A CONCISE PROOF OF A DOUBLE INEQUALITY INVOLVING THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

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Abstract. In this note we provide a concise proof for a double inequality involving the exponential and logarithmic functions, our method is based on the usage of the majorization inequalities and the Schur-convexity of a function.

Keywords: inequality, exponential function, logarithmic function, Schur-convexity, majorization inequality, Stolarsky mean.

1. Introduction

In [4], Guo and Qi presented a double inequality involving the exponential and logarithmic functions, as follows:

$$\ln\frac{e^x-e^y}{x-y}<\frac{(x-1)e^x-(y-1)e^y}{e^x-e^y}<\ln\frac{e^x+e^y}{2}, \tag{1}$$

where x and y are arbitrary real numbers with $x \neq y$.

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In [4], the authors also mentioned that the idea of establishing inequality (1) was motivated by the following two inequalities [1, p. 352]:

$$e^{\frac{x+y}{2}} < \frac{e^x - e^y}{x - y} < \frac{e^x + e^y}{2}, \tag{2}$$

$$\frac{x+y}{2} < \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y}, \tag{3}$$

where x and y are arbitrary real numbers with $x \neq y$.

Guo and Qi [4] proved the inequality (1) by the method of mathematical analysis. In this paper, we give a new proof of inequality (1) using the majorization inequalities (introduced by Hardy et al. [2]) and Schur-convexity (introduced by Schur [3]). As an application of inequality (1), we establish a comparison result of the Stolarsky means for different parameters.

2. Definitions and Lemmas

In this section, we need to introduce some definitions and lemmas relating to the theory of majorization inequalities.

Definition 1. [5, 6] Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

1. $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \preceq \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order. Furthermore, $\boldsymbol{x}$ is said to be strictly majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\boldsymbol{x}$ is not permutation of $\boldsymbol{y}$.

2. Let $\Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ is said to be a strictly Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $f(\boldsymbol{x}) < f(\boldsymbol{y})$. f is said to be a strictly Schur-concave function on Ω if and only if $-f$ is strictly Schur-convex function on Ω.

Definition 2 ([5, 6])**.** Let $\Omega \subseteq \mathbb{R}^n$. Ω is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $0 \leq \alpha \leq 1$ implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

Lemma 1 ([5, 6])**.** *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior convex set. Ω^0 is the interior of Ω. $f : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then f is the strictly Schur-convex (Schur-concave) function, if and only if f is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) > 0(< 0, respectively) \tag{4}$$

holds for any $\boldsymbol{x} \in \Omega^0$ and $x_1 \neq x_2$.

3. The proof of inequality (1)

Let us first deal with the left-hand inequality of (1), which reads as follows:

Proposition 1. *For arbitrary real numbers x, y with $x \neq y$, we have*

$$\ln \frac{e^x - e^y}{x - y} < \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y}. \tag{5}$$

Proof. By the L'Hospital rule, it is easy to find that

$$\lim_{x \to y} \left[\ln \frac{e^x - e^y}{x - y} - \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y} \right] = 0.$$

Thus, we define a function $f(x, y)$ by

$$f(x, y) = \begin{cases} \ln \frac{e^x - e^y}{x - y} - \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y}, & x \neq y \\ 0, & x = y. \end{cases}$$

Note that the inequality (5) is symmetrical with respect to variables x and y. To prove inequality (5), it is sufficient to prove that $f(x, y) < 0$ for $x > y$.

Let us now discuss the Schur-convexity of $f(x, y)$ on $\Omega = \{(x, y) : x > y, x, y \in \mathbb{R}\}$.

Differentiating $f(x, y)$ with respect to x gives

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{e^x}{e^x - e^y} - \frac{1}{x - y} - \frac{xe^x(e^x - e^y) - e^x[(x-1)e^x - (y-1)e^y]}{(e^x - e^y)^2} \\ &= \frac{e^{2x} - e^{x+y} - e^{2x} - (y - x - 1)e^{x+y}}{(e^x - e^y)^2} - \frac{1}{x - y} \\ &= \frac{(x - y)e^{x+y}}{(e^x - e^y)^2} - \frac{1}{x - y}. \end{aligned}$$

Similarly to the above, we have

$$\frac{\partial f}{\partial y} = \frac{(y - x)e^{x+y}}{(e^x - e^y)^2} - \frac{1}{y - x}.$$

Hence,

$$\Delta_1 := (x - y)\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) = 2\left[\frac{(x - y)^2 e^{x+y}}{(e^x - e^y)^2} - 1\right].$$

It is easy to observe that

$$\frac{(x - y)^2 e^{x+y}}{(e^x - e^y)^2} - 1 < 0 \Longleftrightarrow e^{x+y} < \left(\frac{e^x - e^y}{x - y}\right)^2,$$

which is equivalent to, a known result, the left-hand side inequality of (2). Hence, we obtain $\Delta_1 < 0$. By Lemma 1, we conclude that $f(x, y)$ is strictly Schur-concave on Ω. Further, from an evident majorization relationship

$$\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \prec (x, y),$$

along with the definition of Schur-concave function, we deduce that

$$0 = f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) > f(x, y),$$

which implies the desired inequality (5). The proof of Proposition 1 is complete. $\square$

Let us now verify the validity of the right-hand inequality of (1), which is stated by Proposition 2 below.

Proposition 2. *For arbitrary real numbers x, y with $x \neq y$, we have*

$$\frac{(x-1)e^x - (y-1)e^y}{e^x - e^y} < \ln\frac{e^x + e^y}{2}. \tag{6}$$

Proof. By using the L'Hospital rule, it is not difficult to verify that

$$\lim_{x \to y}\left[\ln\frac{e^x + e^y}{2} - \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y}\right] = 0.$$

Thus, we define a function $g(x, y)$ by

$$g(x, y) = \begin{cases} \ln\frac{e^x+e^y}{2} - \frac{(x-1)e^x-(y-1)e^y}{e^x-e^y}, & x \neq y \\ 0, & x = y. \end{cases}$$

Because the inequality (6) is symmetrical with respect to variables x and y, in order to prove inequality (6), it is enough to prove that $g(x, y) > 0$ for $x > y$.

In the following we discuss the Schur-convexity of $g(x, y)$ on $\Omega = \{(x, y) : x > y, x, y \in \mathbb{R}\}$.

Direct computation gives

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{e^x}{e^x + e^y} - \frac{xe^x(e^x - e^y) - e^x[(x-1)e^x - (y-1)e^y]}{(e^x - e^y)^2} \\ &= \frac{e^x}{e^x + e^y} - \frac{e^{2x} - (x - y + 1)e^{x+y}}{(e^x - e^y)^2} \end{aligned}$$

and

$$\frac{\partial g}{\partial y} = \frac{e^y}{e^x + e^y} - \frac{e^{2y} - (y - x + 1)e^{x+y}}{(e^x - e^y)^2}.$$

Therefore,

$$\begin{aligned}
\Delta_2 :&= (x-y)\left(\frac{\partial g}{\partial x}-\frac{\partial g}{\partial y}\right)\\
&= (x-y)\left[\frac{e^x-e^y}{e^x+e^y}+\frac{-e^{2x}+e^{2y}+2(x-y)e^{x+y}}{(e^x-e^y)^2}\right]\\
&= \frac{4e^{x+y}(x-y)^2}{(e^x+e^y)(e^x-e^y)^2}\left(\frac{e^x+e^y}{2}-\frac{e^x-e^y}{x-y}\right)\\
&= \frac{4e^{x+y}(x-y)^2}{(e^x+e^y)(e^x-e^y)^2}\left(\frac{e^x+e^y}{2}-\frac{1}{x-y}\int_y^x e^t dt\right).
\end{aligned}$$

Recall the well-known Hermite-Hadamard inequality for a convex function ψ on the interval $[x, y]$:

$$\frac{1}{x-y}\int_y^x \psi(t)dt \le \frac{\psi(x)+\psi(y)}{2}. \tag{7}$$

If we take $\psi(t) = e^t$, then we have a strict inequality of (7), that is,

$$\frac{1}{x-y}\int_y^x e^t dt < \frac{e^x+e^y}{2}.$$

Hence, we obtain $\Delta_2 > 0$, this implies that $g(x, y)$ is strictly Schur-convex on Ω. Then, from

$$\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \prec (x, y),$$

it follows that

$$0 = g\left(\frac{x+y}{2}, \frac{x+y}{2}\right) < g(x, y),$$

which implies the required inequality (6). The Proposition 2 is proved. □

4. An application to the Stolarsky mean

In order to demonstrate the application of inequality (1), we establish a comparison result of the Stolarsky means for different parameters.

Let $(x, y) \in \mathbb{R}^2_+$. The Stolarsky mean of (x, y) is defined in [7] as

$$E(a,b;x,y) = \begin{cases}
\left(\dfrac{b}{a}\cdot\dfrac{y^a-x^a}{y^b-x^b}\right)^{1/(a-b)}, & ab(a-b)(x-y)\ne 0,\\
\left(\dfrac{1}{a}\cdot\dfrac{y^a-x^a}{\ln y-\ln x}\right)^{1/a}, & a(x-y)\ne 0, b=0;\\
\dfrac{1}{e^{1/a}}\left(\dfrac{x^{x^a}}{y^{y^a}}\right)^{1/(x^a-y^a)}, & a(x-y)\ne 0, a=b;\\
\sqrt{xy}, & a=b=0, x\ne y;\\
x, & x=y.
\end{cases}$$

We have the following inequalities for the Stolarsky mean $E(a, b; x, y)$.

Proposition 3. *Let u, v be arbitrary positive numbers with $u \neq v$. Then*

$$E(1,0;u,v) < E(1,1;u,v) < E(2,1;u,v). \tag{8}$$

Proof. Taking $e^x = u$ and $e^y = v$ in the inequality (1), we obtain

$$\ln\left(\frac{u-v}{\ln u - \ln v}\right) < \frac{(\ln u - 1)u - (\ln v - 1)v}{u-v} < \ln\frac{u+v}{2}$$

$$\iff \frac{u-v}{\ln u - \ln v} < \frac{1}{e}\left(\frac{u^u}{v^v}\right)^{1/(u-v)} < \frac{u+v}{2}. \tag{9}$$

Obviously, the inequality (9) can be equivalently transformed to the desired inequality (8) according to the definition of $E(a, b; x, y)$ described above. This proves Proposition 3.

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