a non-zero solution if and only if $Det(A) = 0$, which implies that $\lambda^4 + 2\lambda^2(E + F + n^2) + FE - G^2 - 2n^2(E + F) + n^4 = 0$.

and so the characteristic equation corresponding to Hamiltonian $H_2$ given in equation(11) is given by $16\lambda^4 + (16 + 2(-36 + 32c - 12\gamma p_1 + 9\sqrt{3}p_3)k_1 - 6(-23 + 4\gamma p_2 - 4\sqrt{3}p_4)k_2)\lambda^2 + 27(1-\gamma^2) + 4(252 + 384c + 540\gamma - 288\gamma p_1 + 18\sqrt{3}(-3 + 22\gamma^2)p_3)k_1 + 4(-819 - 1755\gamma - 288\gamma p_2 + 36\sqrt{3}(-3 + 11\gamma^2)p_4)k_2 = 0$,

The stability of Libration point $L_4$ is assured only when the discriminant of the characteristic equation is greater than zero, implying that $\mu < \mu_c = \mu_0 - (3.76183..)k_1 + (10.825...)k_2$, where $\mu_0 = 0.0385208965\ldots$ When $D > 0$, the roots $\pm i\omega_1'$ and $\pm i\omega_2'$ ($\omega_1'$ and $\omega_2'$ being long/short-periodic frequencies) are related to each other as $\omega_1'^2 + \omega_2'^2 = 1 + p_5k_1 + p_6k_2$, $\omega_1'^2\omega_2'^2 = \frac{27}{16}(1-\gamma^2) + p_7k_1 + p_8k_2$, $(0 < \omega_1' < \omega_2' < \frac{1}{\sqrt{2}})$, where $p_5 = \frac{1}{8}(-36 + 32c - 12\gamma p_1 + 9\sqrt{3}p_3)$, $p_6 = \frac{-3}{8}(-23 + 4\gamma p_2 - 4\sqrt{3}p_4)$, $p_7 = \frac{1}{64}(252 + 384c + 540\gamma - 288\gamma p_1 + 18\sqrt{3}(-3 + 22\gamma^2)p_3)$, $p_8 = \frac{1}{64}(-819 - 1755\gamma - 288\gamma p_2 + 36\sqrt{3}(-3 + 11\gamma^2)p_4)$.

It is observed that the perturbed frequencies $(\omega_1', \omega_2')$ are related to the unperturbed one $(\omega_1, \omega_2)$ as $\omega_1' = \omega_1(1 + pk_1 + p'k_2)$, $\omega_2' = \omega_2(1 + qk_1 + q'k_2)$, where $p = \frac{27p_5 - 27\gamma^2p_5 - 16p_7\omega_2^2}{54(1-\gamma^2)k^2}$, $p' = \frac{27p_6 - 27\gamma^2p_6 - 16p_8\omega_2^2}{54(1-\gamma^2)k^2}$, $q = \frac{27p_5 - 27\gamma^2p_5 - 16p_7\omega_1^2}{54(-1+\gamma^2)k^2}$, $q' = \frac{27p_6 - 27\gamma^2p_6 - 16p_8\omega_1^2}{54(-1+\gamma^2)k^2}$, $k^2 = 2\omega_1^2 - 1 = 1 - 2\omega_2^2$,

Following the method given in Whittaker (1965), we use a canonical transformation from the phase space $(x, y, p_x, p_y)$ into the phase space of the angles $(\phi_1, \phi_2)$ and the actions $(I_1, I_2)$, so that the Hamiltonian $H_2$ be normalized.

$$X = JT, \tag{13}$$

where

$$X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix}, T = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix},$$

$$J = (a'_{ij})_{1\leq i,j\leq 4},\ Q_i = (\frac{2I_i}{\omega'_i})^{\frac{1}{2}} \sin\phi_i,\ P_i = (2I_i\omega'_i)^{\frac{1}{2}} \cos\phi_i,\ \ (i = 1, 2).$$

Now, we have calculated all the elements of $J$ and we obtain $a'_{ij} = a_{ij}(1 + \alpha_{ij}k_1 + \alpha'_{ij}k_2)$, $i, j = 1, 2, 3, 4$, where $a_{11} = a_{12} = 0$, $a_{13} = \frac{l_1}{2\omega_1 k_1}$, $a_{14} = \frac{l_2}{2\omega_2 k_2}$, $a_{21} = \frac{-4\omega_1}{l_1 k}$, $a_{22} = \frac{-4\omega_2}{l_2 k}$, $a_{23} = \frac{3\sqrt{3}\gamma}{2\omega_1 l_1 k}$, $a_{24} = \frac{3\sqrt{3}\gamma}{2\omega_2 l_2 k}$, $a_{31} = \frac{-\omega_1 m_1}{2l_1 k}$, $a_{32} = \frac{-\omega_2 m_2}{2l_2 k}$, $a_{33} = \frac{3\sqrt{3}\gamma}{2\omega_1 l_1 k}$, $a_{34} = \frac{3\sqrt{3}\gamma}{2\omega_2 l_2 k}$, $a_{41} = \frac{3\sqrt{3}\gamma\omega_1}{2l_1 k}$, $a_{42} = \frac{3\sqrt{3}\gamma\omega_2}{2l_2 k}$, $a_{43} = \frac{n_1}{2\omega_1 l_1 k}$, $a_{44} = \frac{n_2}{2\omega_2 l_2 k}$, and all the values of $\alpha_{ij}$ and $\alpha'_{ij}$, $i, j = 1, 2, 3, 4$, are given in Appendix.

The transformation changes the second order part of the Hamiltonian into the normal form $H_2 = \omega'_1 I_1 - \omega'_2 I_2$ and the general solutions of the corresponding equations of motion are $I_i =$ Constant $(i = 1, 2)$, $\phi_1 = \omega'_1 t+$ Constant, $\phi_2 = \omega'_2 t+$ Constant.

## 5. Second order normalization

Moser's conditions are utilized for transforming the Hamiltonian to the Birkhoff's normal form with the help of double D'Alembert's series. Here we wish to perform Birkhoff's normalization for which the co-ordinates $(x, y)$ are to be expanded in double D'Alembert series:

$$x = \sum_{n\geq 1} B_n^{1,0}, \qquad y = \sum_{n\geq 1} B_n^{0,1},$$

where the homogeneous components $B_n^{1,0}$ and $B_n^{0,1}$ of degree $n$ in $\sqrt{I_1}$, $\sqrt{I_2}$ are of the form

$$\sum_{0\leq m\leq n} I_1^{\frac{1}{2(n-m)}} I_2^{\frac{1}{2(n-m)}} \sum_{(i,j)} (C_{n-m,m,i,j} \cos(i\phi_1 + j\phi_2) + S_{n-m,m,i,j} \sin(i\phi_1 + j\phi_2)). \tag{14}$$

The double summation over the indices $i$ and $j$ is such that $(a)$ $i$ runs over those integers in the interval $0 \leq i \leq n - m$ that have the same parity as $n - m$ $(b)$ $j$ runs over those integers in the interval $-m \leq j \leq m$ that have the same parity as $m$. $I_1$ and $I_2$ are to be regarded as constants of integration and $\phi_1$, $\phi_2$ are to be determined as linear functions of time such that

$$\dot{\phi_1} = \omega'_1 + \sum_{n\geq 1} f_{2n}(I_1, I_2),\ \ \dot{\phi_2} = -\omega'_2 + \sum_{n\geq 1} g_{2n}(I_1, I_2),$$

where $f_{2n}$ and $g_{2n}$ are of the form

$$f_{2n} = \sum_{0 \le m \le n} f'_{2(n-m),2m} I_1^{n-m} I_2^m, \quad g_{2n} = \sum_{0 \le m \le n} g'_{2(n-m),2m} I_1^{n-m} I_2^m.$$

The first order components $B_1^{1,0}$ and $B_1^{0,1}$ are the values of $x$ and $y$ given by equation(14). The second order components $B_2^{1,0}$ and $B_2^{0,1}$ are solutions of the partial differential equations $\Delta_1 \Delta_2 B_2^{1,0} = \Phi_2$ and $\Delta_1 \Delta_2 B_2^{0,1} = \Psi_2$, where $\Delta_i = (D^2 + \omega_i'^2)$, $(i = 1, 2)$, $D = \omega_1' \frac{\partial}{\partial \phi_1} - \omega_2' \frac{\partial}{\partial \phi_2}$.

$\Phi_2 = X_2(D^2 - \frac{1}{32}(72 + t_{10}k_1 + t_{11}k_2)) + Y_2(2nD + \frac{3}{32}(8\sqrt{3}\gamma + t_8 k_1 + t_9 k_2))$, $\Psi_2 = Y_2(D^2 - \frac{1}{32}(24 + t_{12}k_1 + t_{13}k_2)) - X_2(2nD - \frac{3}{32}(8\sqrt{3}\gamma + t_8 k_1 + t_9 k_2))$, and $X_2$ and $Y_2$ are homogeneous components of order 2 obtained on substituting $x = B_1^{1,0} + B_2^{1,0}$, $y = B_1^{0,1} + B_2^{0,1}$, in $\frac{\partial \Gamma_3}{\partial x}$, $\frac{\partial \Gamma_3}{\partial y}$.

## 6. Second order coefficients in the frequencies

To make use of Moser's modified version of Arnold's theorem (1961), it is necessary to reduce the Hamiltonian to its normalized form. So, we performed the first and second order normalization. We have found the second order coefficients in the frequencies. For this we have obtained the partial differential equations which are satisfied by the third order homogeneous components of the fourth order part of Hamiltonian $H_4$ and second order polynomials in the frequencies. Following the iterative procedure of Bhatnagar and Hallan (1983), we note that the third order components $B_3^{0,1}$ and $B_3^{1,0}$ can be obtained by solving the partial differential equations

$$\begin{cases} \Delta_1 \Delta_2 B_3^{1,0} = \Phi_3 - 2f_2 P - 2g_2 Q, \\ \Delta_1 \Delta_2 B_3^{0,1} = \Psi_3 - 2f_2 U - 2g_2 V, \end{cases} \tag{15}$$

where

$$\begin{aligned}
\Phi_3 &= X_3(D^2 - \frac{1}{32}(72 + t_{10}k_1 + t_{11}k_2)) + Y_3(2nD + \frac{3}{32}(8\sqrt{3}\gamma + t_8 k_1 + t_9 k_2)), \\
P &= \frac{\partial}{\partial \phi_1}\{(\omega_1' \frac{\partial B_1^{1,0}}{\partial \phi_1} - nB_1^{0,1})(\omega_1'^2 \frac{\partial^2}{\partial \phi_1} - \frac{1}{32}(72 + t_{10}k_1 + t_{11}k_2)) \\
&\quad + (\omega_1' \frac{\partial B_1^{0,1}}{\partial \phi_1} - nB_1^{1,0})(2n\omega_1' \frac{\partial}{\partial \phi_1} + \frac{3}{32}(8\sqrt{3}\gamma + t_8 k_1 + t_9 k_2))\}, \\
Q &= \frac{\partial}{\partial \phi_2}\{(-\omega_2' \frac{\partial B_1^{1,0}}{\partial \phi_2} - nB_1^{0,1})(\omega_2'^2 \frac{\partial^2}{\partial \phi_2} - \frac{1}{32}(72 + t_{10}k_1 + t_{11}k_2)) \\
&\quad + (-\omega_2' \frac{\partial B_1^{0,1}}{\partial \phi_2} - nB_1^{1,0})(-2n\omega_2' \frac{\partial}{\partial \phi_2} + \frac{3}{32}(8\sqrt{3}\gamma + t_8 k_1 + t_9 k_2))\},
\end{aligned}$$

$$\Psi_3 = Y_3(D^2 - \frac{3}{32}(24 + t_{12}k_1 + t_{13}k_2)) - X_3(2nD - \frac{3}{32}(8\sqrt{3}\gamma + t_8k_1 + t_9k_2)),$$

$$U = \frac{\partial}{\partial\phi_1}(\omega_1'\frac{\partial B_1^{0,1}}{\partial\phi_1} - nB_1^{1,0})(D^2 - (\frac{3}{4} + \frac{3k_1}{8} + (\frac{3}{2} + \frac{\sqrt{3}}{8} - \frac{21\gamma}{16})k_2))$$

$$+ (\omega_1'\frac{\partial^2 B_1^{1,0}}{\partial\phi_1} - nB_1^{0,1})(2nD - (\frac{3\sqrt{3}\gamma}{4} - \frac{15\sqrt{3}k_1}{8} + \frac{11k_1\gamma}{8} + \frac{3\sqrt{3}k_2}{16})),$$

$$V = \frac{\partial}{\partial\phi_2}(\omega_2'\frac{\partial B_1^{1,0}}{\partial\phi_1} - nB_1^{0,1})(2nD - (\frac{3\sqrt{3}\gamma}{4} - \frac{15\sqrt{3}k_1}{8} + \frac{11k_1\gamma}{8} + \frac{3\sqrt{3}k_2}{16}))$$

$$- (\omega_2'\frac{\partial^2 B_2^{1,0}}{\partial\phi_2} - nB_1^{1,0})(D^2 - (\frac{3}{4} + \frac{3k_1}{8} + (\frac{3}{2} + \frac{\sqrt{3}}{8} - \frac{21\gamma}{16})k_2)),$$

and $X_3$ and $Y_3$ are homogeneous components of order 3 obtained on substituting $x = B_1^{1,0} + B_2^{1,0}$, $y = B_1^{0,1} + B_2^{0,1}$ in $\frac{\partial\Gamma_3}{\partial x} + \frac{\partial\Gamma_4}{\partial x}$ and $\frac{\partial\Gamma_3}{\partial y} + \frac{\partial\Gamma_4}{\partial y}$ . The components $B_3^{0,1}$ and $B_3^{1,0}$ are not required to be found out. We find the coefficients of $Cos\phi_1$, $Sin\phi_1$, $Cos\phi_2$ and $Sin\phi_2$ in the right-hand sides of equation(15), they are the critical terms. We eliminate these terms by properly choosing the coefficients in the polynomials $f_2 = f'_{2,0}I_1 + f'_{0,2}I_2$, $g_2 = g'_{2,0}I_1 + g'_{0,2}I_2$, where

$$f'_{2,0} = \frac{\text{Coefficient of Cos } \phi_1 in\Phi_3}{2(\text{Coefficient of Cos}\phi_1\text{in}P)} = A(\text{say}),$$

$$f'_{0,2} = g'_{2,0} = \frac{\text{Coefficient of Cos}\phi_2 in\Phi_3}{2(\text{Coefficient of Cos}\phi_2 inQ)} = B(\text{say}),$$

$$g'_{0,2} = \frac{\text{Coefficient of Cos}\phi_2 in\Psi_3}{2(\text{Coefficient of Cos}\phi_2\text{in}Q)} = C(\text{say}).$$

## 7. Non-linear stability

Now the condition is $K_1\omega_1' + K_2\omega_2' \neq 0$, for all pairs $(K_1, K_2)$ of rational integers such that $|K_1| + |K_2| \leq 4$.

We calculate, $K_1\omega_1' + K_2\omega_2' = 0$, $\Leftrightarrow \frac{\omega_1'}{\omega_2'} = -\frac{K_1}{K_2}$.

Here, we have, $0 < \omega_2 < \frac{1}{\sqrt{2}} < \omega_1 < 1$, and so $0 < \omega_2' < \frac{1}{\sqrt{2}} < \omega_1' < 1$ $(|K_1| << 1, |K_2| << 1)$.

So, we have

$$\frac{\omega_1'}{\omega_2'} > 1 \tag{16}$$

equation (16) to be true, $K_1$ and $K_2$ are of opposite signs and $-\frac{K_1}{K_2} > 1$.

Therefore, $K_1$, $K_2$ can have the following values, $K_1 = 1, K_2 = -2$; $K_1 = -1, K_2 = 2$. $K_1 = 1, K_2 = -3$; $K_1 = -1, K_2 = 3$.

**Case-I.** When $K_1 = 1, K_2 = -2$; $K_1 = -1, K_2 = 2$. Equation (16) gives

$$\frac{\omega_1'}{\omega_2'} = 2, i.e. \omega_1' - 2\omega_2' = 0. \tag{17}$$

Solving equations (14) and (17) and putting $\gamma = 1 - 2\mu$, we get

$$\mu_1' = (0.024293897\ldots) + (2.08929\ldots)k_1 + (4.69455\ldots)k_2.$$

**Case-II.** When $K_1 = 1, K_2 = -3$; $K_1 = -1, K_2 = 3$.
Equation (16) gives

$$\frac{\omega_1^{'}}{\omega_2^{'}} = 3, i.e.\omega_1^{'} - 3\omega_2^{'} = 0. \tag{18}$$

Solving equations (14) and (18) and putting $\gamma = 1 - 2\mu$, we get

$$\mu_2' = (0.013516016\ldots) + (2.01103\ldots)k_1 + (4.70867\ldots)k_2.$$

Hence for the values $\mu_1'$ and $\mu_2'$ of the mass ratio condition $(a)$ of Moser's theorem is not satisfied.

The normalized Hamiltonian up to fourth order is written as $H = \omega_1 I_1 + \omega_2 I_2 + \frac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + \ldots$

The determinant $D$ occurring in condition $(a)$ of Moser's theorem is

$$D = \begin{vmatrix} A & B & \omega_1' \\ B & C & -\omega_2' \\ \omega_1' & -\omega_2' & 0 \end{vmatrix} = -(A\omega_2^{'2} + 2B\omega_1^{'} + C\omega_1^{'2}).$$

Following the iterative procedure of Bhatnagar and Hallan (1983), we observed that Moser's second condition is violated for the unperturbed problem (i.e. for $k_1 = k_2 = 0$ ) when $\mu_3 = 0.0109137\ldots$ When $k_1, k_2 \neq 0$, we take $\mu_3^{'} = \mu_3 + Xk_1 + X'k_2$ such that $D = 0$. It is also observed that the condition $(b)$ of Moser's theorem is satisfied i.e. $D \neq 0$, if in the interval$0 < \mu < \mu_c$, the mass ratio does not take the value $\mu_3^{'} = \mu_3 + Xk_1 + X'k_2$, where $X = 40.917\ldots$, $X' = 607.324\ldots$.

## 8. Conclusion

E. P. Esteban and S. Vazquez have studied the rotating stratified heterogeneous oblate spheroid in Newtonian Physics by taking three layers. But we have taken the smaller primary with mass $m_2$ a heterogeneous triaxial rigid body with $N$ layers having different densities $\rho_i$ and axes $(a_i, b_i, c_i), (i = 1, 2, 3, 4, \ldots, N)$ respectively in the restricted three-body problem. We have found that there exist five stationary solutions (called libration points) of the equations of motion. Three of them are collinear and two are triangular equilibrium points.

We also observed that in the non-linear sense, collinear points are unstable for all mass ratios $\mu$, and triangular points are stable in the range of linear stability $0 < \mu < \mu_c$, $\mu < \mu_c = \mu_0 - (3.76183\ldots)k_1 + (10.825\ldots)k_2$, Where, $\mu_0 = 0.0385208965\ldots$ except for three mass ratios

$$\mu_1' = (0.0242939\ldots) + (2.08929\ldots)k_1 + (4.69455\ldots)k_2,$$
$$\mu_2' = (0.013516016\ldots) + (2.01103\ldots)k_1 + (4.70867\ldots)k_2,$$
$$\mu_3' = (0.0109366\ldots) + (40.917\ldots)k_1 + (607.324\ldots)k_2,$$

at which Moser's theorem does not apply.

Here, if we take $k_1 = k_2 = 0$, then the values of $\mu_1^{'}$, $\mu_2^{'}$ and $\mu_3^{'}$ agree with those found by Deprit and Deprit (1967).

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**Appendix.**

$t_1 = 2(2 + 3c + c\gamma^2)$,
$t_2 = \sqrt{3}c + 2p_3$,
$t_3 = c\gamma - 2p_1$,
$t_4 = 3 + 4c\gamma - 3p_1 + 3\sqrt{3}\gamma p_3$,
$t_5 = 3(15 + 4p_2 - 4\sqrt{3}\gamma p_4)$,
$t_6 = (-3\sqrt{3} + 4\sqrt{3}c - 4\sqrt{3}\gamma p_1 + 9p_3)$,
$t_7 = 3(7\sqrt{3} - 4\sqrt{3}\gamma p_2 + 12p_4)$,
$t_8 = 4(-5\sqrt{3} + \sqrt{3}p_1 - 11\gamma p_3)$,
$t_9 = 4\sqrt{3}p_2 - 44\gamma p_4 + 65\sqrt{3}$,
$t_{10} = 33 + 16c + 33\gamma p_1 - 9\sqrt{3}p_3$,
$t_{11} = -47 + 44\gamma p_2 - 12\sqrt{3}p_4$,
$t_{12} = 3 + 16c - 21\gamma p_1 - 3\sqrt{3}p_3$,
$t_{13} = -3(45 + 28\gamma p_3 + 4\sqrt{3}p_4)$,
$t_{14} = -15\sqrt{3} - 15\sqrt{3}\gamma p_1 + p_3)$,
$t_{15} = 40\sqrt{3} - 15\sqrt{3}\gamma p_2 + p_4$,
$t_{16} = 25 - 37p_1 + 25\sqrt{3}\gamma p_3$,
$t_{17} = -37p_2 + 25\sqrt{3}\gamma p_4$,
$t_{18} = 85 - 41p_1 + 45\sqrt{3}\gamma p_3$,
$t_{19} = -40 - 41p_2 + 45\sqrt{3}\gamma p_4$,
$t_{20} = -15\sqrt{3} + 25\sqrt{3}\gamma p_1 + 41p_3)$,
$t_{21} = 25\sqrt{3}\gamma p_2 + 41p_4$,
$t_{22} = 10(-57 + 23\gamma p_1 + 57\sqrt{3}p_3)$,
$t_{23} = 10(23\gamma p_2 + 57\sqrt{3}p_4)$,
$t_{24} = 5(37 + 37\gamma p_1 + 11\sqrt{3}p_3)$,
$t_{25} = 5(-272 + 37\gamma p_2 + 11\sqrt{3}p_4)$,
$t_{26} = -35\sqrt{3} + 23\sqrt{3}p_1 - 37\gamma p_3$,
$t_{27} = 56\sqrt{3} + 23\sqrt{3}p_2 - 37\gamma p_4$,

$t_{28} = -21\sqrt{3} + 57\sqrt{3}p_1 + 43\gamma p_3,$
$t_{29} = 57\sqrt{3}p_2 - 43\gamma p_4,$
$t_{30} = -60(-93 + 43\gamma p_1 + 69\sqrt{3}p_3,$
$t_{31} = 15(1965 + 172\gamma p_2 + 276\sqrt{3}p_4,$

$\alpha_{13} = \frac{-1}{32k^2 l_2(3+4\omega_1^2)}\{-6336c - 864p + 45c_1 + 832p_7 + (1536c + 2304p - 16c_1 - 832p_5 - 256p_7)\omega_1^2 + (5376p - 3072c - 16c_1 + 256p_5)\omega_1^4 + (39 - 12\omega_1^2)c_5c_7\}$, $\alpha'_{13} = \frac{-1}{32k^2 l_1(3+4\omega_1^2)}\{864q + 45c_2 + 832p_8 + (2304q - 16c_2 - 832p_6 - 256p_6)\omega_1^2 + (5376q - 16c_2 + 256p_6)\omega_1^4 + 1024\omega_1^6 + (39 - 12\omega_1^2)c_5c_7\}$,

$\alpha_{21} = \frac{-\omega_1}{8k^3 l_1^2(3+4\omega_1^2)}\{3168c - 864p - 9c_1 + 320p_7 + (6336c - 1152p - 40c_1 + 320p_5 - 727p_7)\omega_1^2 + (-768p - 512c - 16c_1 + 768p_5)\omega_1^4 + (1024c - 1024p)\omega_1^6 - (15 + 36\omega_1^2)c_6c_7\}$,

$\alpha'_{21} = \frac{-\omega_1}{8k^3 l_1^2(3+4\omega_1^2)}\{-864q - 9c_2 - 320p_8 + (-1152q - 40c_2 + 320p_6 - 728p_8)\omega_1^2 + (-768q - 16c_2 + 768p_6)\omega_1^4 - 1024q\omega_1^6 - (15 - 36\omega_1^2)c_6c_7\}$,

$\alpha_{23} = \frac{-1}{64k^3 l_1^3 c_7}\{108864c + 23328p + (101376c - 41472p + 15552p_5)\omega_1^2 - (151296p + 116736c - 14080p_5)\omega_1^4 + (147456c - 48128p - 27648p_5)\omega_1^6 + (-49152c + 20480p + 4096p_5)\omega_1^8 - 16384p\omega_1^{10} + (-729 + 660\omega_1^2 + 1296\omega_1^4 - 192\omega_1^6)c_5c_7 - 67p_7(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) - c_1(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha'_{23} = \frac{-1}{64k^3 l_1^3 c_7}\{-23328q + (41472q - 15552p_6)\omega_1^2 + (151296q - 14080p_6)\omega_1^4 + (48128q + 27648p_6)\omega_1^6 - (20480q + 4096p_6)\omega_1^8 + 16384q\omega_1^{10} - (729 + 660\omega_1^2 - 1296\omega_1^4 + 192\omega_1^6)c_6c_7 + 64p_8(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) + c_2(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha_{31} = \frac{-\omega_1^2}{64k^3 l_1^3}\{38592c - 864p + (-40704c - 11520p + 4928p_5)\omega_1^2 + (5120c - 3840p - 51200p_5)\omega_1^4 + (-4096c + 16384p + 1024p_5)\omega_1^6 + 4096\omega_1^8 + 16384q\omega_1^{10} + (-231 + 240\omega_1^2 + 48\omega_1^4)c_5c_7 + 64p_7(-70 + 80\omega_1^2 + 16\omega_1^4) + c_1(-333 + 284\omega_1^2 + 336\omega_1^4 + 64\omega_1^6)\}$,

$\alpha'_{31} = \frac{-\omega_1^2}{64k^3 l_1^3}\{-864q - (11520q - 4928p_6)\omega_1^2 + (-3840q - 3120p_6)\omega_1^4 + (16384q - 1024p_6)\omega_1^6 + 4096\omega_1^8 + (-231 + 240\omega_1^2 + 48\omega_1^4)c_6c_7 + 64p_8(-77 + 80\omega_1^2 + 16\omega_1^4) + c_2(-333 + 284\omega_1^2 + 336\omega_1^4 + 64\omega_1^6)\}$,

$\alpha_{33} = \frac{1}{64k^3 l_1^3 c_7}\{-85536c - 23328p + (-120384c + 41472p - 15552p_5)\omega_1^2 + (151296p + 64512c - 14080p_5)\omega_1^4 + (-137216c + 48128p + 27648p_5)\omega_1^6 + (24576c - 20480p - 4096p_5)\omega_1^8 + 16384(p - c)\omega_1^{10} + (729 + 660\omega_1^2 - 1296\omega_1^4 + 192\omega_1^6)c_5c_7 - 64p_7(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) + c_2(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha'_{33} = \frac{-1}{64k^3 l_1^3 c_7}\{-23328q + (41472q - 15552p_6)\omega_1^2 - (151296q - 14080p_6)\omega_1^4 + (48128q + 27648p_6)\omega_1^6 + (-20480q - 4096p_6)\omega_1^8 + 16384q\omega_1^{10} + (729 + 660\omega_1^2 - 1296\omega_1^4 + 192\omega_1^6)c_6c_7 + 64p_8(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) + c_2(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha_{41} = \frac{-\omega_1^2}{64k^3 l_1^3 c_7}\{108864c - 23328p + (101376c - 3456p + 15552p_5)\omega_1^2 + (-116736c - 46848p + 14080p_5)\omega_1^4 + (147456c - 68608p - 27648p_5)\omega_1^6 + (-49152c + 69632p + 4096p_5)\omega_1^8 + 16384p\omega_1^{10} + (-729 - 660\omega_1^2 + 1296\omega_1^4 - 192\omega_1^6)c_5c_7 - 64p_7(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) - c_1(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha'_{41} = \frac{-\omega_1^2}{64k^3 l_1^3 c_7}\{23328q + (-3456q + 15552p_6)\omega_1^2 + (-46848q + 14080p_6)\omega_1^4 + (-68608q - 27648p_6)\omega_1^6 + (69632q + 4096p_5)\omega_1^8 + 16384q\omega_1^{10} + (-729 - 660\omega_1^2 +$

$1296\omega_1^4 - 192\omega_1^6)c_6c_7 - 64p_8(243 + 220\omega_1^2 - 432\omega_1^4 + 64\omega_1^6) - c_2(243 + 936\omega_1^2 - 64\omega_1^4 + 384\omega_1^6 + 256\omega_1^8)\}$,

$\alpha_{43} = \frac{1}{64k^3l_1^3}\{4928c + 7776p + (-15552c - 10368p + 7488p_5)\omega_1^2 + (-5376c - 48384p - 1536p_5)\omega_1^4 + (35840c - 24576p - 7168p_8)\omega_1^6 - 4096(c-p)\omega_1^8 + (-351 + 72\omega_1^2 + 336\omega_1^4)c_5c_7 + 64p_7(-117 + 24\omega_1^2 + 112\omega_1^4) + 3c_1(-135 + 12\omega_1^2 + 176\omega_1^4 + 64\omega_1^6)\}$,

$\alpha'_{43} = \frac{1}{64k^3l_1^3}\{7776q - (10368q - 7488p_6)\omega_1^2 + (48384q - 1536p_5)\omega_1^4 + (24576q - 7168p_6)\omega_1^6 + 4096q\omega_1^8 + (-351 + 72\omega_1^2 + 336\omega_1^4)c_6c_7 + 64p_8(-117 + 24\omega_1^2 + 112\omega_1^4) + 3c_2(-135 + 12\omega_1^2 + 176\omega_1^4 + 64\omega_1^6)\}$,

The values of $\alpha_{ij}$ and $\alpha'_{ij}$ for $j = 1, 2$, can be obtained from those for $j = 1, 3$, respectively by replacing $\omega_1$ by $\omega_2$, $l_1$ by $l_2$, $m_1$ by $m_2$ and $n_1$ by $n_2$ whenever they occur, keeping $k$ unchanged.

## References


[1] R. Aggarwal, Z.A. Taqvi, I. Ahmad, *Non-linear stability of L4 restricted three body problem for radiated axes symmetric primaries with resonances*, Bulletin of Astronomical Society of India, 35 (2006), 1-29.

[2] V.I. Arnold, *The stability of the equilibrium position of a Hamiltonian system of ordinary differential equations in the general elliptic case*, Sov. Math. Dokl., 2 (1961), 247.

[3] Bhatnagar, K.B., Hallan, P.P., *The effect of perturbation in restricted problem of three bodies*, Celes. Mech., 30 (1983), 97.

[4] A.A. Ansari, M. Alam, *Dynamics in the circular restricted three body problem with perturbations*, International Journal of Advanced Astronomy, 5 (2017a), 19-25.

[5] A.A. Ansari, *Effect of Albedo on the motion of the infinitesimal body in circular restricted three body problem with variable masses*, Italian Journal of Pure and Applied Mathematics, 38 (2017b), 581-600.

[6] N. Chandra, R. Kumar, *Effect of oblateness on the non-linear stability of the triangular liberation points of the restricted three-body problem in the presence of resonances*, Astrophys. Space Sci., 291 (2004), 1-19.

[7] A. Deprit, A. Deprit-Bartholome, *Stability of the triangular Lagrangian points*, Astronomy and Astrophysics, 72 (1967), 173-179.

[8] E.P. Esteban, S. Vazquez, *Rotating Stratified Heterogeneous Oblate Spheroid In Newtonian Physics*, Celes. Mech. and Dyn. Astron., 81 (2001), 299-312.

[9] J. Gyorgyey, *On the non-linear stability of motions around $L_5$ in the elliptic restricted problem of three bodies*, Celestial Mechanics, 36 (1985), 281-285.

[10] G. Krzysztof, A.J. Maciejewski, N. Zuzanna, *About the libration points in the restricted photogravitational three-body problem*, Celest. Mech. and Dyn. Astron., 52 (1991), 195-201.

[11] B. Ishwar, J.P. Sharma, *Non-linear stability in photo-gravitational non-planer R3BP with oblate smaller primary*, Astrophys. Space Sci., 337 (2012), 563-571.

[12] R. Jain, D. Sinha, *Non-linear stability of $L_4$ in the restricted problem when the primaries are finite straight segment under resonances*, Astrophys. Space Sci., 353 (2014), 73-88.

[13] S. Jain, P.P. Hallan, K.B. Bhatnagar, *Nonlinear stability of $L_4$ in the restricted three-body when the primaries are triaxial rigid bodies*, Indian Journal of Pure and Applied Mathematics, 32 (2001), 413-445.

[14] B.S. Kushvah, J.P. Sharma, B. Ishwar, *Non-linear stability in the generalized photogravitational restricted three-body problem with Poynting-Robertson drag*, Astrophys. Space Sci., 312 (2007), 279-293.

[15] K. Shalini, et al., *Existence and Stability of L4 of the R3BP When the Smaller Primary is a Heterogeneous Triaxial Rigid Body with N Layers*, J. Appl. Environ. Biol. Sci., 6 (2016), 249-257.

[16] K. Shalini, M.S. Suraj, R. Aggarwal, *The Non-linear Stability of $L_4$ in the R3BP when the Smaller Primary is a Heterogeneous Spheroid*, Journal of Astronautical Science, 64 (2017), 18-49.

[17] A. Riaguas, A. Elipe, T.L. Moratalla, *Non-linear stability of the equilibria in the gravity field of a finite straight segment*, Celest. Mech. and Dyn. Astron., 81 (2001), 235-248.

[18] J. Singh, *Non-linear stability in the restricted three-body problem with oblate variable mass*, Astrophys. Space Sci., 333 (2011), 61-68.

[19] P.V. Subba Rao, R.K. Sharma, *Effect of oblateness on the non-linear stability of $L_4$ in the restricted three-body problem,* Celest. Mech. and Dyn. Astron., 65 (1997), 291-312.

[20] V. Szebehely, *Theory of orbits*, Academic Press, New York, 242-264, 1967.

[21] E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, London, Cambridge University Press, 1965.

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