THE CONDITIONS OF EXISTENCE OF A SOLUTION OF THE TWO-POINT IN TIME PROBLEM FOR NONHOMOGENEOUS PDE

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Abstract. The existence of a solution of the problem with local homogeneous two-point in time conditions for nonhomogeneous PDE of the second order in time and generally infinite order in spatial variables was investigated in the classes of entire functions. The case when the characteristic determinant of the problem is identically zero was studied. We proposed the differential-symbol method of constructing the solution of the problem.

Keywords: characteristic determinant of a problem, two-point local conditions, differential-symbol method.

1. Introduction

The problems with n-point in time conditions ($n \in \mathbb{N}\backslash\{1\}$) for PDE are generalization of multipoint problems for ODE which are known in the literature as the Vallee-Poussin problems [1]. The multipoint problems for PDE are ill-posed and their solvability is connected with problem of small denominators (below estimation of the so-called characteristic determinant). Papers (see [2, 3, 4] and bibliography in them) are devoted to research of the multipoint in time problems in the bounded domains based on the metric approach.

The spaces of functions which allow exponential growth as the classes of unique solvability of the multipoint problem for PDE in unbounded layer are studied in the papers [5, 6, 7].

The differential-symbol method of solving the problem with initial and two-point in time conditions for PDE is proposed in the works [8, 9, 10]. In these papers, classes of entire functions and classes of quasipolynomials as the classes of unique solvability of the problems are dedicated.

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The work [11], in particular, is devoted to constructing the polynomial solutions of the system of PDE with constant coefficients.

This paper is the continuation of researches [12, 13, 14, 15]. It is devoted to research of existence of solutions of the problem with local homogeneous two-point in time conditions for nonhomogeneous PDE of the second order with respect to time variables when the characteristic determinant is identically zero.

2. Problem statement

In the domain $(t,x) \in \mathbb{R}^{1+s}$, $x = (x_1, \dots, x_s)$, $s \in \mathbb{N}$, we investigate a solvability of the problem

$$L\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big)U(t,x) \equiv \frac{\partial^2 U}{\partial t^2} + 2\,a\Big(\frac{\partial}{\partial x}\Big)\frac{\partial U}{\partial t} + b\Big(\frac{\partial}{\partial x}\Big)U = f(t,x), \tag{2.1}$$

$$\begin{aligned} &l_{0\partial}U(t,x) \equiv A_1\Big(\frac{\partial}{\partial x}\Big)U(0,x) + A_2\Big(\frac{\partial}{\partial x}\Big)\frac{\partial U}{\partial t}(0,x) = 0, \\ &l_{1\partial}U(t,x) \equiv B_1\Big(\frac{\partial}{\partial x}\Big)U(h,x) + B_2\Big(\frac{\partial}{\partial x}\Big)\frac{\partial U}{\partial t}(h,x) = 0, \quad h > 0. \end{aligned} \tag{2.2}$$

In equation (2.1) $f(t,x)$ is given nonzero function, $a\big(\frac{\partial}{\partial x}\big)$ and $b\big(\frac{\partial}{\partial x}\big)$ are the following differential expressions

$$a\Big(\frac{\partial}{\partial x}\Big) = \sum_{|k|=0}^{\infty} a_k \frac{\partial^k}{\partial x^k}, \quad b\Big(\frac{\partial}{\partial x}\Big) = \sum_{|k|=0}^{\infty} b_k \frac{\partial^k}{\partial x^k},$$

where $a_k, b_k \in \mathbb{C}$, $k = (k_1, \dots, k_s) \in \mathbb{Z}_+^s$, $|k| = k_1 + \ldots + k_s$, $\frac{\partial^k}{\partial x^k} = \frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_s^{k_s}}$, moreover their symbols $a(\nu)$ and $b(\nu)$ are entire functions (in particular, they can be polynomials), $\nu = (\nu_1, \dots, \nu_s) \in \mathbb{C}^s$.

The differential polynomials with complex coefficients $A_1\left(\frac{\partial}{\partial x}\right)$, $A_2\left(\frac{\partial}{\partial x}\right)$, $B_1\left(\frac{\partial}{\partial x}\right)$ and $B_2\left(\frac{\partial}{\partial x}\right)$ in local two-point conditions (2.2) are presented, moreover the corresponding symbols $A_1(\nu)$, $A_2(\nu)$, $B_1(\nu)$ and $B_2(\nu)$ for each $\nu \in \mathbb{C}^s$ satisfy the inequality

$$\big(\,|A_1(\nu)|^2 + |A_2(\nu)|^2\,\big)\big(\,|B_1(\nu)|^2 + |B_2(\nu)|^2\,\big) \neq 0.$$

The solution of problem (2.1), (2.2) is understood as entire function of the following form

$$U(t,x) = \sum_{\widetilde{k} \in \mathbb{Z}_+^{1+s}} u_{\widetilde{k}}\, t^{k_0} x^k, \quad \widetilde{k} = (k_0, k), \quad u_{\widetilde{k}} \in \mathbb{C},$$

of variables t and x, which satisfy equation (2.1) in $\mathbb{R}^{1+s}$ and conditions (2.2) in $\mathbb{R}^s$.

For ODE

$$L\Big(\frac{d}{dt},\nu\Big)T(t,\nu)=0,\quad \nu\in\mathbb{C}^s,$$

we consider the fundamental system of solutions $\big\{T_0\,(t,\nu)\,,\,T_1\,(t,\nu)\,\big\}$ normal at the point $t=0$ and write the determinant:

$$\Delta\,(\nu)=\begin{vmatrix} l_{0\nu}T_0(t,\nu) & l_{0\nu}T_1(t,\nu) \\ l_{1\nu}T_0(t,\nu) & l_{1\nu}T_1(t,\nu) \end{vmatrix}, \tag{2.3}$$

where

$$l_{0\nu}T_0(t,\nu)\equiv A_1(\nu),\qquad l_{0\nu}T_1(t,\nu)\equiv A_2(\nu),$$

$$l_{1\nu}T_j(t,\nu)\equiv B_1\,(\nu)\,T_j(h,\nu)+B_2\,(\nu)\,\frac{dT_j}{dt}(h,\nu),\quad j\in\{0,1\}.$$

The determinant $\Delta\,(\nu)$ is the characteristic determinant of problem (2.1), (2.2).

Let's establish the solvability of problem (2.1), (2.2) in the class of entire functions when characteristic determinant (2.3) of the problem is identically zero.

3. The conditions of existence of solution of the problem

Since $a(\nu)$ and $b(\nu)$ are entire functions then [16] the functions $T_0(t,\nu)$ and $T_1(t,\nu)$ are entire functions in vector-parameter $\nu\in\mathbb{C}^s$ for all $t\in\mathbb{R}$. So the function $\Delta(\nu)$ (as superposition of entire functions) is entire function too.

Let's consider the function

$$\Phi\,(t,\lambda,\nu)=\frac{e^{\lambda t}-T_0\,(t,\nu)-\lambda T_1\,(t,\nu)}{L\,(\lambda,\nu)}, \tag{3.1}$$

which is the solution of Cauchy problem

$$L\left(\frac{d}{dt},\nu\right)\Phi=e^{\lambda t},\quad \Phi\,(0,\lambda,\nu)=0,\quad \left.\frac{\partial\Phi\,(t,\lambda,\nu)}{\partial t}\right|_{t=0}=0.$$

The function (3.1) is the quasipolynomial of variable t, besides $\Phi\,(t,\lambda,\nu)$ is entire function of the first order in parameter λ and entire function of order $\bar p$ in the set of parameters $\nu_1,\ldots,\nu_s$. Here $\bar p=\max\{p_a,\ p_b/2\}$, where p_a and p_b are degrees of $a\,(\nu)$ and $b\,(\nu)$ accordingly if $a\,(\nu)$ and $b\,(\nu)$ are polynomials, and $\bar p=\infty$ if $a\,(\nu)$ or $b\,(\nu)$ is not polynomial.

We introduce some classes of entire functions. These classes depend on value p where $p=\max\{\bar p,\,1\}\in[1;+\infty]$.

$A_{p'}$ is the class of entire functions $\varphi(x)$ the order of which is less than p', where $1/p+1/p'=1$, if $1<p<+\infty$;

$A_{p'}=A_1$ is the class of entire functions $\varphi(x)$ of exponential type if $p=\infty$;

$A_{p'}=A_\infty$ is the class of entire functions $\varphi(x)$ if $p=1$.

By $\mathbb{A}_{p'}$, denote the class of entire functions $U(t,x)$ which for each fixed $t \in \mathbb{R}$ belong to $A_{p'}$.

Let's show that in the case $\Delta(\nu) \equiv 0$ in $\mathbb{C}^s$ the solution of problem (2.1), (2.2) exists under some conditions on the function $f(t,x)$ and it can be found by the formula

$$U(t,x) = f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}\Phi(t,\lambda,\nu)\right\}\Big|_{\lambda=0,\,\nu=O}, \tag{3.2}$$

in which $\nu \cdot x = \nu_1 x_1 + \ldots + \nu_s x_s$.

We consider the function $\Phi_1(\lambda,\nu) = l_{1\nu}\Phi(t,\lambda,\nu)$, where

$$l_{1\nu}\Phi(t,\lambda,\nu) \equiv B_1(\nu)\Phi(h,\lambda,\nu) + B_2(\nu)\frac{\partial\Phi}{\partial t}(h,\lambda,\nu).$$

Theorem 3.1. *Let for two-point problem* (2.1), (2.2), *in which* $\Delta(\nu) \equiv 0$ *in* $\mathbb{C}^s$ *the following condition is satisfied:*

for all $x \in \mathbb{R}^s$ *and* $f \in \mathbb{A}_{p'}$ *the identity*

$$f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}\Phi_1(\lambda,\nu)\right\}\Big|_{\lambda=0,\,\nu=O} \equiv 0 \tag{3.3}$$

is fulfilled.

Then the solution of problem (2.1), (2.2) *in the class* $\mathbb{A}_{p'}$ *exists and it can be obtained by formula* (3.2).

Proof. First we note that the result of action of the differential expression $f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)$ onto the function $e^{\nu\cdot x}\Phi(t,\lambda,\nu)$ in formula (3.2) is entire function of the first order in λ and of the order p in the set of variables $\nu_1, \ldots, \nu_s$.

Further we define the differential expression $f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)$ for entire function $f(t,x)$ of the class $\mathbb{A}_{\overline{p}'}$ by differential expression of infinite order by replacing in the Maclaurin expansion of the function $f(t,x)$ the variables t and the vector-parameter x by $\frac{\partial}{\partial\lambda}$ and $\frac{\partial}{\partial\nu}$ accordingly. Then the expression in the right side of formula (3.2) is the series that defines after setting $\lambda = 0$ and $\nu = O$ entire function $U(t,x)$ which belongs to the class $A_{p'}$ for each fixed t [17], i. e. $U(t,x) \in \mathbb{A}_{p'}$.

Let's prove that the function (3.2) satisfy the equation (2.1):

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)U(t,x) = f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}L\left(\frac{d}{dt},\nu\right)\Phi(t,\lambda,\nu)\right\}\Big|_{\lambda=0,\,\nu=O}$$
$$= f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\lambda t+\nu\cdot x}\right\}\Big|_{\lambda=0,\,\nu=O} = f(t,x).$$

In addition, from the condition $\Phi(0,\lambda,\nu) = \dfrac{\partial\Phi}{\partial t}(0,\lambda,\nu) = 0$ we get

$$\begin{aligned} l_{0\partial}U(t,x) &= f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left(A_1\left(\frac{\partial}{\partial x}\right)\left\{\Phi(0,\lambda,\nu)\,e^{\nu\cdot x}\right\}\right)\Big|_{\lambda=0,\nu=O} \\ &\quad + f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left(A_2\left(\frac{\partial}{\partial x}\right)\left\{\frac{\partial\Phi}{\partial t}(0,\lambda,\nu)\,e^{\nu\cdot x}\right\}\right)\Big|_{\lambda=0,\nu=O} \equiv 0. \end{aligned}$$

Since the identity (3.3) is fulfilled, we show that function (3.2) satisfy the second condition in (2.2):

$$\begin{aligned} l_{1\partial}U(t,x) &= f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\Big\{B_1\Big(\frac{\partial}{\partial x}\Big)\Big\{\Phi(h,\lambda,\nu)\,e^{\nu\cdot x}\Big\}\Big\}\Big|_{\lambda=0,\,\nu=O} \\ &\quad + f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\Big\{B_2\Big(\frac{\partial}{\partial x}\Big)\Big\{\frac{\partial\Phi}{\partial t}(h,\lambda,\nu)\,e^{\nu\cdot x}\Big\}\Big\}\Big|_{\lambda=0,\,\nu=O} \\ &= f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\Big\{B_1(\nu)\Big\{\Phi(h,\lambda,\nu)\,e^{\nu\cdot x}\Big\}\Big\}\Big|_{\lambda=0,\,\nu=O} \\ &\quad + f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\Big\{B_2(\nu)\Big\{\frac{\partial\Phi}{\partial t}(h,\lambda,\nu)\,e^{\nu\cdot x}\Big\}\Big\}\Big|_{\lambda=0,\,\nu=O} \\ &= f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\Big\{e^{\nu\cdot x}\Phi_1(\lambda,\nu)\Big\}\Big|_{\lambda=0,\,\nu=O} \equiv 0. \end{aligned}$$

The theorem is proved. □

Remark 3.2. Solution (3.2) of problem (2.1), (2.2) in the class $\mathbb{A}_{p'}$ is nonunique, because null-space of the problem in the same class is nontrivial [12].

4. Examples

Let's establish the conditions of solvability of two-point problem (2.1), (2.2) for the specific examples.

Example 4.1. In the domain $(t,x)\in\mathbb{R}^2$ we investigate the problem of finding the solutions of the equation

$$\left[\frac{\partial^2}{\partial t^2} + 2\frac{\partial^2}{\partial t\partial x} + 1 + \frac{\partial^2}{\partial x^2}\right]U(t,x) = f(t,x), \tag{4.1}$$

that satisfy local two-point conditions

$$\frac{\partial U}{\partial x}(0,x) + \frac{\partial U}{\partial t}(0,x) = 0, \quad \frac{\partial U}{\partial x}(\pi,x) + \frac{\partial U}{\partial t}(\pi,x) = 0. \tag{4.2}$$

♦ This problem is the problem (2.1), (2.2), in which $a(\nu)=\nu$, $b(\nu)=1+\nu^2$, $h=\pi$, $A_1(\nu)=B_1(\nu)=\nu$, $A_2(\nu)=B_2(\nu)=1$, $\bar{p}=p=1$.

The fundamental system of solutions of ODE

$$\left[\frac{d^2}{dt^2} + 2\nu\frac{d}{dt} + 1 + \nu^2\right] T\left(t,\nu\right) = 0$$

normal at the point $t = 0$ has the form

$$T_0\left(t,\nu\right) = e^{-\nu t}\left[\nu \sin t + \cos t\right], \quad T_1\left(t,\nu\right) = e^{-\nu t}\sin t.$$

The characteristic determinant of problem (4.1), (4.2) yields

$$\Delta\left(\nu\right) = \begin{vmatrix} \nu & 1 \\ -\nu e^{-\pi\nu} & -e^{-\pi\nu} \end{vmatrix} \equiv 0.$$

The condition of existence (3.3) of solution of problem (4.1), (4.2) according to Theorem 3.1 is following:

for $f \in \mathbb{A}_\infty$ such identity

$$f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{-\nu\pi}\left(\lambda+\nu\right)\frac{e^{(\lambda+\nu)\pi}+1}{\left(\lambda+\nu\right)^2+1}e^{\nu x}\right\}\Bigg|_{\lambda=\nu=0} \equiv 0$$

holds in $\mathbb{R}$.

For example, this identity is satisfied for the function of form $f\left(t,x\right) = e^{x-t}$. The solution of problem (4.1), (4.2) for this function can be found by formula (3.2):

$$U\left(t,x\right) = f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\nu x}\Phi\left(t,\lambda,\nu\right)\right\}\Big|_{\lambda=\nu=0} = \left\{e^{\nu x}\Phi\left(t,\lambda,\nu\right)\right\}\Big|_{\lambda=-1,\,\nu=1}$$
$$= \frac{e^{-t} - T_0\left(t,1\right) + T_1\left(t,1\right)}{L\left(-1,1\right)}e^{x} = e^{x-t} - e^{x-t}\cos t.$$

Note that obtained solution of problem (4.1), (4.2) is nonunique. For example, the solution of problem (4.1), (4.2) is function of the form $U\left(t,x\right) = e^{x-t}$, and it is also the sum of this function with arbitrary elements of null-space of the problem. Let's note that elements of the null-space of problem (4.1), (4.2) have the form

$$U\left(t,x\right) = \varphi\left(x-t\right)\cos t,$$

where φ is arbitrary twice continuously differentiable function in $\mathbb{R}$. □

Example 4.2. Let's investigate the existence conditions of the solution of the two-point problem in domain $t \in \mathbb{R}$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ for nonhomogeneous differential-functional equation

$$\frac{\partial^2}{\partial t^2}U\left(t,x\right) + 2\frac{\partial}{\partial t}U\left(t,x+\omega\right) + 2U\left(t,x+\omega\right) - U\left(t,x\right) = f\left(t,x\right) \tag{4.3}$$

with homogeneous local conditions

$$U(0,x) + \frac{\partial U}{\partial t}(0,x) = 0, \;\; U(1,x) + \frac{\partial U}{\partial t}(1,x) = 0, \tag{4.4}$$

where $\omega = (1,1,-1)$ is the displacement vector in spatial coordinates.

♦ Differential-functional equation (4.3) we can write as the differential equation of infinite order

$$\left[\frac{\partial^2}{\partial t^2} + 2\, e^{\omega\cdot\frac{\partial}{\partial x}}\frac{\partial}{\partial t} + 2\, e^{\omega\cdot\frac{\partial}{\partial x}} - 1\right] U\,(t,x) = f(t,x).$$

For this problem, we have $a\,(\nu) = e^{\omega\cdot\nu}$, $b\,(\nu) = 2\,e^{\omega\cdot\nu} - 1$, $\nu = (\nu_1,\nu_2,\nu_3)$, $A_1\,(\nu) = A_2\,(\nu) = B_1\,(\nu) = B_2\,(\nu) = 1$, $s = 3$, $h = 1$, $\bar{p} = p = \infty$.

The fundamental system of solutions of ODE

$$\left[\frac{d^2}{dt^2} + 2\, e^{\omega\cdot\nu}\frac{d}{dt} + 2\, e^{\omega\cdot\nu} - 1\right] T\,(t,\nu) = 0$$

normal at the point $t = 0$ has the form

$$\begin{aligned} T_0(t,\nu) &= e^{-te^{\omega\cdot\nu}}\left\{e^{\omega\cdot\nu}\frac{\sinh\left[t(e^{\omega\cdot\nu}-1)\right]}{e^{\omega\cdot\nu}-1} + \cosh\left[t(e^{\omega\cdot\nu}-1)\right]\right\}, \\ T_1(t,\nu) &= e^{-te^{\omega\cdot\nu}}\frac{\sinh\left[t(e^{\omega\cdot\nu}-1)\right]}{e^{\omega\cdot\nu}-1} \end{aligned} \tag{4.5}$$

(in particular, if $e^{\omega\cdot\nu} = 1$ we obtain $T_0(t,\nu) = e^{-t}(t+1)$, $T_1(t,\nu) = te^{-t}$).

For problem (4.3), (4.4), we have:

$$\Delta\,(\nu) = \begin{vmatrix} 1 & 1 \\ e^{-e^{\omega\cdot\nu}}\dfrac{\sinh\left[e^{\omega\cdot\nu}-1\right]}{e^{\omega\cdot\nu}-1} & e^{-e^{\omega\cdot\nu}}\dfrac{\sinh\left[e^{\omega\cdot\nu}-1\right]}{e^{\omega\cdot\nu}-1} \end{vmatrix} \equiv 0.$$

The condition of existence of solutions of problem (4.3), (4.4) according to theorem 3.1 is following:

for $f \in \mathbb{A}_1$ such identity

$$f\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\right)\left\{\frac{e^{\lambda} - e^{-2e^{\omega\cdot\nu}+1}}{\lambda - 1 + 2e^{\omega\cdot\nu}}e^{\nu\cdot x}\right\}\Big|_{\lambda=0,\,\nu=O} \equiv 0 \tag{4.6}$$

holds in $\mathbb{R}^3$.

Condition (4.6) is satisfied, in particular, if the right-hand side of equation (4.3) has the form:

$$f(t,x) = \cos[2\pi t]e^{-t+x_2+x_3}.$$

Really,

$$\begin{aligned} &\cos\left[2\pi\frac{\partial}{\partial\lambda}\right]e^{-\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\nu_2}+\frac{\partial}{\partial\nu_3}}\left\{e^{\nu\cdot x}\Phi_1(\lambda,\nu)\right\}\Big|_{\lambda=0,\,\nu=O} \\ &= \frac{1}{2}\left\{e^{\nu\cdot x}\Phi_1(\lambda,\nu)\right\}\Big|_{\lambda=2\pi i-1,\,\nu=(0,1,1)} + \frac{1}{2}\left\{e^{\nu\cdot x}\Phi_1(\lambda,\nu)\right\}\Big|_{\lambda=-2\pi i-1,\,\nu=(0,1,1)} \\ &= \frac{1}{2}e^{x_2+x_3}\left\{\frac{e^{\lambda}-e^{-1}}{\lambda+1}\right\}\Big|_{\lambda=2\pi i-1} + \frac{1}{2}e^{x_2+x_3}\left\{\frac{e^{\lambda}-e^{-1}}{\lambda+1}\right\}\Big|_{\lambda=-2\pi i-1} \equiv 0. \end{aligned}$$

So for function $f(t,x) = \cos[2\pi t]e^{-t+x_2+x_3}$ problem (4.3), (4.4) has solution in $\mathbb{A}_1$, which can be found by formula (3.2):

$$\begin{aligned}
U(t,x) &= f\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}\Phi(t,\lambda,\nu)\right\}\Big|_{\lambda=0,\,\nu=O} \\
&= \frac{1}{2}\left\{e^{\nu\cdot x}\Phi(t,\lambda,\nu)\right\}\Big|_{\lambda=2\pi i-1,\nu=(0,1,1)} + \frac{1}{2}\left\{e^{\nu\cdot x}\Phi(t,\lambda,\nu)\right\}\Big|_{\lambda=-2\pi i-1,\,\nu=(0,1,1)} \\
&= \frac{1}{4\pi^2}e^{-t+x_2+x_3}\left\{1-\cos[2\pi t]\right\}.
\end{aligned}$$

Let's note that obtained solution of problem (4.3), (4.4) is only partial solution, because it is found to within elements of the null-space of the problem of form

$$U(t,x) = \varphi(x)\,e^{-t},$$

where φ is arbitrary continuously function in $\mathbb{R}^3$. □

5. Conclusions

We found the condition of existence of solution of the problem in the class of entire functions for nonhomogeneous PDE of second order with respect to time variable, in which homogeneous local two-point conditions are imposed, and infinite order with respect to spatial variables in the case when the characteristic determinant identically equals to zero. We showed examples for which the solutions of two-point problems exist. These solutions are constructed by using the differential-symbol method.

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