CRITERION FOR NONEXISTENCE HORSESHOE-LIKE IN C^1 TOPOLOGY

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Abstract. In this paper we show that if $\Lambda \subset M$ is a closed invariant set and $p \in \Lambda$ is a hyperbolic saddle periodic point satisfying condition A with real and positive eigenvalues, then Λ is not horseshoe-like.
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1. Introduction

Bowen in his remarkable survey on Anosov diffeomorphism has proved that C^{1+}-diffeomorphisms do not have fat horseshoes, these are horseshoes of positive Lebesgue measure. In contrast, he gave an example of a totally disconnected horseshoe on sphere $\mathbb{S}^2$ of positive volume. On the other hand, Bowen has proved that a basic set (locally maximal hyperbolic set with a dense orbit) of a C^2 diffeomorphism which attracts a set with positive volume, necessarily attracts a neighborhood of itself [3 Theorem 4.11]. In particular, the unstable manifolds through points of this set must be contained in it, and consequently C^2 diffeomorphisms have no horseshoes with positive volume. In this context A.Fakhari and M.Soufi proved that any partially hyperbolic horseshoe-like attractor of a C^1 -generic diffeomorphism has zero volume [4]. As well they constructed a C^1–diffeomorphism with a partially hyperbolic horseshoe-like attractor of positive volume. In this paper we show that under some conditions there is no horseshoe-like in the context of C^1-diffeomorphisms. Indeed we show that if $\Lambda \subset M$ is a closed invariant set and $p \in \Lambda$ is a hyperbolic saddle periodic point satisfying condition A with real and positive eigenvalues, then Λ is not horseshoe-like.

Let $f : M \to M$ be a diffeomorphism of a compact connected Riemannian manifold M. A set Λ is said to be invariant relative to f if $f(\Lambda) = \Lambda$.

For a point $x \in M$ the stable set of x is

$$W^s(x) = \{y \in M : d(f^k(x), f^k(y)) \to 0 \quad as \quad k \to +\infty\}$$

and the unstable of x is

$$W^u(x) = \{y \in M : d(f^k(x), f^k(y)) \to 0 \quad as \quad k \to -\infty\}.$$

Let $\mathcal{O}(p)$ be a hyperbolic periodic orbit of f, then the dimension of unstable manifold of p is called index of p.

A compact invariant set Λ is said to be *horseshoe-like* if there are local stable and local unstable manifolds through all its points which intersect Λ in a Cantor set.

A splitting $T_\Lambda M = E \oplus F$ of the tangent bundle restricted to an invariant set Λ is dominated splitting if there is a constant $0 < \lambda < 1$ such that for some choice of a Riemannian metric on M

$$\|Df \mid E_x\|.\|Df^{-1} \mid F_{f(x)}\| \leq \lambda, \quad \text{for every } x \in \Lambda.$$

Λ is *partially hyperbolic*, if additionally E is *uniformly contracting* or F is *uniformly expanding*, i.e there exists $0 < \lambda < 1$ such that

$$\|Df \mid E_x\| \leq \lambda \quad \text{or} \quad \|Df^{-1} \mid F_{f(x)}\| \leq \lambda.$$

A compact invariant set Λ is called *hyperbolic* if there is a Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ of the tangent bundle restricted to Λ and a constant $\lambda < 1$ such that (for some choice of a Riemannian metric on M) for every $x \in \Lambda$

$$\|Df \mid E^s_x\| < \lambda$$

and

$$\|Df^{-1} \mid E^u_x\| < \lambda.$$

Alves and Pinheiro have studied nonuniformly expanding partially hyperbolic sets for C^{1+} diffeomorphisms [1]. They have proved that if non-uniformly expanding condition holds for a positive Lebesgue set of points, then Λ contains some local unstable disk. As a corollary, they deduced the non-existence of partially hyperbolic horseshoe like sets of positive volume. Also, Pacifico *et al.* have tried to construct Lorenz attractor of positive volume in the C^1-topology. The same result have obtained in the context of the volume preserving diffeomorphism. Indeed Xia proved in [2] that if an invariant set Λ of a volume-preserving C^{1+}-diffeomorphism f with positive volume has a dominated splitting $E \oplus F$, with E is uniformly contractive, then Λ contains stable leaves of almost every point. This argument leads to another proof of the classical result toward the ergodicity of C^{1+} volume-preserving Ansosov diffeomorphisms without using the Hopf argument.

2. Main theorems

In this section we present a condition that an invariant set satisfying it, is not horseshoe-like.

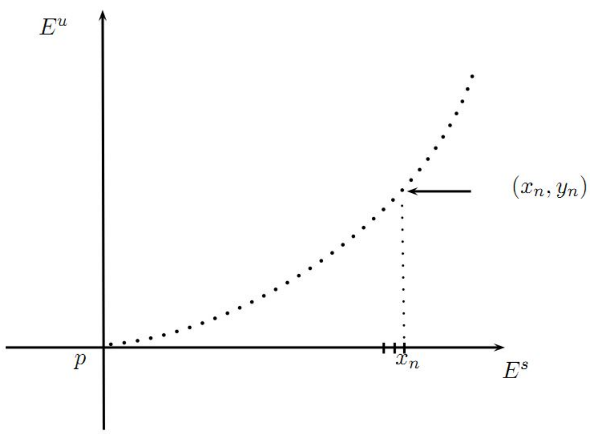
Definition. Let Λ be an invariant set. We say that a point $p \in M$ satisfies condition A if there are a local chart h at p and sequences $\{x_n\}$ and $\{w_n\} \subset T_pM$, $w_n = \sum_{i=1}^{m} \lambda_i^{w_n} v_i$ such that for $1 \leq i \leq m$,

$$\begin{cases} \lim_{n\to\infty} \frac{\lambda_i^{w_n} - \lambda_i^{w_{n+1}}}{\lambda_i^{w_{n+1}}} = 0 \\ \lambda_i^{w_{n+1}} < \lambda_i^{w_n} \\ \lim_{n\to\infty} \lambda_i^{w_n} = 0 \\ h^{-1}(w_n) = x_n \in \Lambda, \end{cases}$$

where $\{v_1, v_2, ..., v_m\}$ is a basis of T_pM.

Remark. In the above definition $\lambda_i^{w_n}$ is a notation relative to w_n as a scaler. Indeed for any $\alpha \in T_pM$, since $\{v_1, v_2, ..., v_m\}$ is a basis of T_pM, so one can write $\alpha = \sum_{i=1}^{m} \lambda_i^{\alpha} v_i$ where λ_i^{α} for $1 \leq i \leq m$, are scalers.

Example 1. Let $f : M \to M$ be a C^1-diffeomorphism on a C^∞-manifold M with $dimM = 2$ and $p \in M$ be a hyperbolic fixed point of f. Let f at p in local chart be as $f(x, y) = (4x, \frac{1}{8}y)$ and $\{(x_n, y_n)\}$ be a sequence such that $x_n = \frac{1}{n}$, $y_n \to 0$ as the following figure.



If Λ is a closed invariant set containing p and $\{(x_n, y_n)\} \subset \Lambda$, then p satisfies condition A, since $x_{n+1} < x_n$, $\frac{x_n - x_{n+1}}{x_{n+1}} \to 0$ and $x_n \to 0$.

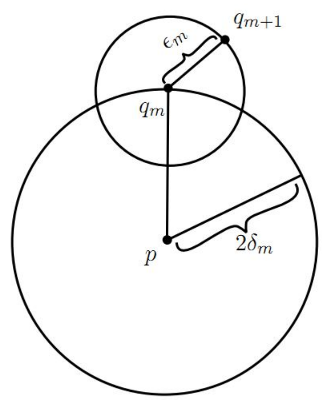
Definition. Let Λ be a close invariant subset of the compact manifold M. A point $p \in \Lambda$ is said to be *topologically dense point* if

$$\lim_{\delta \to 0} \frac{1}{\delta} \max\{\epsilon > 0 \mid B_\epsilon(x) \cap \Lambda = \emptyset, \ \forall x \in B_\delta(p)\} = 0$$

where $B_r(z) = \{x \in M \mid d(z, x) < r\}$.

Example 2. Let $f \in Diff^1(M)$ and $\Lambda \subset M$ be a closed invariant set which is not a perodic point containing a saddle fixed point $p \in \Lambda$ which is topologically dense and whose eigenvalues are real and positive.

p is topologically dense therefore for every $m \in \mathbb{N}$ there are positive integers ϵ_m, δ_m such that $\frac{\epsilon_m}{2\delta_m} \to 0$ as $m \to +\infty$ and $B_{\delta_m}(x) \cap \Lambda \neq \emptyset$ for every $x \in B_{2\delta_m}(p)$. Thus by induction we find sequence $\{q_m\} \subseteq \Lambda$ such that $d(q_m, q_{m+1}) = \epsilon_m$ and $d(q_m, p) = 2\delta_m$ (see the following Figure). So by taking suitable charts we can suppose that $p = 0$ and $\lim_{m\to\infty} \frac{q_m - q_{m+1}}{q_m} = 0$. Therefore p satisfies condition A.



The following theorem shows that Example 2 is a prototype structures for an invariant set to be not horseshoe-like.

Theorem 1. *Let $f \in Diff^1(M)$ and $\Lambda \subset M$ be a closed invariant set which is not a periodic point. Suppose Λ contains a saddle fixed point p satisfying condition A with real and positive eigenvalues. Then Λ is not horseshoe-like.*

Proof. We show that there is a connected component in Λ which is not consist of a single point. So Λ is not a Cantor set and hence it is not horseshoe-like. Since p is a hyperbolic point, there is an $\epsilon_0 > 0$ and a homeomorphism $h : B_{\epsilon_0}(p) \to T_pM$ such that

$$h(p) = 0 \tag{1}$$

$$D_pfoh = hof. \tag{2}$$

There exists $\epsilon' > 0$ such that

$$\{v \in T_pM \mid \|v\| < \epsilon'\} = h(B_{\epsilon_0}(p)).$$

Let $\{\tilde{\lambda}_i \mid 1 \leq i \leq s\}$ be the set of all eigenvalues of D_pf which norm greater than 1. Denote by $\{\tilde{\lambda}_i \mid s+1 \leq i \leq m\}$ the set of all eigenvalues of D_pf which norm less than 1 and let $\{v_1, \ldots, v_s\}$ and $\{v_{s+1}, \ldots, v_m\}$ be the set of eigenvectors of $\{\tilde{\lambda}_i \mid 1 \leq i \leq s\}$ and $\{\tilde{\lambda}_i \mid s+1 \leq i \leq m\}$ respectively. Put

$$\bar{L} = \left\{ \sum_{i=1}^{m} \lambda_i v_i \mid 0 < \lambda_i < \frac{\epsilon'}{2m} \right\}$$

and

$$L = \left\{ \sum_{i=1}^{s} \lambda_i v_i \mid 0 < \lambda_i < \frac{\epsilon'}{2s} \right\}.$$

We can see that $h^{-1}(\bar{L})$ and $h^{-1}(L) \subset B_{\epsilon_0}(p)$. Since p is a saddle point with condition A, there are sequences $\{x_n\}$ and $\{w_n\} \subset T_pM$, $w_n = \sum_{i=1}^{m} \lambda_i^{w_n} v_i$ such that for $1 \le i \le s$, we have

$$(1.3) \qquad \begin{cases} \lim_{n\to\infty} \frac{\lambda_i^{w_n} - \lambda_i^{w_{n+1}}}{\lambda_i^{w_{n+1}}} = 0 \\ \lambda_i^{w_{n+1}} < \lambda_i^{w_n} \\ \lim_{n\to\infty} \lambda_i^{w_n} = 0 \\ h^{-1}(w_n) = x_n \in \Lambda \end{cases}$$

Let $z \in L$. For every $s+1 \le i \le m$ there exists $N_i \in \mathbb{N}$ such that for any $n \ge N_i$, $0 < \lambda_i^{w_n} < \frac{\delta}{2M(m-s)}$ where

$$M = \max\{\tilde{\lambda}_i \mid 1 \le i \le m\}.$$

Put $N_0 = \max\{N_i \mid s+1 \le i \le m\}$. Since $z \in L$, we can consider $z = \sum_{i=1}^{s} \lambda_i^z v_i$ such that $0 < \lambda_i < \frac{\epsilon'}{2s}$. For every $k > N_0$ put

$$m_k = \min\left\{ m \mid \lambda_i^{w_{m+1}} < \frac{\lambda_i^z + \frac{\delta}{2s}}{\tilde{\lambda}_i^k} \right\}$$

for $1 \le i \le s$. So

$$\lambda_i^{w_{m_{k+1}}} < \frac{\lambda_i^z + \frac{\delta}{2s}}{\tilde{\lambda}_i^k} \le \lambda_i^{w_{m_k}}$$

for $1 \le i \le s$.
We claim that there exists $k_0 > N_0$ such that for $1 \le i \le s$,

$$\lambda_i^{w_{m_{k_0+1}}} > \frac{\lambda_i^z - \frac{\delta}{2s}}{\tilde{\lambda}_i^{k_0}}.$$

Suppose our claim is not true. Hence for every $k > N_0$ and some $1 \le i \le s$

$$\lambda_i^{w_{m_{k+1}}} \le \frac{\lambda_i^z - \frac{\delta}{2s}}{\tilde{\lambda}_i^k} < \frac{\lambda_i^z + \frac{\delta}{2s}}{\tilde{\lambda}_i^k} \le \lambda_i^{w_{m_k}}.$$

So

$$\frac{\lambda_i^{w_{m_k}} - \lambda_i^{w_{m_{k+1}}}}{\lambda_i^{w_{m_{k+1}}}} \ge \frac{\frac{\frac{2\delta}{2s\tilde{\lambda}_i^k}}{\frac{\lambda_i^z + \frac{\delta}{2s}}{}}}{2s\tilde{\lambda}_i^k} = \frac{2\delta}{\lambda_i^z + \frac{\delta}{2s}} > 0$$

that contradicts (1.3). Hence there exists $n_0 > N_0$ such that

$$\frac{\lambda_i^z - \frac{\delta}{2s}}{\tilde{\lambda}_i^{n_0}} < \lambda_i^{w_{m_{n_0+1}}} < \frac{\lambda_i^z + \frac{\delta}{2s}}{\tilde{\lambda}_i^{n_0}}$$

for $1 \leq i \leq s$. This shows that

$$\sum_{i=1}^{s} \left| \tilde{\lambda}_i^{n_0} \lambda_i^{y_{m_{n_0}}+1} - \lambda_i^z \right| < \frac{\delta}{2}.$$

Hence we have

$$\begin{aligned} &\left\| Df^{n_0} \left(\sum_{i=1}^{m} \lambda_i^{w_{m_{n_0}}+1} v_i \right) - z \right\| \\ &= \left\| \sum_{i=1}^{s} \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_{n_0}}+1} v_i + \sum_{i=s+1}^{m} \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_{n_0}}+1} v_i - \sum_{i=1}^{s} \lambda_i^z v_i \right\| \\ &\leq \left\| \sum_{i=1}^{s} \tilde{\lambda}_i^{n_0} \lambda^{w_{m_{n_0}}+1} v_i - \sum_{i=1}^{s} \lambda_i^z v_i \right\| + \left\| \sum_{s+1}^{m} \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_{n_0}}+1} v_i \right\| =: B \end{aligned}$$

Since for $s+1 \leq i \leq m$, $\tilde{\lambda}_i^{n_0} < 1$ and $0 < \lambda^{w_{m_{n_0}}+1} < \frac{\delta}{2M(m-s)}$. Hence

$$\begin{aligned} B &\leq \sum_{i=1}^{s} \left| \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_{n_0}}+1} - \lambda_i^z \right| + \sum_{i=s+1}^{m} \left| \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_{n_0}}+1} \right| \\ &< \frac{\delta}{2} + \frac{(m-s)\delta}{2(m-s)M} < \delta. \end{aligned}$$

This shows that $Df^{n_0} \left(\sum_{i=1}^{m} \lambda_i^{w_{m_{n_0}}+1} v_i \right) \in B_\delta(z)$. This shows that for every $x \in h^{-1}(L)$ there is sequence such that

$$\begin{aligned} &D_p f^m(v_{n_m}) \longrightarrow h(x) \\ &\{h(v_{n_m}) = x_{n_m}\} \subset \Lambda \end{aligned}$$

since $h^{-1} \circ D_p f \circ h = f$ so

$$f^m(x_{n_m}) \longrightarrow x.$$

Λ is closed and invariant so we have $x \in \Lambda$. Hence $h^{-1}(L) \subset \Lambda$. Note that $h^{-1}(L)$ is connected component. Hence Λ is not like horseshoe. □

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