MULTIGROUPS AND MULTICOSETS

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Abstract. The present Cantor's set theory has limitations. In various ways, it cannot well represent realities because an element x, in Cantor's sense, is either in or not in X. Even, when $x \in X$, it can only occur once; no repetition is allowed. But so many real life problems are only well represented by sets which allow repetition(s), such as multiset. Such cases arise in, though not limited to, database query, chemical structures and computer proramming.

In this paper, we have some results on the algebraic structure of multisets and some properties of their multicosets.

Keywords: multisets, multigroups, submultiset, multicoset.

1. Introduction

So many real life problems are only well represented by sets which allow repetition(s), such as multiset. Such cases occur very frequently in chemical sciences and computer programming.

The notion can be traced to Dedekind[4]. In the recent time, Nazmul *et al* has put algebraic group structure on multisets[8]. Related algebraic properties as in the classical group can now be studied.

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In this paper, we have some results on the algebraic structure of multisets and some characterisations.

2. Preliminaries

In this paper, we shall use X to denote a non-empty set.

Definition 2.1. A multiset M drawn from a set X is denoted by the count function $C_M : X \longrightarrow \mathbb{N} \cup \{0\}$ defined by $C_M(x) = n \in \mathbb{N}$, the (multiplicity) or number of occurrence of x in M, where $\mathbb{N}$ is the set of positive integers.

Example 2.2. Let set $X = \{1, 2, 3, 4\}$. Then $M = \{1, 1, 1, 2, 2, 3, 3, 3\}$ is a multiset over X with $C_M(1) = 3$, $C_M(2) = 2$, $C_M(3) = 3$ and $C_M(4) = 0$.

Definition 2.3. Let multisets A and B be drawn from X. A is said to be a submultiset of B and is denoted $A \subseteq B$ if $C_A(x) \leq C_B(x)$.

Definition 2.4. The root set or support of a multiset M, which is denoted by M^*, is the set which contains the distinct elements in the multiset. Hence, M^* is the set of $x \in M$ such that $C_M(x) > 0$.

Definition 2.5. Consider a multiset M over a set X.

(1) It is simple if the cardinality of its root is 1;

(2) It is regular if $C_M(x) = C_M(y),\ \forall x, y \in M$;

(3) The peak element $x \in M$ is such that $C_M(x) \geq C_M(y),\ \forall y \in M$;

(4) The intersection of two multisets A and B is denoted by $C_A(x) \cap C_B(x) = \min\{C_A(x), C_B(x)\}$ and their union is denoted by

$$C_A(x) \cup C_B(x) = \max\{C_A(x), C_B(x)\};$$

(5) A and B are equal if and only if $C_A(x) = C_B(x)$.

$[X]^\alpha$ is the set of all the multisets whose elements have the multiplicity of not more than α. $MS(X)$ is the set of all multisets over X. An empty multiset ϕ is such that $C_\phi(x) = 0,\ \forall x \in X$. Cardinality of a multiset M is denoted by $|M| = \sum C_M(x),\ \forall x \in M$.

Definition 2.6. Let X be a group and $e \in X$ its identity. Then, $\forall x, y \in X$, a multiset M drawn from X is called a multigroup if

(1) $C_M(xy) \geq C_M(x) \wedge C_M(y)$;

(2) $C_M(x^{-1}) \geq C_M(x)$.

The implication of Definition 2.6 is that $C_M(x) = C_M((x^{-1})^{-1}) \geq C_M(x^{-1})$. Thus, $C_M(x) = C_M(x^{-1})$. So, $C_M(e) \geq C_M(x) \wedge C_M(x^{-1}) = C_M(x)$. We shall call $MG(X)$ the set of all multigroups over X.

Example 2.7. Let $G = \{e, a, b, ab\}$ be Klein 4 group. Then

(1) $A = \{e, e, e, a, a\}$ is a multigroup;

(2) $B = \{e, e, e, a, a, b, b\}$ is not a multigroup since $0 = C_B(ab) \ngeq C_B(a) \wedge C_B(c) = 2$.

Definition 2.8. A multigroup M over X is called abelian if $C_M(xy) = C_M(yx)$, $\forall x, y \in X$.

Definition 2.9. Let $A, B \in MS(X)$.

(1) $A \circ B$ is a multiset associated with

$$C_{A \circ B}(x) = \bigvee \{C_A(y) \wedge C_B(z) : y, z \in X, x = yz\};$$

(2) A^{-1} is a multiset associated with

$$C_{A^{-1}}(x) = C_A(x^{-1}),\ \forall x \in X;$$

(3) $A_n = \{x : C_A(x) \geq n\}$;

(4) $[n]_x$ is a multiset containing only x in n times;

(5) The complement of the multiset $M \in [X]^\alpha$ denoted by M' is such that

$$C_{M'}(x) = \alpha - C_M(x);$$

(6) $nA = \{x^n : x \in A\}$, where n is the multiplicity of each element that appears from A.

Proposition 2.10 ([8])**.** *Let $A, B \in MS(X)$ and $m, n \in I\!N$.*

(1) *If $A \subseteq B$, then $A_n \subseteq B_n$;*

(2) *If $m \leq n$, then $A_m \supseteq A_n$;*

(3) *$(A \cap B)_n = A_n \cap B_n$;*

(4) *$(A \cup B)_n = A_n \cup B_n$;*

(5) *$A = B$ if and only if $A_n = B_n$, $\forall n \in I\!N$.*

Proposition 2.11 ([8])**.** *Let X be a group and $A \in MG(X)$. Then, A_n, for $n \in I\!N$, is a subgroup of X.*

Proposition 2.12 ([8]). *Let X be a group and $A \in MS(X)$. Then, $A \in MG(X)$ if and only if the following conditions are satisfied:*

(1) $A \circ A \subseteq A$ *and* $A^{-1} \subseteq A$ (*or* $A \subseteq A^{-1}$ *or* $A^{-1} = A$) *or*

(2) $A \circ A^{-1} \subseteq A$.

Remark 2.13. It has been established in [8] that the intersection of multigroups is again a multigroup. It was also illustrated that the union of multigroups needs not be a multigroup. It is later shown in this paper the condition necessary and sufficient for the union of multigroups to be a multigroup.

We shall state the following proposition by Nazmul et al in [8] and make some comments on its inadequacy. Later in this work, we shall state the correct form.

Proposition 2.14 ([8]). *Let $A \in MS(X)$.Then the following are equivalent:*

(1) $C_A(xy) = C_A(yx)$;

(2) $C_A(xyx^{-1}) = C_A(y)$;

(3) $C_A(xyx^{-1}) \geq C_A(y)$;

(4) $C_A(xyx^{-1}) \leq C_A(y)$.

Remark 2.15. Note that with Proposition 2.14(2), properties (3) and (4) are very trivial. So it is only necessary to state (1) and (2) and devise a better proof.

Definition 2.16 ([8]). Let X be a group and e its identity. Also, let $H \in MG(X)$ and $x \in X$.

(1) $[C_H(e)]_x \circ H$ is called a left multicoset of H in X denoted by xH;

(2) $H \circ [C_H(e)]_x$ is called a right multicoset of H in X denoted by Hx.

Proposition 2.17 ([8]). *Let $A \in MS(X)$. The following assertions are equivalent.*

(1) $C_A(xy) = C_A(yx)$, $\forall x, y \in X$;

(2) $A \circ B = B \circ A$, $\forall A, B \in MS(X)$.

Proposition 2.18 ([8]). *Let $H \in AMG(X)$ and define $X/H = \{xH : x \in X\}$. Then the following assertions hold:*

(1) $(xH) \circ (yH) = (xy)H$, $\forall x, y \in X$;

(2) *If* $xH = x_1H$ *and* $yH = y_1H$ *then* $(xy)H = (x_1y_1)H$;

(3) $(X/H, \circ)$ *is a group;*

(4) $X/H \simeq X/H^*$.

Proposition 2.19 ([8]). *Let* $H \in AMG(X)$. *If* $xH = yH$, *then* $C_H(x) = C_H(y)$, $\forall x, y \in X$.

Remark 2.20. It will be shown later that the results in Propositions 3.25 and 3.33 of [8] fail.

3. Some properties of multigroup and abelian multigroup

We now begin to introduce some new results in the following sections.

Proposition 3.1. *Let* $A \in MS(X)$. *Then,* $A \in MG(X)$ *if and only if* $A \circ A^{-1} = A$.

Proof. Let $A \in MG(X)$. Then for any $x, y \in X$,

$$C_A(yz) \geq \{C_A(y) \wedge C_A(z)\} \Rightarrow C_A(yz) = \bigvee\{C_A(y) \wedge C_A(z)\}.$$

Note that

$$\begin{aligned} C_{A \circ A^{-1}}(x) &= \bigvee\{C_A(y) \wedge C_{A^{-1}}(z) : x = yz\} \\ &= \bigvee\{C_A(y) \wedge C_A(z^{-1}) : x = yz\} \\ &= \bigvee\{C_A(y) \wedge C_A(z) : x = yz\} \\ &= C_A(yz) \\ &= C_A(x). \end{aligned}$$

Conversely, let $A = A \circ A^{-1}$.

$$\begin{aligned} C_A(xy^{-1}) &= C_{A \circ A^{-1}}(xy^{-1}) \\ &= \bigvee\{C_A(x) \wedge C_{A^{-1}}(y^{-1})\} \\ &= \bigvee\{C_A(x) \wedge C_A((y^{-1}))^{-1}\} \\ &= \bigvee\{C_A(x) \wedge C_A((y)\}. \end{aligned}$$

But

$$\bigvee\{C_A(x) \wedge C_A((y)\} \geq \{C_A(x) \wedge C_A((y)\}.$$

□

Nazmul *et al* in [8] have shown that the intersection of multigroups is also a multigroup and have also illustrated with an example that the union of a multigroup is not a multigroup. In what follows, the condition necessary for the union of two multigroups to be a multigroup is stated and proved.

Proposition 3.2. *Let* $A, B \in MG(X)$*, Then* $A \cup B \in MG(X)$ *if* $C_A(a) = C_A(b) = C_A(ab)$ *or* $C_B(a) = C_B(b) = C_B(ab),\ \ \forall a \in A, b \in B$.

Proof.

$$\begin{aligned} C_{A\cup B}(ab) &= C_A(ab) \vee C_B(ab) \\ &\geq [(C_A(a) \wedge C_A(b)) \vee (C_B(a) \wedge C_B(b))] \\ &= C_A(a) \vee (C_B(a) \wedge C_B(b)) \\ &= C_{A\cup B}(a) \wedge C_{A\cup B}(b). \end{aligned}$$

□

In what follows, a revision of Proposition 3.24 of [8] is made as earlier remarked in Remark 2.15.

Proposition 3.3. *Let* $A \in MG(X)$*. Then* $C_A(xy) = C_A(yx)$ *if and only if* $C_A(xyx^{-1}) = C_A(y)$.

Proof. Assume $C_A(xy) = C_A(yx)$. Then

$$C_A(xyx^{-1}) = C_A(x(yx^{-1})) = C_A((x^{-1})xy) = C_A(y).$$

Also, assume $C_A(xyx^{-1}){=}C_A(y)$. Then $C_A(yx){=}C_A(x(yx)x^{-1}){=}C_A(xy)$. □

Recall the definition of an *abelian multigroup* from Definition 2.8. Aslo, Example 3.27 in [8] has shown that every multigroup over an abelian group X is an abelian multigroup. The following example shows that the group X needs not be an abelian for it to have an abelian multigroup over it.

Example 3.4. Let $X = S_3 = \{e, (12), (13), (23), (123), (132)\}$, where e is the identity of X.

$$A = \{e, e, e, (12), (12), (13), (13), (23), (23), (123), (123), (132), (132)\}$$

is a multigroup. Indeed, it is an abelian multigroup since

$$C_A((12)(13)) = C_A((123)) = C_A((132)) = C_A((13)(12)).$$

But

$$(123) = (12)(13) \neq (13)(12) = (132).$$

4. Some properties of multicosets

In this section, X is a group, e its identity and $H \in MG(X)$. Also, an alternative and a rather easier approach is provided to the concept of multicoset. Note that the multiset $[C_H(e)]_x$ is a *simple* multiset in which x is counted as much as e is counted in $H \in MG(X)$. Recalling the Definition 2.9 of $A \circ B$, the left multicoset $[C_H(e)]_x \circ H$ can be worked out, though a bit tedious than the alternative presented in this section.

Definition 4.1. Let $(X, *)$ be a group and $H = \{y_1, y_2, y_3 \cdots, y_n\} \in MG(X)$. Then

$$x * H = \{x * y_1, x * y_2, x * y_3, \cdots, x * y_n\}$$

is called the left multicoset $[C_H(e)]_x \circ H$.

Example 4.2. Consider the multiplicative group of units. Let $H = \{1, 1, 1, -1, -1, i, i, -i, -i\}$. $[C_H(e)]_i = \{i, i, i\}$ and $[C_H(e)]_i \circ H = x * H = \{1, 1, -1, -1, -i, -i, i, i, i\}$. Obviously, $x * H$ needs not be a multigroup.

Remark 4.3. Note that

$$C_{xH}(x) = C_H(e) = C_H(x^{-1}x).$$

Then,

$$C_{xH}(y) = C_H(x^{-1}y).$$

We simply use xH for $x * H$ henceforth except otherwise is necessary.

Proposition 4.4. *$H = yH$ if and only if $C_{xH}(x) = C_H(y)$, $\forall x, y \in X$.*

Proof. Assume that $H = yH$. Then

$$C_{xH}(x) = C_H(e) = C_H(y^{-1}y) = C_{yH}(y) = C_H(y).$$

Conversely, assume that $C_{xH}(x) = C_H(y)$. Then

$$C_H(y) = C_{xH}(x) = C_H(e) = C_H(y^{-1}y) = C_{yH}(y).$$

□

Example 4.5. Let $X = S_3$. If $y = (12)$ and

$$H = \{e, e, e, (12), (12), (12), (13), (13), (123), (123)\},$$

it is obvious that $H = yH$ and Proposition 4.4 can be verified, noting that the choice of x and y can be varied.

We recall that Proposition 3.32 of [8] (herein as Proposition 2.19) requires that H should be an abelian multigroup and that $xH = yH$ for it to be regular. But the following result shows that H needs not be abelian.

Proposition 4.6. *$xH = yH$ if and only if $C_H(x^{-1}y) = C_{xH}(x)$, $\forall x, y \in X$, in which case, xH, and indeed H, is regular.*

Proof. Assume that $xH = yH$. Then, $H = x^{-1}yH$ and let $z = x^{-1}y$. $C_{xH}(x) = C_H(z) = C_H(x^{-1}y)$ by Proposition 4.4. Hence, $C_{xH}(x) = C_{xH}(y)$. Thus, both xH and H are regular.

Conversely, assume $C_{xH}(x) = C_H(x^{-1}y)$, $\forall x, y \in X$. $C_{yH}(y) = C_H(e) = C_{xH}(x) = C_H(x^{-1}y) = C_{xH}(y)$. □

Remark 4.7. With the foregoing properties, relation can be defined on the elements of multigroups over x.

Proposition 4.8. *If we define $x \sim y$ as $C_H(x^{-1}y) = C_{xH}(x)$, for any $x, y \in X$, then $\sim$ is an equivalence relation on X.*

Proof. (1) *Reflexivity:* $x \sim x$ since $C_H(x^{-1}x) = C_H(e) = C_{xH}(x)$.

(2) *Symmetricity:* Let $x \sim y$. Then, $C_H(x^{-1}y) = C_{xH}(x) = C_H(e) = C_H(y^{-1}y)C_{yH}(y) = C_H(y^{-1}x)$. Thus, $y \sim x$.

(3) *Transitivity:* Let $x \sim y$ and $y \sim z$. Then, $C_H(x^{-1}y) = C_{xH}(x)$ and $C_H(y^{-1}z) = C_{yH}(y)$. Let $x^{-1}y = h_1, y^{-1}z = h_2 \in H$, then, $y = xh_1, z = yh_2 = xh_1h_2 = xh \in xH$. Thus, $x^{-1}z = h \in H$. Hence, $C_H(x^{-1}z) = C_{xH}(x) \Rightarrow x \sim z$. □

The following result shows that multicosets are invariant for regular multigroup H over a group X the support of H is X.

Proposition 4.9. *Let $H \in MG(X)$ be regular such that $H^* = X$. Then, for any $x, y \in X$, the following are equivalent:*

(1) *$xH = H = Hx$;*

(2) *$xH = Hy$;*

(3) *$xHy = xyH = yxH = H$.*

Proof. (1) $\Rightarrow$ (2): $C_{xH}(x) = C_H(x) = C_{yH}(x)$.

(2) $\Rightarrow$ (3): Since $x, y \in H, xy, yx \in H$. By (2), $xyH = yxH$ and by (1), $xyH = H = yxH$. $C_{xHy}(x) = C_H(x^{-1}xy^{-1}) = C_H(y^{-1}) = C_{yH}(e) = C_H(e) = C_{xH}(x) = C_H(x)$.

(3) $\Rightarrow$ (1): Since $xyH = yxH = H$, $xH = H$. $C_{xH}(x) = C_H(e) = C_H(xx^{-1}) = C_{xH}(x) = C_{Hx}(x)$, where e is the identity of X. □

Remark 4.10. If $xH = Hx$, then H is normal.

Proposition 4.11. *Every regular multigroup is abelian but the converse is not true in general.*

Proof. Let H be regular multiset over a group X. Since x and y are arbitrary in H, if $xy = z \in H$ so is $yx = w \in H$. Then, $C_H(xy) = C_H(z) = C_H(w) = C_H(yx)$. H is abelian. But if $H = \{e, e, e, (132), (132), (123), (123)\}$, where e is the identity of X, it is abelian but $C_H(x) \neq C_H(y)$, $\forall x, y \in H$. □

Proposition 4.12. *Let $H \in MG(X)$ and $X = \{x_i\}_{i=1}^{k}$. Then,*

$$\cup x_iH = \cup n\{x_i\},\ \forall x_i \in X,$$

where $n = C_H(e)$, e is the identity of X and $|\cup x_iH| = n|X|$. Besides, $\cup x_iH$ is regular.

Proof. $\forall x \in X, C_{\cup x_i H}(x) = \vee C_{x_i H}(x) = C_H(e) = n = C_{n\{x\}}(x) = \vee C_{n\{x_i\}}(x) = C_{\cup n\{x_i\}}(x)$. Note that since $C_{\cup x_i H}(x) = n \ \forall x \in X$, $\cup x_i H$ is regular. Furthermore,

$$|\cup x_i H| = \sum_{j=1}^{k} C_{\cup x_i H}(z_j) = \sum_{j=1}^{k} \vee C_{\cup x_i H}(w_i) = C_H(e)|X| = n|X|.$$

□

Example 4.13. Let $X = S_3$ and $H = \{e, e, (12), (12)\} = eH$, where e is the identity of X.

$$(13)H = \{(13), (13), (132), (132)\};$$

$$(23)H = \{(23), (23), (123), (123)\};$$

$$(12)H = \{e, e, (12), (12)\};$$

$$(132)H = \{(132), (132), (13), (13)\}$$

and $(123)H = \{(123), (123), (23), (23)\}$. Then,

$$\cup xH = \{e, e, (12), (12), (13), (13), (23), (23), (132), (132), (123), (123)\}.$$

In classical group theory, $xH \neq yH \Rightarrow xH \cap yH = \emptyset$, but, in multigroup theory, $xH \neq yH \nRightarrow xH \cap yH = \emptyset$ but rather implies that $C_H(x) \neq C_H(y)$. In what follows, we show by means of counter examples that the result of Proposition 3.25 of [8] fails. Subsequently, we also show that the result of Proposition 3.33 of [8] fails, since its proof is based on the former.

Example 4.14. Consider a multiset

$$A = \{e, e, (12), (12), (12), (13), (13), (123), (123), (132), (132)\}.$$

This is such that $C_A(xy) = C_A(yx)$. Then Consider a multiset $B = \{(13), (13)\}$. $A \circ B = \{e, e, (13), (13), (123), (123)\}$ and $B \circ A = \{e, e, (13), (13), (132), (132)\}$. Thus, $A \circ B \neq B \circ A$ as claimed by [8].

It can be observed that the result can be true if X is an abelian group. Alternatively, if X is not an abelian group, $\forall b \in B$, $C_A(b) \neq 0$.

Example 4.15. Let $H = \{e, e, (12), (12)\}$ as in Example 4.13. This is regular and also abelian. But $\{(12), (12), (23), (23), (123), (123)\} = (13)H \circ (23)H \neq [(13)(23)]H = (123)H = \{(23), (23), (123), (123)\}$. Thus, $(xH) \circ (yH) \neq (xy)H$ as claimed by [8]. Furthermore, $(13)H = (132)H$ and $(23)H = (123)H$ but $(13)(23)H = (123)H \neq eH = (132)(123)H$. Also, $(X/H, \circ)$ is not a group since

$$(13)H \circ (23)H = \{(12), (12), (23), (23), (123), (123)\} \notin \{xH : x \in X\}.$$

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