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REVERSES OF THE TRIANGLE INEQUALITY FOR ABSOLUTE VALUE IN HILBERT C^*-MODULES

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Abstract. In this paper we obtain some inequalities related to the reverse triangle inequalities for vectors in the framework of Hilbert C^*-modules. Also we improve a celebrated reverse triangle inequality due to Diaz and Metcalf. As a consequence, we apply our results to get some operator inequalities.

Keywords: triangle inequality, Hilbert C^*-module, C^*-algebra, positive element.

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1. Introduction and preliminaries

If $(\mathcal{X};\|\cdot\|)$ is a normed linear space, then

$$\left\|\sum_{i=1}^{n} x_i\right\| \leq \sum_{i=1}^{n} \|x_i\|, \tag{1.1}$$

for any vectors $x_i \in \mathcal{X}, i \in \{1,...,n\}$. Inequalities of this kind have been called triangle inequality. A number of mathematicians have investigated the inequality (1.1) in various settings. Farenick [13] have investigated the triangle inequality over matrix algebras in Hilbert C^*-modules. We also refer to interesting papers by Shrawan et al. [15] and Dadipour et al. [6]. Some versions of the triangle inequality with simple conditions for the case of equality are presented in [5, 14].

The first to consider the problem of obtaining reverses for the triangle inequality in the more general case of Hilbert and Banach spaces were Diaz and Metcalf [7] who showed that in an inner product space $\mathcal{H}$ over the real or complex number field, the following reverse of the triangle inequality holds

$$r\sum_{i=1}^{n} \|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{1.2}$$

provided

$$0 \leq r \leq \|x_i\| \leq \operatorname{Re}\langle x_i, e\rangle$$

for $k \in \{1,\dots,n\}$, where $e \in \mathcal{H}$ is a unit vector, i.e. $\|e\| = 1$.

Another reverse of the generalized triangle inequality in Hilbert space was given in [10, Theorem 5] as follows:

Theorem 1.1. *Let $(\mathcal{H};\langle\cdot,\cdot\rangle)$ be an inner product space, $x_i \in \mathcal{H}$, for all $i \in \{1,...,n\}$ and $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$ (probability distribution). If there exists constants $r_i > 0, i \in \{1,...,n\}$, so that*

$$\left\|x_i - \sum_{j=1}^{n} p_j x_j\right\| \leq r_i$$

for all $i \in \{1,...,n\}$, then

$$\sum_{i=1}^{n} p_i\|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2 \leq \sum_{i=1}^{n} p_i r_i^2. \tag{1.3}$$

Some other interesting reverses of the triangle inequality for the case of Hilbert space can be found in [12]. For related results, see also [1, 2, 3, 4, 8, 16].

The motivation of this paper is to extend some generalizations of the reverse triangle inequality like (1.3), in the framework of Hilbert C^*-modules (see Theorem 2.1). We also improve inequality (1.2) in a similar framework (this will be considered in Theorem 3.1).

At the end of this section, we would like to recall some notions, which will be used in the forthcoming sections. Let $\mathcal{A}$ be a C^*-algebra. A *pre-Hilbert $\mathcal{A}$-module* is a linear space $\mathcal{X}$ which is a right $\mathcal{A}$-module together with an $\mathcal{A}$-valued mapping $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ with following properties:

(a) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(b) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$;

(c) $\langle x, ya \rangle = \langle x, y \rangle a$;

(d) $\langle x, y \rangle^* = \langle y, x \rangle$;

for all $x, y, z \in \mathcal{X}, a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. It is straightforward that a C^*-algebra valued inner product is conjugate-linear in the first variable. We can define a norm on $\mathcal{X}$ by $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. If $\mathcal{X}$ is complete with respect to this norm, then $\mathcal{X}$ is called a *Hilbert $\mathcal{A}$-module*. The absolute value of $x \in \mathcal{X}$ is defined as the square root of $\langle x, x \rangle$, and it is denoted by $|x|$. It is worthwhile to point out that this is not actually an extension of a norm, in general, since it may happen that the triangle inequality does not hold.

Throughout the article, $\mathcal{A}$ and $\mathcal{X}$ are C^*-algebra and Hilbert $\mathcal{A}$-module respectively. A C^*-algebra is called unital if $\mathcal{A}$ has a unit $1_{\mathcal{A}}$ and for each $a \in \mathcal{A}$ we have $a.1_{\mathcal{A}} = a$. For convenience, in unital C^*-algebra $\mathcal{A}$ we write a instead of $a.1_{\mathcal{A}}$.

2. On the generalized reverses of the triangle inequality

We start our work by presenting a reverse of the triangle inequality for Hilbert C^*-modules.

Theorem 2.1. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $x_i \in \mathcal{X}$ for all $i \in \{1, ..., n\}$, and p_i are positive elements in real number field such that $\sum_{i=1}^n p_i = 1$. If there exist positive elements r_i, $i \in \{1, ..., n\}$ in $\mathcal{A}$, so that*

$$\left| x_i - \sum_{j=1}^n p_j x_j \right|^2 \leq r_i^2 \tag{2.1}$$

for $i \in \{1, .., n\}$, then

$$\sum_{i=1}^n p_i |x_i|^2 - \left| \sum_{i=1}^n p_i x_i \right|^2 \leq \sum_{i=1}^n p_i r_i^2. \tag{2.2}$$

Proof. According to (2.1) we have

$$\langle x_i, x_i \rangle - 2\mathrm{Re} \left\langle x_i, \sum_{j=1}^n p_j x_j \right\rangle + \left\langle \sum_{j=1}^n p_j x_j, \sum_{j=1}^n p_j x_j \right\rangle \leq r_i^2. \tag{2.3}$$

Multiply (2.3) by $p_i \geq 0$, and sum over i from 1 to n, to get

$$\sum_{i=1}^{n} p_i \langle x_i, x_i \rangle - 2\mathrm{Re} \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \left\langle \sum_{j=1}^{n} p_j x_j, \sum_{j=1}^{n} p_j x_j \right\rangle \leq \sum_{i=1}^{n} p_i r_i^2.$$

This says that

$$\sum_{i=1}^{n} p_i |x_i|^2 - 2\mathrm{Re} \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + \left| \sum_{j=1}^{n} p_j x_j \right|^2 \leq \sum_{i=1}^{n} p_i r_i^2,$$

this inequality is equivalent with

$$\sum_{i=1}^{n} p_i |x_i|^2 - \left| \sum_{i=1}^{n} p_i x_i \right|^2 \leq \sum_{i=1}^{n} p_i r_i^2,$$

which is inequality (2.2). □

As a consequence of Theorem 2.1 we have the following generalization of the reverse triangle inequality in the framework of Hilbert C^*-modules.

Proposition 2.1. *Let p_i,r_i and x_i for all $i \in \{1, ..., n\}$ be as in the statement of Theorem* 2.1*, then*

$$\mathrm{Re} \left(\sum_{i=1}^{n} p_i |x_i| \right) \left| \sum_{j=1}^{n} p_j x_j \right| \leq \left| \sum_{i=1}^{n} p_i x_i \right|^2 + \frac{1}{2} \sum_{i=1}^{n} p_i r_i^2. \tag{2.4}$$

Proof. From (2.3) we obviously have

$$|x_i|^2 + \left| \sum_{j=1}^{n} p_j x_j \right|^2 \leq 2\,\mathrm{Re} \left\langle x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + r_i^2, \tag{2.5}$$

for all $i \in \{1, ..., n\}$. Whence

$$2\,\mathrm{Re}\,|x_i| \left| \sum_{j=1}^{n} p_j x_j \right| \leq |x_i|^2 + \left| \sum_{j=1}^{n} p_j x_j \right|^2.$$

Here we exploited the fact that for each $a, b \in \mathcal{A}$, $2\,\mathrm{Re}\, ab^* \leq |a|^2 + |b|^2$. Therefore

$$2\,\mathrm{Re}\,|x_i| \left| \sum_{j=1}^{n} p_j x_j \right| \leq 2\,\mathrm{Re} \left\langle x_i, \sum_{j=1}^{n} p_j x_j \right\rangle + r_i^2$$

for all $i \in \{1, ..., n\}$. Arguments similar to the ones used in the proof of Theorem 2.1 give us (2.4). □

Remark 2.1. In particular, if $\mathcal{A}$ be a commutative C^*-algebra, by utilizing the inequality $2\left|a\right|\left|b\right| \le \left|a\right|^2 + \left|b\right|^2$, we can obtain from (2.5) the following result:

$$\sum_{i=1}^{n} p_i \left|x_i\right| \left|\sum_{j=1}^{n} p_j x_j\right| \le \left|\sum_{i=1}^{n} p_i x_i\right|^2 + \frac{1}{2}\sum_{i=1}^{n} p_i r_i^2.$$

One more consequence of Theorem 2.1 is the following result:

Proposition 2.2. *Let p_i,r_i and x_i for all $i \in \{1, ..., n\}$ be as in the statement of Theorem 2.1 with the additional assumption that $\mathcal{A}$ is commutative. Then*

$$\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\left(\sqrt{p_i}\left|x_i\right|\left|\sum_{j=1}^{n} p_j x_j\right|\right) \le 2\left|\sum_{j=1}^{n} p_j x_j\right|^2 + \sum_{i=1}^{n} p_i r_i^2.$$

Proof. If we multiply (2.5) by $p_i > 0$ and sum over i from 1 to n, we get

$$\sum_{i=1}^{n} p_i|x_i|^2 + \left|\sum_{j=1}^{n} p_j x_j\right|^2 \le 2\left|\sum_{j=1}^{n} p_j x_j\right|^2 + \sum_{i=1}^{n} p_i r_i^2.$$

We now use the fact that $2\left|a\right|\left|b\right| \le \left|a\right|^2 + \left|b\right|^2$. Thus,

$$\begin{aligned}\sum_{i=1}^{n} p_i|x_i|^2 + \sum_{i=1}^{n}\frac{1}{n}\left|\sum_{j=1}^{n} p_j x_j\right|^2 &= \sum_{i=1}^{n}\left(p_i|x_i|^2 + \frac{1}{n}\left|\sum_{j=1}^{n} p_j x_j\right|^2\right) \\ &\ge \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\left(\sqrt{p_i}\left|x_i\right|\left|\sum_{j=1}^{n} p_j x_j\right|\right)\end{aligned}$$

for all $i \in \{1, ..., n\}$. This is the same as saying that

$$2\left|\sum_{j=1}^{n} p_j x_j\right|^2 + \sum_{i=1}^{n} p_i r_i^2 \ge \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\left(\sqrt{p_i}\left|x_i\right|\left|\sum_{j=1}^{n} p_j x_j\right|\right).$$

□

3. The case of a unit vector

The following refinement of the Diaz-Metcalf result may be stated as well:

Theorem 3.1. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module. Suppose that $x_i \in \mathcal{X}$ for all $i \in \{1, \ldots, n\}$ satisfy the condition*

(3.1)
$$\left(\sum_{i=1}^{n} r_1\left|x_i\right|\right)^2 \le \left(\sum_{i=1}^{n}\operatorname{Re}\left\langle e, x_i\right\rangle\right)^2, \quad \left(\sum_{i=1}^{n} r_2\left|x_i\right|\right)^2 \le \left(\sum_{i=1}^{n}\operatorname{Im}\left\langle e, x_i\right\rangle\right)^2,$$

for each $i \in \{1, \ldots, n\}$, *where* e *be a unit vector in* $\mathcal{X}$ *and* r_1, r_2 *are positive elements in* C^**-algebra* $\mathcal{A}$. *Then*

$$\sqrt{r_1^2 + r_2^2} \sum_{i=1}^{n} |x_i| \leq \left| \sum_{i=1}^{n} x_i \right|. \tag{3.2}$$

Proof. We can simply exploit the Cauchy-Schwarz inequality and find the upper bound

$$\left| \left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right|^2 \leq \|e\|^2 \left| \sum_{i=1}^{n} x_i \right|^2 = \left| \sum_{i=1}^{n} x_i \right|^2. \tag{3.3}$$

We can rewrite the first term as

$$\begin{aligned} \left| \left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right|^2 &= \left| \sum_{i=1}^{n} \operatorname{Re} \langle e, x_i \rangle + i \left(\sum_{i=1}^{n} \operatorname{Im} \langle e, x_i \rangle \right) \right|^2 \\ &= \left(\sum_{i=1}^{n} \operatorname{Re} \langle e, x_i \rangle \right)^2 + \left(\sum_{i=1}^{n} \operatorname{Im} \langle e, x_i \rangle \right)^2. \end{aligned}$$

On the other hand, from (3.1) we infer that

$$r_1^2 \left(\sum_{i=1}^{n} |x_i| \right)^2 \leq \left(\sum_{i=1}^{n} \operatorname{Re} \langle e, x_i \rangle \right)^2$$

and

$$r_2^2 \left(\sum_{i=1}^{n} |x_i| \right)^2 \leq \left(\sum_{i=1}^{n} \operatorname{Im} \langle e, x_i \rangle \right)^2.$$

Adding these two inequalities to inequality (3.3), we deduce the desired inequality (3.2). □

Remark 3.1. If $\mathcal{A}$ is a commutative C^*-algebra, then we can replace conditions (3.1) with

$$0 \leq r_1 |x_i| \leq \operatorname{Re} \langle e, x_i \rangle, \ 0 \leq r_2 |x_i| \leq \operatorname{Im} \langle e, x_i \rangle.$$

We can apply Theorem 3.1 to derive some new operator inequalities. We only give the following such results. Notice that, if $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, then $\mathbb{B}(\mathcal{H})$ becomes a $\mathbb{B}(\mathcal{H})$-module if the inner product of elements $A, B \in \mathbb{B}(\mathcal{H})$ is defined by $\langle A, B \rangle = A^* B$.

Corollary 3.1. *Let* $A_i \in \mathbb{B}(\mathcal{H})$ *for all* $i \in \{1, \ldots, n\}$ *satisfy the condition*

$$0 \leq B_1 |A_i| \leq \operatorname{Re} A_i, \ 0 \leq B_2 |A_i| \leq \operatorname{Im} A_i,$$

for each $i \in \{1, \dots, n\}$ *and* B_1, B_2 *are positive operators in* $\mathbb{B}(\mathcal{H})$, *then*

$$\sqrt{B_1^2 + B_2^2} \sum_{i=1}^{n} |A_i| \le \left| \sum_{i=1}^{n} A_i \right|.$$

In particular, for $i \in \{1, 2\}$ *we have*

$$\sqrt{B_1^2 + B_2^2}\left(|A_1| + |A_2|\right) \le |A_1 + A_2|. \tag{3.4}$$

The following reverse of the generalized triangle inequality also holds. Before we proceed, we need the following lemma:

Lemma 3.1. *Let* $\mathcal{A}$ *be a* C^**-algebra and let* $a \in \mathcal{A}$.

(a) *If* a *is self adjoint, then* $a \le |a|$.

(b) *If* a *is normal, then* $|\mathrm{Re}\, a| \le |a|$.

Theorem 3.2. *Let* $\mathcal{X}$ *be a Hilbert* $\mathcal{A}$*-module, and* e *be a unit vector in* $\mathcal{X}$. *If* $\langle e, \sum_{i=1}^{n} x_i \rangle$ *and* r_i *are normal and positive elements in* $\mathcal{A}$ *for* $i \in \{1, ..., n\}$ *respectively, and* $x_i \in \mathcal{X}$ *for all* $i \in \{1, ..., n\}$, *such that*

$$|x_i| - \mathrm{Re}\,\langle e, x_i \rangle \le r_i, \tag{3.5}$$

for each $i \in \{1, ..., n\}$, *then*

$$\sum_{i=1}^{n} |x_i| - \left| \sum_{i=1}^{n} x_i \right| \le \sum_{i=1}^{n} r_i. \tag{3.6}$$

Proof. If we sum in (3.5) over i from 1 to n, then we get

$$\sum_{i=1}^{n} |x_i| \le \mathrm{Re} \left\langle e, \sum_{i=1}^{n} x_i \right\rangle + \sum_{i=1}^{n} r_i. \tag{3.7}$$

A little calculation shows that

$$\begin{aligned} \mathrm{Re} \left\langle e, \sum_{i=1}^{n} x_i \right\rangle &\le \left| \mathrm{Re} \left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right| && \text{(by Lemma 3.1 (a))} \\ &\le \left| \left\langle e, \sum_{i=1}^{n} x_i \right\rangle \right| && \text{(by Lemma 3.1 (b))} \\ &\le \|e\| \left| \sum_{i=1}^{n} x_i \right| && \text{(by Cauchy-Schwarz inequality)} \\ &= \left| \sum_{i=1}^{n} x_i \right|. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we get (3.6). □

Theorem 3.2 immediately yields:

Corollary 3.2. *If we consider $\mathcal{H}$ as a $\mathbb{C}$-module, then from* (3.6) *we can obtain the following reverse trangle inequality*

$$\sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\| \leq \sum_{i=1}^{n} r_i,$$

where r_i are positive elements in $\mathbb{R}$ for $\{1,\ldots,n\}$ (see [11] and also [9, Theorem 44]).

Remark 3.2. If $\mathcal{A}$ is a commutative C^*-algebra, then the assumption $\langle e, \sum_{i=1}^{n} x_i\rangle$ are normal is not necessary.

Another consequence of our discussion is the following.

Corollary 3.3. *Let $A_i \in \mathbb{B}(\mathcal{H})$, for each $i \in \{1,\ldots,n\}$ and $\sum_{i=1}^{n} A_i$ be normal. If B_i are positive operators in $\mathbb{B}(\mathcal{H})$ for all $i \in \{1,\ldots,n\}$ such that*

$$|A_i| - \operatorname{Re} A_i \leq B_i,$$

for each $i \in \{1,\ldots,n\}$, then

$$\sum_{i=1}^{n}|A_i| - \left|\sum_{i=1}^{n} A_i\right| \leq \sum_{i=1}^{n} B_i.$$

In particular, for $i \in \{1,2\}$ we have

$$|A_1| + |A_2| - |A_1 + A_2| \leq B_1 + B_2.$$

Now we present a useful lemma, which is applied in the next theorem.

Lemma 3.2. *Let $\mathcal{A}$ be a C^*-algebra and a, b in $\mathcal{A}$ be positive elements and $ab = ba$, then*

$$\sqrt{ab} \leq \frac{a+b}{2}. \tag{3.9}$$

The next theorem is known; see [9, Theorem 50]. The proof given here is different, and in the spirit of our discussion.

Theorem 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and let $e \in \mathcal{X}$ be such that $|e| = 1$ and $x_i \in \mathcal{X},\ i \in \{1,...,n\}$. If $M_i > m_i > 0$ for all $i \in \{1,...,n\}$, are such that*

$$\left|x_i - \frac{M_i + m_i}{2} e\right|^2 \leq (M_i + m_i)^2, \tag{3.10}$$

then

$$\sum_{i=1}^{n}|x_i| - \left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n}\frac{(M_i - m_i)^2}{M_i + m_i}.$$

Proof. It follows from left side of inequality (3.10) that

$$\left\langle x_i - \frac{M_i+m_i}{2}e, x_i - \frac{M_i+m_i}{2}e \right\rangle$$
$$= |x_i|^2 - (M_i+m_i)\operatorname{Re}\langle x_i, e\rangle + \left|\frac{M_i+m_i}{2}\right|^2.$$

Using the substitutions $a = |x_i|^2$ and $b = \left|\frac{M_i+m_i}{2}\right|^2$ in (3.9), this can be rewritten as

$$2\,|x_i|\left|\frac{M_i+m_i}{2}\right| \le |x_i|^2 + \left|\frac{M_i+m_i}{2}\right|^2$$

or, after rearranging terms,

$$|x_i| - \operatorname{Re}\langle x_i, e\rangle \le \frac{(M_i-m_i)^2}{M_i+m_i}.$$

Hence by Theorem 3.2 we obtain

$$\sum_{i=1}^n |x_i| - \left|\sum_{i=1}^n x_i\right| \le \sum_{i=1}^n \frac{(M_i-m_i)^2}{M_i+m_i}.$$

The validity of this inequality is just Theorem 3.3. □

Another result of this type is the following one:

Theorem 3.4. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and let $e \in \mathcal{X}$ be such that $|e| = 1$ and $x_i \in \mathcal{X}$, $i \in \{1, ..., n\}$. If $M_i \ge 0$ for all $i \in \{1, ..., n\}$, are such that*

$$\left|x_i - \frac{M_i}{2}e\right|^2 \le M_i^2, \tag{3.11}$$

then

$$\sum_{i=1}^n |x_i|^2 - \operatorname{Re}\left\langle \sum_{i=1}^n M_i x_i, e\right\rangle \le \frac{3}{4}\sum_{i=1}^n M_i^2. \tag{3.12}$$

Proof. A short calculation reveals that

$$\left\langle x_i - \frac{M_i}{2}e, x_i - \frac{M_i}{2}e\right\rangle = |x_i|^2 + \left|\frac{M_i}{2}\right|^2 |e| - 2\operatorname{Re}\left\langle x_i, \frac{M_i}{2}e\right\rangle. \tag{3.13}$$

According to (3.13) validity of (3.11) implies

$$|x_i|^2 + \left|\frac{M_i}{2}\right|^2 |e| - 2\operatorname{Re}\left\langle x_i, \frac{M_i}{2}e\right\rangle \le M_i^2$$

which on simplification reduces to

$$|x_i|^2 - \operatorname{Re}\langle M_i x_i, e\rangle \le \frac{3}{4}M_i^2.$$

Summing over all terms then yields (3.12). □

The following particular case is of interest:

Theorem 3.5. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $e_1, e_2, ..., e_n$ be a sequence of unit vectors in $\mathcal{X}$ such that $\langle e_i, e_j\rangle = 0$ for $i \neq j \leq n$, and let $x_i \in \mathcal{X}$ for all $i \in \{1, ..., n\}$, and p_i are positive elements in real number field such that $\sum_{i=1}^n p_i = 1$. If there exist constants positive elements r_i in $\mathcal{A}$ so that*

$$\left| x_i - \sum_{j=1}^n p_j \langle e_j, x_j\rangle e_j \right|^2 \leq r_i^2,$$

for all $i \in \{1, ..., n\}$, then

$$\sum_{i=1}^n p_i |x_i|^2 - \sum_{j=1}^n |p_j \langle e_j, x_j\rangle|^2 \leq \sum_{i=1}^n p_i r_i^2. \tag{3.14}$$

Proof. A straightforward computation shows that

$$\begin{aligned}
&\left\langle x_i - \sum_{j=1}^n p_j \langle e_j, x_j\rangle e_j, x_i - \sum_{j=1}^n p_j \langle e_j, x_j\rangle e_j \right\rangle \\
&= \langle x_i, x_i\rangle + \left\langle \sum_{i=1}^n p_i e_i \langle e_i, x_i\rangle, \sum_{j=1}^n p_j e_j \langle e_j, x_j\rangle \right\rangle - 2\sum_{j=1}^n |p_j \langle e_j, x_j\rangle|^2 \\
&= \langle x_i, x_i\rangle + \sum_{i=1}^n \sum_{j=1}^n p_i p_j \langle e_i, x_i\rangle^* \langle e_j, e_j\rangle \langle e_j, x_j\rangle - 2\sum_{j=1}^n |p_j \langle e_j, x_j\rangle|^2 \\
&= |x_i|^2 + \sum_{i=1}^n p_j^2 \langle e_j, x_j\rangle^* \langle e_j, e_j\rangle \langle e_j, x_j\rangle - 2\sum_{j=1}^n |p_j \langle e_j, x_j\rangle|^2 \\
&= |x_i|^2 + \sum_{j=1}^n p_j^2 \langle e_j, x_j\rangle^* \langle e_j, x_j\rangle - 2\sum_{j=1}^n |p_j \langle e_j, x_j\rangle|^2 \\
&= |x_i|^2 - \sum_{j=1}^n |p_j \langle e_j, x_j\rangle|^2.
\end{aligned}$$

Using this one can see that

$$|x_i|^2 - \sum_{j=1}^n |p_j \langle e_j, x_j\rangle|^2 \leq r_i^2. \tag{3.15}$$

If we multiply (3.15) by $p_i \geq 0$ and sum over i from 1 to n, we obtain

$$\sum_{i=1}^n p_i |x_i|^2 - \sum_{i=1}^n |p_i \langle e_i, x_i\rangle|^2 \leq \sum_{i=1}^n p_i r_i^2$$

which finishes the proof. □

Corollary 3.4. *With the substitution* $p_i = \frac{1}{n}$, $i \in \{1, ..., n\}$, (3.14) *becomes*

$$\sum_{i=1}^{n} |x_i|^2 - \frac{1}{n} \sum_{i=1}^{n} |\langle e_i, x_i \rangle|^2 \leq \sum_{i=1}^{n} r_i^2.$$

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