THE BICLIQUE PARTITION NUMBER OF SOME IMPORTANT GRAPHS

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Abstract. The biclique partition number of a graph G, $bp(G)$ is the minimum number of complete bipartite subgraphs needed to partition the edge set of G. Let $r(G) = \max\{n_+(G), n_-(G)\}$ where $n_+(G), n_-(G)$ are the number of positive and the number of negative eigenvalues of the adjacency matrix of G, respectively. A graph G satisfying, $bp(G) = r(G)$ is called an eigensharp graph. In this paper we apply Pollak and Graham Theorem to find the biclique partition number of the line graph of complete graph and its complement, the line graph of complete bipartite graph and its complement and the line graph of a tree graph and we discuss the eigensharp property of these graphs. Also we identify the biclique partition number of the $kth-$power graph of paths and cycles.
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1. Introduction

All graphs in this paper are finite undirected simple graphs. For a graph $G = (V(G), E(G))$, the set $V(G)$ denotes the set of vertices and $E(G)$ denotes the set of edges. The order of a graph G is equal to the cardinality of $V(G)$ and is denoted by $|G|$. The distance between two vertices u and v in G, denoted by $d(u, v)$, is the length of a shortest path between u and v in G. A clique is a complete subgraph. A biclique is a complete bipartite subgraph. The set of eigenvalues of $A(G)$, the adjacency matrix of G, is called the spectrum of G and is written spec(G). If λ_i ,$1 \leq i \leq k$, are the distinct eigenvalues of $A(G)$ with multiplicity m_i, then we write

$$spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_k \\ m_1 & m_2 & ... & m_k \end{pmatrix}.$$

Covering of a graph by a certain type of subgraphs is an important concept. It has been studied by different authors. There are several types of graph covering, including path covering, tree covering, clique covering and biclique

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covering. In this paper, we'll focus on the biclique partition covering of a graph. The biclique partition covering of a graph has been studied by several authors, see [2], [3], [4], [6] and [7].

A biclique partition of a graph G is a collection of bicliques of G that partition the edge set of G. Similarly, a biclique cover of G is a collection of bicliques that cover the edge set of G. The minimum cardinality of a biclique partition of a graph G is called the biclique partition number, denoted by $bp(G)$. Note that $bp(G) \leq n-1$. This holds because for any graph G with n vertices stars on $n-1$ vertices form a biclique partition for G. In 1971, Graham and Pollak [6] proved that $bp(K_n) = n-1$. Witsenhausen ($cf.$ [6]), showed that for a graph G with n vertices, the biclique partition number $bp(G)$ is bounded below as follows:

$$bp(G) \geq \max\{n_+(G), n_-(G)\}$$

where $n_+(G), n_-(G)$ are the number of positive and the number of negative eigenvalues of the adjacency matrix of G, respectively. Suppose that $\max\{n_+(G), n_-(G)\} = r(G)$. A graph G satisfying, $bp(G) = r(G)$ is called an eigensharp graph, see [7].

Given a graph G, an independent set is a subset of the vertex set of G such that no two vertices are adjacent. The independence number $\alpha(G)$ is the cardinality of a largest set of independent vertices. A maximum independent set with the largest number of vertices in a given graph G is denoted by $I(G)$, (i.e. $\alpha(G) = |I(G)|$). A star S_n is a tree with $(n+1)$ vertices with one vertex having degree n, and the other n vertices having degree 1. In S_n the vertex of degree n is called the center of the star. The star graph S_n is therefore isomorphic to the complete bipartite graph $K_{1,n}$. Because every star is a complete bipartite graph, the vertex cover number of G is an upper bound of $bp(G)$, and moreover one can easily prove that if G is a graph on n vertices then $bp(G) \leq n - \alpha(G)$. For a graph G when the biclique partition covering of minium cardinality is a collection of stars we use star center to represent the star, $K_{1,n}$. In this paper we will consider the edges of K_n partitioned into $n-1$ bicliques using edge-disjoint stars.

In this paper, Graham and Pollak Theorem is the cornerstone of our results. We study the biclique partition number for several classes of graphs. In section 2, we study the biclique partition number of the line of any complete graph and its complement, the line graph of any complete bipartite graph and its complement. We characterize when these graphs are eigensharp depending on the set of edges incident with v generate a clique in $L(G)$ of order degree(v). The cliques of $L(G)$ in this way partition the edge set of $L(G)$, and we apply Graham and Pollak Theorem to calculate the biclique partition number of these graphs. In section 3, we discuss the biclique partition number of the line graph of a tree graph, by noting that the line graph of a tree graph is a connected block graph in which each cutpoint is on exactly two blocks. Also we applied Graham and Pollak Theorem on these blocks to calculate the biclique partition number of the line graph of a tree graph.

Finally, the biclique partition number of the $kth-$power of paths and cycles are characterized.

2. The biclique partition number for some families of line graphs

In this section, the biclique partition number of the line graphs of complete graphs and their complements, and that of line graphs of complete bipartite graphs and their complements are completely characterized. Moreover, the biclique partition number of the line graph of trees is calculated.

Definition. *Let G be a simple graph with n* vertices *and m edges. The line graph $L(G)$ of G is the simple graph whose vertex set is the set of edges of G and for any $a, b \in V(L(G))$* the *vertices a and b are adjacent in $L(G)$ if and only if a and b have a common vertex in G.*

For a graph G and a vertex v in G with degree$(v) \geq 2$, the set of edges incident to v generate a clique in $L(G)$ of order degree(v). The cliques of $L(G)$ in this way partition the edges of $L(G)$, and this partition helps us to find the biclique partition number of the line graph using Graham and Pollak Theorem. Each vertex of $L(G)$ belongs to exactly two cliques which are the two cliques corresponding to the two endpoints of the corresponding edge in G. We will study the biclique partition number of line graph of some families of graphs and investigate when they have the eigensharp property. It is easy to show that the line graph of a star graph, $L(S_n) = K_n$, and therefore $bp(L(S_n)) = n - 1$, and so $L(S_n)$ is eigensharp. Trees are eigensharp, see [7]. Also paths P_n are eigensharp and the complements of paths are eigensharp too, see [2]. We note that $L(P_n) = P_{n-1}$, so line graph of path and its complement are eigensharp. The line graph of a cycle C_n is isomorphic to C_n, which is eigensharp when $n \neq 4k$ with $k > 2$, see [7]. Moreover, its complement is eigensharp, see [2].

2.1 The graphs $L(K_n)$ and their complement $\overline{L(K_n)}$

The line graph of K_n, $L(K_n)$ is a graph with $\binom{n}{2}$ vertices and $n\binom{n-1}{2}$ edges. Suppose that the vertices $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. In K_n, we denote the edge between u_i and u_j by $u_{i,j}$. Therefore $V(L(K_n)) = \{u_{i,j} : 1 \leq i < j \leq n\}$. Then two distinct vertices in $L(K_n)$are adjacent if their labels share exactly one digit. So, when n is even, then the set $I(L(K_n)) = \{u_{1,2}, u_{3,4}, u_{5,6}, \ldots, u_{n-3,n-2}, u_{n-1,n}\}$ is a maximum independent set in $L(K_n)$, and if n is odd then the set $I(L(K_n)) = \{u_{1,2}, u_{3,4}, u_{5,6}, \ldots, u_{n-2,n-1}\}$ is a maximum independent set in $L(K_n)$, which is of order $\left\lfloor \frac{n}{2} \right\rfloor$. The vertex u_i has degree $(n-1)$ in K_n, so we have in $L(K_n)$, n cliques of order $(n-1)$, say $B_i : 1 \leq i \leq n$, where $V(B_i) = \{u_{i,j} : 1 \leq j \leq n, i \neq j, \}$. Take any edge in $L(K_n)$, then this edge comes from two adjacent edges in K_n, suppose that the common vertex is u_i and the ends of these two edges in K_n are u_j and u_k respectively, the edge in $L(K_n)$ that is incident with vertex $u_{i,j}$ and $u_{i,k}$, so this edge belongs to exactly one member from the family $\{B_i\}_{i=1}^n$. So the family $\{B_i\}_{i=1}^n$ forms a partition of the edges of $L(K_n)$, and clearly $V(B_i) \cap V(B_j) = \{u_{i,j}\}$.

To get a better understanding of the computation of $bp(L(K_n))$, we give the following example of the characterization of $L(K_4)$ and $L(K_5)$.

Example 1. As we explained earlier, the family $\{B_i\}_{i=1}^4$ forms a partition of the edge set of $L(K_4)$. Each B_i is a clique of order 3, and also $V(B_1) = \{u_{1,2}, u_{1,3}, u_{1,4}\}, V(B_2) = \{u_{2,1}, u_{2,3}, u_{2,4}\}, V(B_3) = \{u_{3,1}, u_{3,2}, u_{3,4}\}$ and $V(B_4) = \{u_{4,1}, u_{4,2}, u_{4,3}\}$. It is clear that $I(L(K_4)) = \{u_{1,2}, u_{3,4}\}$. By Graham and Pollak Theorem, $bp(B_i) = 2$. So we need two vertices to cover each B_i by biclique partiton of minimum cardinality. Let H be the induced subgraph of $L(K_4)$ such that $V(H) = V(L(K_4)) - I(L(K_4)) = \{u_{1,3}, u_{1,4}, u_{2,3}, u_{2,4}\}$. Then for B_1, the number of vertices having label 1 is 2, which we need to cover $B_1$by stars, and the same for other cliques B_2, B_3 and B_4, and therefore $bp(L(K_4)) = |H| = 4$.

Example 2. The family $\{B_i\}_{i=1}^5$ forms a partition of the edge set of $L(K_5)$. Each B_i is a clique of order 4, and also $V(B_1) = \{u_{1,2}, u_{1,3}, u_{1,4}, u_{1,5}\}, V(B_2) = \{u_{2,1}, u_{2,3}, u_{2,4}, u_{2,5}\}, V(B_3) = \{u_{3,1}, u_{3,2}, u_{3,4}, u_{3,5}\}, V(B_4) = \{u_{4,1}, u_{4,2}, u_{4,3}, u_{4,5}\}$ and, $V(B_5) = \{u_{5,1}, u_{5,2}, u_{5,3}, u_{5,4}\}$. It is clear that $I(L(K_5)) = \{u_{1,2}, u_{3,4}\}$. By Graham and Pollak Theorem, $bp(B_i) = 3$. So we need three vertices to cover each B_i by biclique partiton of minimum cardinality. Let H be the induced subgraph of $L(K_5)$ such that $V(H) = V(L(K_5)) - I(L(K_5)) = \{u_{1,3}, u_{1,4}, u_{1,5}, u_{2,3}, u_{2,4}, u_{2,5}, u_{3,5}, u_{4,5}\}$. Then for $B_1$the number of vertices having label 1 is 3 vertices, which covers $B_1$by stars, and the same for other cliques B_2, B_3, B_4. But for B_5 there are 4 vertices having label 5, it is not possible to dispense with any one of them, because we need $u_{1,5}, u_{2,5}, u_{3,5}$ and $u_{4,5}$to cover B_1, B_2, B_3 and B_4 respectively by stars covering and therefore, $bp(L(K_5))$=$|H|$=8.

In general for $L(K_n)$, we will get the following theorem.

Theorem 3. *For the graph* $L(K_n)$, $bp(L(K_n)) = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. In $L(K_n)$ the family $\{B_i\}_{i=1}^n$ forms a partition of $E(L(K_n))$, where B_i is clique of order $(n-1)$. Let H be the induced subgraph of $L(K_n)$ such that $V(H) = V(L(K_n)) - I(L(K_n))$. We claim that the vertices of H give a biclique partition of minimum cardinality. Take any B_i in $L(K_n)$, by Graham and Pollak Theorem, $bp(B_i) = n-2$, so we need exactly $(n-2)$ vertices from B_i (i.e. the vertices have label i) to cover it by biclique partition of minimum cardinality. Since $V(B_i) = \{u_{i,j} : 1 \le j \le n,\ i \ne j\}$. There are four cases for $B_i, 1 \le j \le n$.

Case 1. If i is odd, and $i < n$, then $u_{i,i+1} \in I(L(K_n))$, and therefore $u_{i,i+1} \notin H$.

Case 2. If i is even, and $i < n$, then $u_{i-1,i} \in I(L(K_n))$, and therefore $u_{i-1,i} \notin H$.

Case 3. If n is even, and $i = n$, then $u_{n-1,n} \in I(L(K_n))$, and therefore $u_{n-1,n} \notin H$.

Thus for the three cases we conclude that for any B_i in $L(K_n)$, only remained exactly $(n-2)$ vertices have label i that belong to H and so B_i is covered by stars which is a biclique partition of minium cardinality.

Case 4. If n is odd, and $i = n$, then H contains the vertices $u_{n,1}, u_{n,2}, \ldots, u_{n,n-1}$ having label n, and it is not possible to dispense with any one of them, because we need $u_{n,1}, u_{n,2}, \ldots, u_{n,n-1}$ to cover $B_1, B_2, \ldots, B_{n-1}$respectively by biclique partition of minimum cardinality. Hence

$$bp(L(K_n)) = |H| = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor .$$

□

On the other hand,

$$spec(L(K_n)) = \begin{pmatrix} 2n-4 & n-4 & -2 \\ 1 & n-1 & \frac{n(n-3)}{2} \end{pmatrix},$$

see [1], and hence when $n = 5$, then $r(L(K_n)) = n$ and if $n \geq 6$, then $r(L(K_n)) = \frac{n(n-3)}{2}$. So, $L(K_n)$ is not eigensharp for $n \geq 4$ and it is eigensharp when $n < 4$.

Now we show that the complement of $L(K_n)$ is eigensharp, consider the following theorem.

Theorem 4. *The complement of* $L(K_n)$, $\overline{L(K_n)}$, *is eigensharp.*

Proof. We will show that $bp(\overline{L(K_n)}) = r(\overline{L(K_n)})$, we know that

$$spec(L(K_n)) = \begin{pmatrix} 2n-4 & n-4 & -2 \\ 1 & n-1 & \frac{n(n-3)}{2} \end{pmatrix},$$

see [1]. Since $L(K_n)$ is $(2n-4)-$regular,

$$spec(\overline{L(K_n)}) = \begin{pmatrix} \binom{n}{2} - (2n-4) - 1 & 3-n & 1 \\ 1 & n-1 & \frac{n(n-3)}{2} \end{pmatrix},$$

see[1]. Hence $r(\overline{L(K_n)}) = n_+(\overline{L(K_n)}) = \binom{n-1}{2}$, and therefore $bp(\overline{L(K_n)}) \geq \binom{n-1}{2}$. In fact equality is achieved. The graph $\overline{L(K_n)}$ has maximum independent set of order $n-1$ (i.e. $\alpha(\overline{L(K_n)}) = n-1$), since in K_n each vertex is incident to $n-1$ edges, and these edges are pairwise adjacent vertices in $L(K_n)$ and so they are nonadjacent vertices in $\overline{L(K_n)}$. These edges are the largest set of pairwise nonadjacent vertices in $\overline{L(K_n)}$. And so $bp(\overline{L(K_n)}) \leq \binom{n}{2} - (n-1) = \binom{n-1}{2}$. Thus $bp(\overline{L(K_n)}) = \binom{n-1}{2}$ and therefore $\overline{L(K_n)}$ is eigensharp. □

2.2 The graph $L(K_{n,m})$ and their complement $\overline{L(K_{n,m})}$

First, we will characterize the graph $L(K_{n,m})$ and we assume that $2 \le n \le m$. The complete bipartite graph $K_{n,m}$, has two maximal independent sets X_n, Y_m where $X_n = \{x_1, \dots, x_n\}$ and $Y_m = \{y_1, \dots, y_m\}$. In $K_{n,m}$ each edge is incident with one vertex from X_n and one vertex from Y_m, we may say that the vertex set is $V(L(K_{n,m})) = \{u_{i,j} : 1 \le i \le n\ , 1 \le j \le m\ \}$ and $u_{i,j}$ represents the edge in $K_{n,m}$ between x_i and y_j. Now, in $L(K_{n,m})$ the vertex $u_{i,j}$ is adjacent to $u_{k,h}$ if and only if $i = k$ or $j = h$. In X_n each x_i has degree m, so we have in $L(K_{n,m})$, n cliques of order m, denoted by $B(H_i) : 1 \le i \le n$, induced by $H_i = \{u_{i,1}, u_{i,2}, \dots, u_{i,m}\}$. Since X_n is an independent set in $K_{n,m}$, we have $H_i \cap H_j = \varphi$, for all $i \ne j$, $1 \le i, j \le n$. On the other hand Y_m is an independent set, so we have m cliques of order n say $B(V_i)$, induced by $V_i = \{u_{1,i}, u_{2,i}, \dots, u_{n,i}\}$ and $V_i \cap V_j = \varphi$, $i \ne j$, $1 \le i, j \le m$. Since each edge connects one vertex from X_n and one vertex from Y_m, we have $H_i \cap V_j = u_{i,j}$.

Now we want to compute the biclique partition number of $L(K_{n,m})$.

Theorem 5. *For the graph* $L(K_{n,m})$, $bp(L(K_{n,m})) = n(m-1)$, *where* $m \ge n \ge 2$.

Proof. In $L(K_{n,m})$ the family $\{B(H_i)\}_{i=1}^n$ is a family of disjoint cliques each of order m. Suppose $A_i : 1 \le i \le n$ is a biclique partition of $B(H_i)$, we need $(m-1)$ distinct vertices for each member of $\{B(H_i)\}_{i=1}^n$to cover it by stars, therefore $|A_i| = m - 1$. Taking A_i in this way, $A_1 = \{u_{1,2}, u_{1,3}, \dots, u_{1,m}\}, A_2 = \{u_{2,1}, u_{2,3}, \dots, u_{2,m}\}, \dots,\ A_i = \ \{u_{i,1}, u_{i,2}, u_{i,3}, \dots, u_{i,i-1}, u_{i,i+1}, \dots, u_{i,m}\}\ , \dots,$ $A_n = \{u_{n,1}, u_{n,2}, \dots, u_{n,n-1}, u_{n,n+1}, \dots, u_{n,m}\}$, (i.e. we take all vertices of $B(H_i)$ and we leave the vertex $u_{i,i}$). We will prove that $\{A_i\}_{i=1}^n$ is a biclique partition of minium cardinality for $L(K_{n,m})$. First, A_i is biclique partition of $B(H_i)$ for each $1 \le i \le n$, because $|A_i| = m - 1$ and $A_i \cap A_j = \ \varphi : 1 \le i, j \le n$, $i \ne j$. Second, for any $B(V_j) : 1 \le j \le m$, we have $bp(B(V_j)) = n - 1$. Take the induced subgraph by the vertices $\{u_{i,j} : 1 \le i \le n, i \ne j\} \subseteq B(V_j)$ which contains $n - 1$ vertices, so it covers all $B(V_j)$ by stars covering, and for all $1 \le i \le n$, $u_{i,j} \in A_i$, so $\{u_{i,j} : 1 \le i \le n, i \ne j\} \subseteq \bigcup_{i=1}^n A_i$. This gives us $\{A_i\}_{i=1}^n$ is a family of stars which cover $B(V_j)$: $1 \le j \le m$. So, we have $\{A_i\}_{i=1}^n$ is a biclique partition of minimum cardinality for $L(K_{n,m})$. Therefore we have $bp(L(K_{n,m})) = \sum_{i=1}^n\ |A_i| = n(m-1) = n(m-1)$. □

Question. Is $L(K_{n,m})$ eigensharp?

To answer this question we will use the spectrum of the cartesian product $K_n \times K_m$, see [1]. In fact, $K_n \times K_m$ is isomorphic to the line graph of the complete bipartite graph $K_{n,m}$, $L(K_{n,m})$. Now, the spectrum of $K_n \times K_m$ is equal to

$$spec(L(K_{n,m})) = \begin{pmatrix} m+n-2 & m-2 & n-2 & -2 \\ 1 & n-1 & m-1 & (m-1)(n-1) \end{pmatrix}.$$

Thus $r\left(L\left(K_{n,m}\right)\right)=(m-1)(n-1)$. So, $L\left(K_{n,m}\right)$ is not eigensharp.

In the following theorem we determine the biclique partition number of the graph $\overline{L\left(K_{n,m}\right)}$.

Theorem 6. *For the graph* $\overline{L\left(K_{n,m}\right)}$, $bp(\overline{L\left(K_{n,m}\right)})=m\left(n-1\right)$, *where* $m\geq n\geq 2$.

Proof. Recall that for the graph $L\left(K_{n,m}\right), V(L\left(K_{n,m}\right))=\{u_{i,j}: 1\leq i\leq n, 1\leq j\leq m\}$, it easy to show that in $\overline{L\left(K_{n,m}\right)}$ the vertices $u_{i,j}$ and $u_{k,l}$ are adjacent if and only if $k\neq i$ and $l\neq j$. Also $E(\overline{L\left(K_{n,m}\right)})=\{u_{i,j}u_{k,l}: k\neq i \text{ and } l\neq j\}$, therefore the sets $H_i=\{u_{i,1},u_{i,2},\ldots,u_{i,m}\}: 1\leq i\leq n$, is an independent set of vertices in $\overline{L\left(K_{n,m}\right)}$.

Now to calculate $bp(\overline{L\left(K_{n,m}\right)})$, consider the pairwise disjoint family $S=\{H_i\}_{i=1}^{n}$. Let F be the induced subgraph in $L\left(K_{n,m}\right)$ such that $V(F)=F=H_k\cup H_j$ for some $k\neq j$,$1\leq j,k\leq n$. The adjacency matrix of F is $A\left(F\right)=\begin{bmatrix} O & B\\ B^T & O\end{bmatrix}$, it is clear that B is the adjacenay matrix of K_m, (i.e. $B=A(K_m)$). Since F is a bipartite graph, λ is an eigenvalue of multiplicity p if and only if $-\lambda$ is an eigenvalue of multiplicity p. So,

$$spec(F)=\begin{pmatrix} m-1 & -1 & 1 & 1-m\\ 1 & m-1 & m-1 & 1\end{pmatrix}$$

and therefore $r(F)=m$. So, $bp(F)\geq m$. The induced subgraph F is partitioned into $K_{1,m-1},K_{2,m-1},\ldots,K_{m,m-1}$. Take one set of H_k or H_j it will cover the induced subgraph F by m stars. So, $bp(F)=m$. Since any induced subgraph of $\overline{L\left(K_{n,m}\right)}$ that contains two disjoint members of S, the biclique partition number that covers by stars must be m. Therefore, it can be proved by induction on $H_i: 1\leq i\leq n$ that $bp(\overline{L\left(K_{n,m}\right)})=\sum_{i-1}^{n-1}|H_i|=m(n-1)$, by showing that we need $\{H_i\}_{i=1}^{n-1}$ to cover $\overline{L\left(K_{n,m}\right)}$ by stars. Hence we get the result. $\square$

For eigensharpness property of $\overline{L\left(K_{n,m}\right)}$, we find the spectrum of $\overline{L\left(K_{n,m}\right)}$. Since $L\left(K_{n,m}\right)$ is $(m+n-2)$-regular graph,

$$spec(\overline{L\left(K_{n,m}\right)})=\begin{pmatrix} mn-m-n+1 & 1-m & 1-n & 1\\ 1 & n-1 & m-1 & \left(m-1\right)\left(n-1\right)\end{pmatrix},$$

therefore $r\left(\overline{L\left(K_{n,m}\right)}\right)=(m-1)(n-1)+1$, and this shows that $\overline{L\left(K_{n,m}\right)}$ is not eigensharp.

3. The biclique partition number of the line graph of a tree graph $L(T)$

In this section, we obtain the biclique partition number of the line graph of a tree of order n, $L(T)$.

Definition. *A block graph is a graph in which every biconnected component (block) is a maximal clique.*

Theorem 7 ([5]). *A graph is the line graph of a tree if and only if it is a connected block graph in which each cutpoint is exactly on two blocks.*

Suppose that $B_1, B_2, \dots, B_k$ are the blocks of $L(T)$ of orders $n_i : 1 \leq i \leq k$, and $n_i \geq 2$. Let H be an induced subgraph containing two blocks from $L(T)$ say B_i and B_j such that B_i and B_j are not disjoint. The block B_i and B_j have a unique cut vertex in common, say h. Then to compute $bp(H)$, we will look to $bp(B_i)$ and $bp(B_j)$. We know that $bp(B_i) = n_i - 1$ and $bp(B_j) = n_j - 1$, suppose that the vertex set of biclique partition of minium cardinality for $B_i = \{u_1, \dots, u_{n-1}\}$ and the vertex set of biclique partition of minium cardinality for $B_j = \{v_1, \dots, v_{n-1}\}$, so to cover H by biclique partition, we can take star cover of $\{u_1, \dots, u_{n-1}\} \cup \{v_1, \dots, v_{n-1}\}$. To find a biclique partition of minium number of H. Take$V(B_i) \cap V(B_j) = \{h\}$, $h \in \{u_1, \dots, u_{n-1}\}$ and $h \in \{v_1, \dots, v_{n-1}\}$; because h covers all edges incident with h in B_i and all edges incident with h in B_j. Therefore the biclique partition number of H equal to $(n_i - 1) + (n_j - 1) - 1$. The result easily extended to all blocks in $L(T_n)$, because each cutpoint is exactly in two blocks. Thus we have the following theorem.

Theorem 8. *If $L(T)$ has r cutpoints, such that degree $(r) > 3$, then $bp(L(T)) = bp(B_1) + ... + bp(B_k) - r = \sum_{i=1}^{k}(n_i - 1) - r$.*

We do not know whether the graph $L(T)$ is an eigensharp graph or not.

4. The biclique partition number of kth-power of path and cycle graphs

This section focuses on the biclique partition numbers of $kth-$power graph of paths and cycles. For any *positive* integer k and a connected graph G, a new graph, G^k, called the $kth-$power of G, can be defined as follows: $V(G^k) = V(G)$ and two distinct vertices u and v in G^k are adjacent if the distance between u and v in G is less than or equal to k. In this section, we will calculate the biclique partition numbers of the $kth-$power of a path, P_n^k, and the $kth-$power of a cycle, C_n^k.

The kth-power of path graph P_n^k. First, we describe the $kth-$power graph of a path, P_n^k. Then we will determine the biclique partition number of the $kth-$power graph of a path P_n^k, using Graham and Pollak Theorem. Suppose that the vertices of $V(P_n^k) = \{i : 1 \leq i \leq n\}$. The vertices i and j are adjacent if and only if $d(i,j) \leq k$. Let H be the induced subgraph containing $(k+1)$ consecutive vertices. Clearly, the induced subgraph of these vertices in P_n^k is a maximal clique of order $k+1$.

First, we will assume that $(k+1)$ divides n. Suppose $\frac{n}{k+1} = m$, so we can divide the vertices of P_n^k into pairwise disjoint family of subgraphs $\{H_i\}_{i=1}^{m}$, each of them has $(k+1)$ consecutive vertices $H_1 = \{1, 2, \ldots, k+1\}, H_2 = \{k+2, \ldots, 2(k+1)\}, \ldots, H_m = \{(m-1)(k+1)+1, \ldots, m(k+1)\}$. So, $\{H_i\}_{i=1}^m$ is a family of pairwise disjoint cliques each of them has order $(k+1)$. We rename the vertices of H_i in order to be used to calculate $bp\left(P_n^k\right)$. Suppose that $H_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,k+1}\}$, and we call $v_{i,1}$ the first element. Now we will study the relation between H_i and $H_{i+1} : 1 \leq i < m$. Let $v_{i,j}$ be a vertex in H_i, then $v_{i,j}$ is adjacent to all vertices in H_i, and $v_{i+1,p} \in H_{i+1} \quad : p < j \leq k+1$, because $d\left(v_{i,j}, v_{i+1,p}\right) \leq k$. So, the first element $v_{i,1}$ is not adjacent to any vertex in H_{i+1}.

In the following theorem we determine the biclique partition number of, P_n^k, assuming that $(k+1)$ divides n.

Theorem 9. *For the graph P_n^k, if $(k+1) \mid n$, then $bp\left(P_n^k\right) = \frac{k}{k+1}n$.*

Proof. Suppose $\frac{n}{k+1} = m$. So, we can divide the vertices of P_n^k into disjoint family, $\{H_i\}_{i=1}^m$ of cliques of order $(k+1)$. By Graham and Pollak Theorem, $bp(H_i) = k$. Let $X_i = \{v_{i,2}, \ldots, v_{i,k+1}\}$ be a biclique partition of minimum cardinality of H_i. Now we claim that $\bigcup_{i=1}^m X_i$ is a biclique partition of minium cardinality for P_n^k. Since $\{H_i\}_{i=1}^m$ is a family of disjoint cliques. Clearly, we need X_i to partition the edges of each H_i as star covering. Also these stars cover the edges between H_i and H_{i+1}. Then $\bigcup_{i=1}^m X_i$ partition the edges in $E(P_n^k)$. Also, using Graham and Pollak Theorem this is the least number needed. Thus $bp(P_n^k) = \sum_{i=1}^m |X_i| = \frac{kn}{k+1}$. □

On the other hand when n is not a multiple of $k+1$, by division algorithm $n = (k+1)m + r$, $0 < r < k+1$. Hence we get the following theorem.

Theorem 10. *For the graph P_n^k, $bp\left(P_n^k\right) = km + (r-1)$, where $n = (k+1)m + r$, $1 \leq r \leq k$.*

Proof. From the assumption, $\left\lfloor \frac{n}{k+1} \right\rfloor = m$. So, as in the previous theorem the first $m(k+1)$ vertices can be covered by disjoint family of m cliques each of order $k+1$. The edges can be covered by $\bigcup_{i=1}^m X_i$ as a star covering. While the last r vertices induce a clique, R, of order r. By, Graham and Pollak Theorem, we need at least $r-1$ vertices to partition the edges of this clique. The edges between the clique H_m and R are covered by stars of H_m. Hence we get the result. □

The kth-power of cycle graph C_n^k. First, we describe the $kth-$power of a cycle, C_n^k. Then we will determine the biclique partition number of $kth-$power of cycle, C_n^k, using Graham and Pollak Theorem. Let O be the induced subgraph containing $(k+1)$ consecutive vertices. Clearly, the induced subgraph of these vertices in C_n^k is a maximal clique of order $k+1$.

Using division algorithm $n = (k+1)m + r$, $0 \leq r < k+1$. Therefore we can divide the vertices of C_n^k into disjoint family of vertices each consists of

$k+1$ except the last one with r vertices. These member of this family induce a maximal cliques. So, we have $(m+1)$ disjoint cliques. By Graham and Pollak Theorem we need at least km vertices to cover all edges in the first m cliques, we need $r-1$ vertices to cover the last clique of order r. But stars with these vertices as their centers do not cover all edges in C_n^k, when $r \neq 0$; because edges between O_1 and the last clique are not covered. So, we need all vertices of the last clique, (i.e. r vertices), to cover all edges. Hence $bp\left(C_n^k\right) = km + r$. Thus we get the following theorem

Theorem 11. *For the graph C_n^k, $bp\left(C_n^k\right) = km + r$, where $n = (k+1)m + r$, $1 \leq r \leq k$.*

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