ON THE ELEMENTARY AND BASIC CHARACTERS OF $G_n(q)$

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Abstract. We discuss in this paper the elementary and basic characters of $G_n(q)$. The paper highlights the main idea that every irreducible character of $G_n(q)$ appears uniquely in the basic characters. In particular we determine the elementary and basic characters of $G_3(q)$ for any q. As examples, the theory is also applied to $G_3(2)$, $G_3(3)$ and $G_4(2)$.

Keywords: general linear group, Sylow subgroup, Fischer matrices, irreducible characters, elementary and basic characters.

1. Introduction and notations

The study of the irreducible characters of $G_n(q)$, a Sylow p-subgroup of $GL(n, q)$ has attracted much attention over the years. By inducing linear characters of some special subgroups of $G_n(q)$, Andre in [2] introduced the notion of elementary characters. These characters were also known to Lehrer (see [8]). The basic characters are a special product of some of these elementary characters and their constituents form a partition of all the irreducible characters of $G_n(q)$ (see for example [2]).

The group $G_n(q)$ is given by

$$G_n(q) = \{(a_{ij}) \in GL(n, q) | a_{ij} = 0, j < i, a_{ii} = 1 \text{ and } a_{ij} \in \mathbb{F}_q, 1 \leq i < j \leq n\},$$

that is a group of upper triangular matrices with $1's$ in the major diagonal and other entries coming from $\mathbb{F}_q$, where F_q is the Galois field of q elements with $\mathbb{F}_q^*$ and $\mathbb{F}_q^+$ being the multiplicative and additive groups respectively. The

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group $G_n(q)$ can be written as a split extension $N{:}G$ where N is an elementary abelian p-group $p^{k(n-1)}$ and G is the group $G_{n-1}(q)$. We use this fact to apply the method of Fischer matrices in constructing the character tables. The method of Coset Analysis by Moori [10] is used for the determination of the conjugacy classes.

By $\lambda_{ij}(\alpha)^{G_n(q)}$, we will mean the induction of $\lambda_{ij}(\alpha)$ to $G_n(q)$ where $\alpha \in \mathbb{F}_q^*$ and $\lambda_{ij}(\alpha) : G_{ij} \longrightarrow \mathbb{C}^*$ is a linear character of G_{ij}. The G_{ij} are special subgroups of $G_n(q)$ where $(i,j) \in \Phi(n) = \{(i,j)|1 \leq i < j \leq n\}$. We identify a particular type of subsets of $\Phi(n)$ denoted by D as the basic subsets. To each basic subset D we can associate a function $\varphi : D \longrightarrow \mathbb{F}_q^*$ such that $\varphi((i,j)) \in \mathbb{F}_q^*$ (see for instance Subsection 3.2) and thereby denote the basic characters by $\xi_D(\varphi)$. This is the standard notation as used in [2].

In Section 2 we briefly describe the method of constructing the elementary and basic characters of $G_n(q)$. Using Fischer matrices we obtain the character tables of $G_3(2)$ and $G_3(3)$ in Sections 3 and 4. The elementary and basic characters of these groups are discussed in Subsections 3.1, 3.2 and 4.1. We have the main Theorem 4.2 on the elementary and basic characters of $G_3(q)$ proved in Subsection 4.2. The character table of $G_4(2)$, its elementary and basic characters are discussed in Section 5. For general notation we use ATLAS [4] and Isaacs [6].

2. Elementary and basic characters

In this section we briefly describe the elementary and basic characters of $G_n(q)$. We use the notation as in [2]. Basic characters have also been described in [7] as super characters.

Definition 2.1. *Let*

$$G_n(q) = \{(a_{ij}) \in GL(n,q)|a_{ij} = 0, j < i, a_{ii} = 1 \text{ and } a_{ij} \in \mathbb{F}_q, 1 \leq i < j \leq n\},$$

be a Sylow $p-$ *subgroup of* $GL(n,q)$. *We set* $\Phi(n) = \{(i,j)|1 \leq i < j \leq n\}$ *be the set of pairs for the position of* $a_{ij} \in \mathbb{F}_q$ *in a matrix of* $G_n(q)$. *The elements of* $\Phi(n)$ *are called* ***positive roots***.

Definition 2.2. *By fixing* i *for* $1 \leq i \leq n$, *define the* $i^{th}-$ *row of* $G_n(q)$ *to be the set*

$$r_i(n) = \{(i,j) \in \Phi(n)|i < j \leq n\}.$$

Similarly by fixing j *the* $j^{th}-$ *column of* $G_n(q)$ *is the set*

$$c_j(n) = \{(i,j) \in \Phi(n)|1 \leq i < j\}.$$

Using the elements of $\Phi(n)$, we define

$$G_{ij} = \{(x_{ab}) \in G_n(q)|x_{ib} = 0 \ i < b < j\},$$

G_{ij} are subgroups of $G_n(q)$. The irreducible characters of $G_n(q)$ that were constructed by Lehrer [8] are described in Proposition 2.1 following here below.

Proposition 2.1. *Let $\alpha \in \mathbb{F}_q^*$ and ψ_o be a non-trivial irreducible character of $\mathbb{F}_q^+$ considered as an additive group. Let $\lambda_{ij}(\alpha) : G_{ij} \longrightarrow \mathbb{C}^*$ be a function such that $\lambda_{ij}(\alpha)(x) = \psi_o(\alpha(x_{ij}))$ for all $x \in G_{ij}$ where $x = (x_{ab})$. Then $\lambda_{ij}(\alpha)$ is a linear character of G_{ij} and furthermore $\xi_{ij}(\alpha) = \lambda_{ij}(\alpha)^{G_n(q)}$ is an irreducible character of $G_n(q)$.*

Proof. See [8]. □

The following definitions are from [2].

Definition 2.3. *The characters $\lambda_{ij}(\alpha)^{G_n(q)}$, as given in Proposition 2.1, are called the (i, j)th **elementary characters** associated with α.*

Definition 2.4. *Let $D \subseteq \Phi(n)$ such that $|D \cap r_i(n)| \leq 1$ and $|D \cap c_j(n)| \leq 1$. Let $\varphi_{ij} : D \longrightarrow \mathbb{F}^*$ be a function, then*

$$\xi_D(\varphi) = \prod_{(i,j)\in D} \xi_{ij}(\varphi(i, j))$$

*is a character of $G_n(q)$ called the **basic character** of $G_n(q)$ and the subset D is called a **basic subset** of $\Phi(n)$.*

3. Elementary and basic characters of $G_3(2)$

We first construct the character table of $G_3(2)$ by using Fischer matrices. For details on coset analysis and Fischer matrices the readers are referred to ([1], [9], [10], [11]). The elementary and basic characters are discussed in Subsections 3.1 and 3.2. Table 1 gives the conjugacy classes of $G_3(2)$ computed using the coset analysis technique.

classes of $G_2(2)$	classes of $G_3(2)$	$\lvert C_{G_3(2)}(g)\rvert$
$(1a)$	$(1A)$	2^3
	$(2A)$	2^3
	$(2C)$	2^2
$(2a)$	$(2C)$	2^2
	$(4A)$	2^2

Table 1: The Conjugacy Classes of $G_3(2)$

The inertia factor groups are; $H_1 = H_2 = G \cong \mathbb{Z}_2$ and $H_3 = \{1\}$.

The Fischer matrices on the representatives of the classes of G are given below

$$M(1a) = \begin{matrix} & & |C_{\bar{G}}(1A)| & |C_{\bar{G}}(2A)| & |C_{\bar{G}}(2B)| \\ |C_{H_1}(1a)| & 2 \\ |C_{H_2}(1a)| & 2 \\ |C_{H_3}(1a)| & 1 \end{matrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix},$$

$$M(2a) = \begin{array}{c} \\ |C_{H_1}(2a)| \\ |C_{H_2}(2a)| \end{array} \begin{array}{c} \\ 2 \\ 2 \end{array} \begin{array}{c} \quad |C_{\bar{G}}(2C)| \quad |C_{\bar{G}}(4A)| \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}.$$

Note that character tables of the inertia factor groups H_1 and H_2 are 2×2 invertible matrices. Thus, by multiplying (usual matrix multiplication) the partial character tables of the inertia factor groups H_1, H_2 and H_3 by the corresponding rows of the Fischer matrices above, we obtain

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \end{bmatrix}.$$

Similarly we obtain

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The character table of $G_3(2)$ is given below as Table 2.

$[cl(g)]$	$1A$	$2A$	$2B$	$2C$	$4A$
$C_G(g)$	8	8	4	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 2: The Character Table of $G_3(2)$

3.1 Elementary characters of $G_3(2)$

We apply in this Subsection the ideas of [2] to our group $G_3(2)$ to identify its elementary and basic characters. We will later apply the same theory to $G_3(3)$, $G_4(2)$ and $G_3(q)$ in general (see Subsections 4.1, 4.2, 4 and 5). For $\alpha = 1$ and using the character table of $\mathbb{F}_2$, we have ψ_o, the fixed non-trivial irreducible character of $\mathbb{F}_2$, given by say χ_2.

We also have that

$$\Phi(3) = \{(i, j) | 1 \leq i < j \leq 3\} = \{(1, 2), (1, 3), (2, 3)\}.$$

The subgroups G_{ij} are $G_{12} = G_{23} = G_3$ and

$$G_{13} = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{F}_2 \right\}$$

which is the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Thus $\lambda_{ij}(\alpha) : G_{ij} \longrightarrow \mathbb{C}^*$, such that $\lambda_{ij}(\alpha)(x) = \psi_o(\alpha x_{13})$ for all $x \in G_{13}$. We then have that

$$\begin{aligned} \lambda_{ij}(1)(x) = \psi_o(x_{13}) &= 1 \text{ if } x_{13} = 0 \\ &- 1 \text{ if } x_{13} = 1. \end{aligned}$$

The character $\lambda_{13}(1)^{G_3(2)}$ is an irreducible character of $G_3(2)$ of degree

$$[G_3(2) : G_{13}] = 2.$$

Using the character table of $G_3(2)$ in Table 2, we identify $\lambda_{13}(1)^{G_3(2)} = \chi_5$. Since we need

$$(q-1)(\frac{n(n-1)}{2}) = (2-1)(\frac{3(3-1)}{2}) = 3$$

elementary characters, we have two more elementary characters to be induced from G_{12} and G_{23} and clearly they are both of degree 1. We also have that $\lambda_{12}(1)(x),\ \lambda_{23}(1)(x) \in \{1, -1\}$. Using the character table and the structure of the conjugacy class representatives of $G_3(2)$, we easily identify $\lambda_{12}(1)^{G_3} = \chi_3$ and $\lambda_{23}(1)^{G_3} = \chi_2$. Hence the three elementary characters of $G_3(2)$ are χ_2, χ_3 and χ_5 of degrees 1, 1 and 2 respectively.

3.2 Basic characters of $G_3(2)$

To calculate the basic characters of $G_3(2)$, we first identify the basic subsets of $\Phi(3)$. Since $\Phi(3) = \{(1,2),(1,3),(2,3)\}$, there are 8 subsets of $\Phi(3)$. The following subsets are basic $D_1 = \emptyset$, $D_2 = \{(1,2)\}$, $D_3 = \{(1,3)\}$, $D_4 = \{(2,3)\}$, $D_5 = \{(1,2),(2,3)\}$.

Remark 3.1. For the group $G_3(2)$, we have $r_1(3) = \{(1,2),(1,3)\}$, $r_2(3) = \{(2,3)\}$, $c_1(3) = \emptyset$, $c_2(3) = \{(1,2)\}$ and $c_3(3) = \{(1,3),(2,3)\}$. It is clear that $|D_s \cap r_i(3)| \leq 1$ and $|D_s \cap c_j(3)| \leq 1$ for all $s \in \{1,2,3,4,5\}$, $i \in \{1,2\}$ and $j \in \{2,3\}$. Thus D_s is basic.

We now use the formula

$$\xi_D(\varphi) = \prod_{(i,j)\in D} \xi_{i,j}(\varphi(i,j)) \tag{1}$$

for the basic characters where $\varphi : D \longrightarrow \mathbb{F}_q^*$. By definition $\xi_\emptyset(\varphi) = \chi_1$ is the trivial character of $G_3(2)$. In this way we have the basic characters of $G_3(2)$, namely

$$\begin{aligned}
\xi_{D_1}(\varphi) &= \chi_1,\\
\xi_{D_2}(\varphi) &= \xi_{12}(\varphi(1,2)) = \xi_{12}(1) = \lambda_{12}(1)^{G_3} = \chi_3,\\
\xi_{D_3}(\varphi) &= \xi_{13}(\varphi(1,3)) = \xi_{13}(1) = \lambda_{13}(1)^{G_3} = \chi_5,\\
\xi_{D_4}(\varphi) &= \xi_{23}(\varphi(2,3)) = \xi_{23}(1) = \lambda_{23}(1)^{G_3} = \chi_2,\\
\xi_{D_5}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \xi_{12}(1) \times \xi_{23}(1)\\
&= \lambda_{12}(1)^{G_3} \times \lambda_{23}(1)^{G_3} = \chi_2 \times \chi_3 = \chi_4.
\end{aligned}$$

Table 3 below gives the basic characters of $G_3(2)$ decomposed in terms of its irreducible characters viz. $\chi_1,\ \chi_2,\ \chi_3,\ \chi_4$ and χ_5.

basic characters	χ_1	χ_2	χ_3	χ_4	χ_5
$\xi_\emptyset(\varphi)$	1	0	0	0	0
$\xi_{D_2}(\phi)$	0	0	1	0	0
$\xi_{D_3}(\phi)$	0	0	0	0	1
$\xi_{D_4}(\phi)$	0	1	0	0	0
$\xi_{D_5}(\phi)$	0	0	0	1	0

Table 3: The Constituents of the Basic Characters of $G_3(2)$

We observe from Table 3 above that all the basic characters of $G_3(2)$ are irreducible.

4. Elementary and basic characters of $G_3(3)$

We have that $G_3(3) = 3^2{:}3$ and $\mathbb{F}_3 = \{0, 1, -1\}$. From the character table of $G_3(q)$ in [3], we have that

- $2q - 1 = 2(3) - 1 = 5$ inertia groups,
- $q^2 + q - 1 = 3^2 + 3 - 1 = 11$ conjugacy classes,
- $q^2 = 3^2 = 9$ irreducible characters of degree 1,
- $q - 1 = 3 - 1 = 2$ irreducible characters of degree 3.

To find the conjugacy classes of $G_3(3)$, we use the method of coset analysis. Let $G_3(3) = N{:}G$, where $G \cong \mathbb{Z}_3$ and the conjugacy class representatives of G are $\{1a, 3a, 3b\}$. Table 4 following here below, gives the conjugacy classes of $G_3(3)$.

Theorem 4.1. *The group $G_3(3)$ has the following conjugacy classes as listed in Table 4 below, where the upper cases label conjugacy classes of $G_3(3)$ and lower cases are reserved for its subgroups (inertia factor groups).*

Proof. An application of the coset analysis method. □

classes of G	classes of G_3	$\|C_{G_3}(g)\|$
$(1a)$	$(1A)$	3^3
	$(3A)$	3^3
	$(3B)$	3^3
	$(3C)$	3^2
	$(3D)$	3^2
$(3a)$	$(3E)$	3^2
	$(3F)$	3^2
	$(3G)$	3^2
$(3a)$	$(3H)$	3^2
	$(3I)$	3^2
	$(3J)$	3^2

Table 4: The Conjugacy Classes of $G_3(3)$

The structure of the inertia factor groups is $H_1 = \mathbb{Z}_3 = H_2 = H_3$ and $H_4 = \{1\} = H_5$. We obtain the following character table for $G \cong \mathbb{Z}_3$ as in Table 5, where $a = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$ and $\bar{a}$ is the complex conjugate.

$[g]$	$1a$	$3a$	$3b$
$C_G(g)$	3	3	3
χ_1	1	1	1
χ_2	1	a	$\bar{a}$
χ_3	1	$\bar{a}$	a

Table 5: The Character Table of $G \cong \mathbb{Z}_3$

Thus using the general form of the Fischer matrix in [3], we have

$$M(1a) = \begin{array}{c} \\ |C_{H_1}(1a)|\ 3 \\ |C_{H_2}(1a)|\ 3 \\ |C_{H_3}(1a)|\ 3 \\ |C_{H_4}(1a)|\ 1 \\ |C_{H_5}(1a)|\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} |C_{\bar{G}_3}(1A)| & |C_{\bar{G}}(3A)| & |C_{\bar{G}}(3B)| & |C_{\bar{G}}(3C)| & |C_{\bar{G}}(3D)| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & a & \bar{a} \\ 1 & 1 & 1 & \bar{a} & a \\ 3 & b & \bar{b} & 0 & 0 \\ 3 & \bar{b} & b & 0 & 0 \end{pmatrix} \end{array}.$$

In the above Fischer matrix, $a = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$ and $b = \frac{-3}{2} - \frac{3\sqrt{3}}{2}i$.

The other Fischer matrices are given below, where a and $\bar{a}$ are as in the Fischer matrix above

$$M(3a) = \begin{array}{c} \\ |C_{H_1}(3a)|\ 3 \\ |C_{H_2}(3a)|\ 3 \\ |C_{H_3}(3a)|\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} |C_{\bar{G}_3}(3E)| & |C_{\bar{G}}(3F)| & |C_{\bar{G}}(3G)| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & \bar{a} \\ 1 & \bar{a} & a \end{pmatrix} \end{array},$$

$$M(3b) = \begin{matrix} & |C_{\bar{G}_3}(3H)| & |C_{\bar{G}}(3I)| & |C_{\bar{G}}(3J)| \\ |C_{H_1}(3b)|\ 3 & 1 & 1 & 1 \\ |C_{H_2}(3b)|\ 3 & 1 & a & \bar{a} \\ |C_{H_3}(3b)|\ 3 & 1 & \bar{a} & a \end{matrix}.$$

Thus, for instance, on the classes $(1A)$, $(3A)$, $(3B)$, $(3C)$ and $(3D)$, the character values are obtained by multiplying the partial character tables of the inertia factor groups by the corresponding rows of the Fischer matrices and we obtain

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & a & \bar{a}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & a & \bar{a}\\1 & 1 & 1 & a & \bar{a}\\1 & 1 & 1 & a & \bar{a}\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & \bar{a} & a\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & \bar{a} & a\\1 & 1 & 1 & \bar{a} & a\\1 & 1 & 1 & \bar{a} & a\end{bmatrix},$$

$$\begin{bmatrix}1\end{bmatrix}\begin{bmatrix}3 & b & \bar{b} & 0 & 0\end{bmatrix} = \begin{bmatrix}3 & b & \bar{b} & 0 & 0\end{bmatrix},$$

$$\begin{bmatrix}1\end{bmatrix}\begin{bmatrix}3 & \bar{b} & b & 0 & 0\end{bmatrix} = \begin{bmatrix}3 & \bar{b} & b & 0 & 0\end{bmatrix}.$$

In this manner we can obtain the character table of $G_3(3)$ given as in Table 6.

$[g]$	$1A$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$3H$	$3I$	$3J$
$\|C_G(g)\|$	3^3	3^3	3^3	3^2	3^2	3^2	3^2	3^2	3^2	3^2	3^2
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	a	a	a	$\bar{a}$	$\bar{a}$	$\bar{a}$
χ_3	1	1	1	1	1	$\bar{a}$	$\bar{a}$	$\bar{a}$	a	a	a
χ_4	1	1	1	a	$\bar{a}$	1	a	$\bar{a}$	1	a	$\bar{a}$
χ_5	1	1	1	a	$\bar{a}$	a	$\bar{a}$	1	$\bar{a}$	1	a
χ_6	1	1	1	a	$\bar{a}$	$\bar{a}$	1	a	a	$\bar{a}$	1
χ_7	1	1	1	$\bar{a}$	a	1	$\bar{a}$	a	1	$\bar{a}$	a
χ_8	1	1	1	$\bar{a}$	a	a	1	$\bar{a}$	$\bar{a}$	a	1
χ_9	1	1	1	$\bar{a}$	a	$\bar{a}$	a	1	a	1	$\bar{a}$
χ_{10}	3	b	$\bar{b}$	0	0	0	0	0	0	0	0
χ_{11}	3	$\bar{b}$	b	0	0	0	0	0	0	0	0

Table 6: The Character Table of $\bar{G} = G_3(3)$

$[cl(g)]$	0	1	-1
$\lvert C_{\mathbb{F}_2}(g)\rvert$	3	3	3
χ_1	1	1	1
χ_2	1	a	$\bar{a}$
χ_3	1	$\bar{a}$	a

Table 7: The Character Table of $\mathbb{F}_3$

4.1 Elementary characters of $G_3(3)$

We now consider the field $\mathbb{F}_3 = \{0, 1, -1\}$. Then $\alpha \in \{1, -1\}$ and using the character table of $\mathbb{F}_3$ as in Table 7, we fix $\psi_o = \chi_2$.

In Table 7 we have $a = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$.

We also have that

$$\Phi(3) = \{(i, j)|1 \leq i < j \leq 3\} = \{(1, 2), (1, 3), (2, 3)\}.$$

The subgroups G_{ij} are $G_{12} = G_{23} = G_3$ and

$$G_{13} = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{F}_3 \right\},$$

which is the group $\mathbb{Z}_3 \times \mathbb{Z}_3$. Thus $\lambda_{ij}(\alpha) : G_{ij} \longrightarrow \mathbb{C}^*$, such that $\lambda_{ij}(\alpha)(x) = \psi_o(\alpha x_{13})$ for all $x \in G_{13}$.

For instance, for $\alpha = 1$ we have

$$\begin{aligned} \lambda_{ij}(1)(x) = \psi_o(x_{13}) &= 1 \text{ if } x_{13} = 0, \\ &= a \text{ if } x_{13} = 1, \\ &= \bar{a} \text{ if } x_{13} = -1, \end{aligned}$$

where $a = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$ and $\bar{a}$ is its complex conjugate. The character $\lambda_{13}(\alpha)^{G_3(3)}$ is an irreducible character of $G_3(3)$ of degree

$$[G_3(3) : G_{13}] = 3.$$

Using the character table of $G_3(3)$ in Table 6, we identify $\lambda_{13}(1)^{G_3(3)} = \chi_{11}$ and $\lambda_{13}(-1)^{G_3(3)} = \chi_{10}$.

Since we need

$$(q - 1)(\frac{n(n - 1)}{2}) = (3 - 1)(\frac{3(3 - 1)}{2}) = 6$$

elementary characters, we have four more elementary characters to be induced from G_{12} and G_{23} and they are all of degree 1. We also have that $\lambda_{12}(\alpha)(x)$, $\lambda_{23}(\alpha)(x) \in \{1, a, \bar{a}\}$. For instance, by using the character table and the structure

$[g]$	$1A$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$3H$	$3I$	$3J$
$\lambda_{12}(1)^{G_3}$	1	1	1	1	1	a	a	a	$\bar{a}$	$\bar{a}$	$\bar{a}$

Table 8: The Values of $\lambda_{12}(1)(x)$

of the conjugacy class representatives of $G_3(3)$ we have the values of $\lambda_{12}(1)^{G_3(3)}$ on the conjugacy class representatives of $G_3(3)$ as in Table 8.

We easily identify $\lambda_{12}(1)^{G_3(3)} = \chi_2$, $\lambda_{12}(-1)^{G_3(3)} = \chi_3$, $\lambda_{23}(1)^{G_3(3)} = \chi_4$ and $\lambda_{23}(-1)^{G_3(3)} = \chi_7$. Hence the six elementary characters of $G_3(3)$ are χ_2, χ_3, χ_4, χ_7, χ_{10} and χ_{11}.

4.2 Basic characters of $G_3(3)$

As in Subsection 3.2, the basic subsets of $\Phi(3)$ are $D_1 = \emptyset$, $D_2 = \{(1,2)\}$, $D_3 = \{(1,3)\}$, $D_4 = \{(2,3)\}$, $D_5 = \{(1,2),(2,3)\}$. By definition $\xi_\emptyset(\varphi) = \chi_1$ is the trivial character of $G_3(3)$. In this way we have the basic characters of $G_3(3)$, namely

$$\begin{aligned}
\xi_{D_1}(\varphi) &= \chi_1,\\
\xi_{D_2}(\varphi) &= \xi_{12}(\varphi(1,2)) = \xi_{12}(1) = \lambda_{12}(1)^{G_3} = \chi_2,\\
\xi_{D_2}(\varphi) &= \xi_{12}(\varphi(1,2)) = \xi_{12}(-1) = \lambda_{12}(-1)^{G_3} = \chi_3,\\
\xi_{D_3}(\varphi) &= \xi_{13}(\varphi(1,3)) = \xi_{13}(1) = \lambda_{13}(1)^{G_3} = \chi_{10},\\
\xi_{D_3}(\varphi) &= \xi_{13}(\varphi(1,3)) = \xi_{13}(-1) = \lambda_{13}(-1)^{G_3} = \chi_{11},\\
\xi_{D_4}(\varphi) &= \xi_{23}(\varphi(2,3)) = \xi_{23}(1) = \lambda_{23}(1)^{G_3} = \chi_4,\\
\xi_{D_4}(\varphi) &= \xi_{23}(\varphi(2,3)) = \xi_{23}(-1) = \lambda_{23}(1)^{G_3} = \chi_7,\\
\xi_{D_5}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \xi_{12}(1) \times \xi_{23}(1)\\
&= \lambda_{12}(1)^{G_3} \times \lambda_{23}(1)^{G_3} = \chi_2 \times \chi_4 = \chi_5,\\
\xi_{D_5}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \xi_{12}(-1) \times \xi_{23}(-1)\\
&= \lambda_{12}(-1)^{G_3} \times \lambda_{23}(-1)^{G_3} = \chi_3 \times \chi_7 = \chi_9,\\
\xi_{D_5}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \xi_{12}(-1) \times \xi_{23}(1)\\
&= \lambda_{12}(-1)^{G_3} \times \lambda_{23}(1)^{G_3} = \chi_3 \times \chi_4 = \chi_6,\\
\xi_{D_5}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \xi_{12}(1) \times \xi_{23}(-1)\\
&= \lambda_{12}(1)^{G_3} \times \lambda_{23}(-1)^{G_3} = \chi_2 \times \chi_7 = \chi_8.
\end{aligned}$$

Thus all the irreducible characters of $G_3(3)$ are basic characters. In general, we have the following Theorem 4.2.

Theorem 4.2. *All the irreducible characters of $G_3(q)$ are basic characters.*

Proof. From $\Phi(3)$, we have the groups $G_{12} = G_3(q)$, $G_{13} = \mathbb{Z}_q \times \mathbb{Z}_q$ and $G_{23} = G_3(q)$. The characters $\lambda_{13}(\alpha)^{G_3(q)}$ are irreducible characters of $G_3(q)$ of degree

$|G_3(q) : G_{13}| = q$. Thus associated with α, we have $q-1$ elementary characters of $G_3(q)$ of degree q. Since $\xi_{D_3}(\varphi) = \lambda_{13}(\alpha)^{G_3(q)}$, we have $q-1$ basic characters arising this way. This accounts for the $q-1$ irreducible characters of $G_3(q)$ of degree q. The basic subsets D_2 and D_4 both give rise to $\lambda_{12}(\alpha)^{G_3(q)} = \lambda_{12}(\alpha)$ and $\lambda_{23}(\alpha)^{G_3(q)} = \lambda_{23}(\alpha)$ respectively. Thus we have $2q-2$ basic characters of degree 1. For the basic subset D_5 we have $\xi_{D_5}(\varphi) = \lambda_{12}(\alpha)^{G_3(q)} \times \lambda_{23}(\alpha)^{G_3(q)}$. This gives us q^2-2q+1 basic characters of degree 1. The basic subset D_1 contributes 1 irreducible character that is χ_1. We then have $(q^2-2q+1)+(2q-2)+1 = q^2$, therefore accounting for the q^2 linear characters of $G_3(q)$. □

5. The group $G_4(2)$

We determine the character table of $G_4(2)$, its elementary and basic characters.

5.1 The character table of $G_4(2)$

Applying coset analysis we obtain the following conjugacy classes of $G_4(2)$ listed in Table 9.

classes of G	classes of $G_4(2)$	$\lvert C_{G_4}(g)\rvert$
$(1a)$	$(1A)$	2^6
	$(2A)$	2^6
	$(2B)$	2^5
	$(2C)$	2^4
$(2a)$	$(2D)$	2^5
	$(2E)$	2^5
	$(4A)$	2^4
$(2b)$	$(2F)$	2^4
	$(2G)$	2^4
	$(4B)$	2^3
$(2c)$	$(2H)$	2^4
	$(2I)$	2^4
	$(4C)$	2^4
	$(4D)$	2^4
$(4a)$	$(4E)$	2^3
	$(4F)$	2^3

Table 9: The Conjugacy Classes of $G_4(2)$

We now calculate the Fischer matrices. Since G has four orbits on the conjugacy classes of N, it also has four orbits on $\mathrm{Irr}(N)$. We check the lengths of these orbits on $\mathrm{Irr}(N)$. Note that the trivial character is fixed. We now have the lengths $w+u+s=7$, thus from the maximal subgroups of G, we get that $w=1$, $u=2$ and $s=4$. In this case, the lengths are the same as those on the conjugacy classes of G. Note that $N \cong V_3(2)$, the vector space of dimension

3 over 2 elements. We know that $\mathrm{Irr}(2^3)$ is the dual of $V_3(2)$ denoted by V^*. In this case $V_3(2) \cong V^*$ is a G-module. Hence the inertia factor groups are $H_1 = G = H_2$, $H_3 = \mathbb{Z}_2$, and for H_4 we note that there are two subgroups, V_4 and $\mathbb{Z}_4$ of G of index 2. Hence we ought to determine which one stabilizes the representative from the orbit with 2 elements.

By writing, $G =< a, b >= \{1, a, a^2, a^3, b, ab, a^2b, a^3b\} \cong D_8$, we have $(V_4)_1 = \{1, a^2, b, a^2b\}$ and $(V_4)_2 = \{1, a^2, ab, a^3b\}$. We identify H_4 to be $(V_4)_1$, that is the point stabilizer of the orbit with 2 elements. The character table of H_4 and the fusion of H_4 into G are given in Table 10 and Table 11 respectively.

$[g]$	1	a^2	b	a^2b
$C_G(g)$	4	4	4	4
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Table 10: The Character Table of H_4

classes of H_4	classes of $G_3(2)$
1	$1a$
a^2	$2a$
b	$2c$
a^2b	$2c$

Table 11: The Fusion of the Classes from H_4 to G

The character table of $H_1 = G = G_3(2) \cong D_8$ is given in Table 12 following below.

$[cl(g)]$	$1a$	$2a$	$2b$	$2c$	$4a$
$C_G(g)$	8	8	4	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 12: The Character Table of H_1

Note that $H_3 =< x >\cong \mathbb{Z}_2$, where x comes from the $2b$ class of $G \cong D_8$.

In the following, we discuss the calculation of Fischer matrices. The Fischer matrix $M(1a)$ has the form:

$$M(1a) = \begin{array}{c} \\ |C_{H_1}(1a)|\ 8 \\ |C_{H_2}(1a)|\ 8 \\ |C_{H_3}(1a)|\ 2 \\ |C_{H_4}(1a)|\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} |C_{\bar{G}}(1A)| & |C_{\bar{G}}(2A)| & |C_{\bar{G}}(2B)| & |C_{\bar{G}}(2C)| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & a & d & g \\ 4 & b & e & h \\ 2 & c & f & i \end{pmatrix} \end{array},$$

We use the column orthogonalities to obtain the values of a, b and c. We have $8 + 8|a|^2 + 2|b|^2 + 4|c|^2 = 64$ so that $4|a|^2 + |b|^2 + 2|c|^2 = 28$. Similarly $8 + 8a + 8b + 8c = 0$, we obtain $a = 1$, $b = -4$ and $c = 2$. we can compute the other unknown entries of $M(1a)$ in the same way and we obtain $d = 1$, $e = 0$, $f = -2$, $g = -1$, $h = 0$ and $i = 0$.

By similar computations, we obtain

$$M(2a) = \begin{array}{c} \\ |C_{H_1}(2a)|\ 8 \\ |C_{H_2}(2a)|\ 8 \\ |C_{H_4}(2a)|\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} |C_{\bar{G}}(2D)| & |C_{\bar{G}}(2E)| & |C_{\bar{G}}(4A)| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix} \end{array},$$

$$M(2b) = \begin{array}{c} \\ |C_{H_1}(2b)|\ 4 \\ |C_{H_2}(2b)|\ 4 \\ |C_{H_3}(2b)|\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} |C_{\bar{G}}(2F)| & |C_{\bar{G}}(2G)| & |C_{\bar{G}}(4B)| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix} \end{array},$$

$$M(2c) = \begin{array}{c} \\ |C_{H_1}(2c)|\ 4 \\ |C_{H_2}(2c)|\ 4 \\ |C_{H_4}(2c)|\ 4 \\ |C_{H_4}(2c)|\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} |C_{\bar{G}}(2H)| & |C_{\bar{G}}(2I)| & |C_{\bar{G}}(4C)| & |C_{\bar{G}}(4D)| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \end{array},$$

$$M(4a) = \begin{array}{c} \\ |C_{H_1}(4a)|\ 4 \\ |C_{H_2}(4a)|\ 4 \end{array} \begin{array}{c} \begin{array}{cc} |C_{\bar{G}}(4E)| & |C_{\bar{G}}(4F)| \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & \text{-1} \end{pmatrix} \end{array}.$$

To obtain the character table of $G_4(2)$ we multiply the appropriate partial character tables of the inertia factor groups by the appropriate rows of the Fischer matrices. Thus for the classes $(1A)$, $(2A)$, $(2B)$ and $(2C)$ of $\bar{G} = G_4(2)$, by using rows of $M(1a)$ and first columns of the character tables of inertia factor groups, we have

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix},$$

$$\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & -1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & -1\\1 & 1 & 1 & -1\\1 & 1 & 1 & -1\\1 & 1 & 1 & -1\\2 & 2 & 2 & -2\end{bmatrix},$$

$$\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}4 & -4 & 0 & 0\end{bmatrix}=\begin{bmatrix}4 & -4 & 0 & 0\\4 & -4 & 0 & 0\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}2 & 2 & -2 & 0\end{bmatrix}=\begin{bmatrix}2 & 2 & -2 & 0\\2 & 2 & -2 & 0\\2 & 2 & -2 & 0\\2 & 2 & -2 & 0\end{bmatrix}.$$

Similarly for the classes $2D$, $2E$ and $4A$, we have

$$\begin{bmatrix}1\\1\\1\\1\\-2\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\\-2 & -2 & -2\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\1\\1\\-2\end{bmatrix}\begin{bmatrix}1 & 1 & -1\end{bmatrix}=\begin{bmatrix}1 & 1 & -1\\1 & 1 & -1\\1 & 1 & -1\\1 & 1 & -1\\-2 & -2 & 2\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\begin{bmatrix}2 & -2 & 0\end{bmatrix}=\begin{bmatrix}2 & -2 & 0\\2 & -2 & 0\\-2 & 2 & 0\\-2 & 2 & 0\end{bmatrix}.$$

Thus continuing in this manner, we obtain the full character table of $G_4(2)$ as shown in Table 13.

5.2 The Elementary characters of $G_4(2)$

As in section 3.1, we take $\mathbb{F}_2 = \{0, 1\}$. Then $\alpha = 1$ and $\psi_o = \chi_2$, the non-trivial irreducible character of $\mathbb{F}_2$. We have that

$$\Phi(4) = \{(i, j)|1 \leq i < j \leq 4\} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}.$$

The subgroups G_{ij} are $G_{12} = G_{23} = G_{34} = G_4(2)$ and

$$G_{13} = \left\{ \begin{pmatrix}1 & 0 & b & c\\0 & 1 & d & e\\0 & 0 & 1 & f\\0 & 0 & 0 & 1\end{pmatrix} \mid b, c, d, e, f \in \mathbb{F}_2 \right\}. \tag{2}$$

$[g]$	$1A$	$2A$	$2B$	$2C$	$2D$	$2E$	$4A$	$2F$	$2G$	$4B$	$2H$	$2I$	$4C$	$4D$	$4E$	$4F$
$\|C_G(g)\|$	2^6	2^6	2^5	2^4	2^5	2^5	2^4	2^4	2^4	2^3	2^4	2^4	2^4	2^4	2^3	2^3
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
χ_3	1	1	1	1	1	1	1	−1	−1	−1	1	1	1	1	−1	−1
χ_4	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	1	1
χ_5	2	2	2	2	−2	−2	−2	0	0	0	0	0	0	0	0	0
χ_6	1	1	1	−1	1	1	−1	1	1	−1	1	−1	1	−1	1	−1
χ_7	1	1	1	−1	1	1	−1	1	1	−1	−1	1	−1	1	−1	1
χ_8	1	1	1	−1	1	1	−1	−1	−1	1	1	−1	1	−1	−1	1
χ_9	1	1	1	−1	1	1	−1	−1	−1	1	−1	1	−1	1	1	−1
χ_{10}	2	2	2	−2	−2	−2	2	0	0	0	0	0	0	0	0	0
χ_{11}	4	−4	0	0	0	0	0	2	−2	0	0	0	0	0	0	0
χ_{12}	4	−4	0	0	0	0	0	−2	2	0	0	0	0	0	0	0
χ_{13}	2	2	−2	0	2	−2	0	0	0	0	2	0	−2	0	0	0
χ_{14}	2	2	−2	0	2	−2	0	0	0	0	−2	0	2	0	0	0
χ_{15}	2	2	−2	0	−2	2	0	0	0	0	0	2	0	−2	0	0
χ_{16}	2	2	−2	0	−2	2	0	0	0	0	0	−2	0	2	0	0

Table 13: The Character Table of $\bar{G} = G_4(2)$

Using GAP [5], this group is isomorphic to a split extension of the form $2^4{:}2$ and using the IdSmallGroup(G_{13}) function, this is the group number 27 on the GAP list for groups of order 32.

(3) $$G_{14} = \left\{ \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid c, d, e, f \in \mathbb{F}_2 \right\}$$

which is the group $\mathbb{Z}_2 \times D_8$ and

(4) $$G_{24} = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b, c, e \in \mathbb{F}_2 \right\},$$

this is of the form $(\mathbb{Z}_2 \times D_8){:}\mathbb{Z}_2$, it is the group number 49 on the GAP list for groups of order 32.

We get

$$(q-1)(\frac{n(n-1)}{2}) = (2-1)(\frac{4(4-1)}{2}) = 6$$

elementary characters. Three of these are the elementary characters $\lambda_{12}(1)(x)^{G_4(2)}$, $\lambda_{23}(1)(x)^{G_4(2)}$, $\lambda_{34}(1)(x)^{G_4(2)}$ all of degree 1. The character table and structure of the conjugacy class representatives of $G_4(2)$ allow us to identify these three elementary characters as $\lambda_{12}(1)(x)^{G_4(2)} = \chi_6$, $\lambda_{23}(1)(x)^{G_4(2)} = \chi_3$ and $\lambda_{34}(1)(x)^{G_4(2)} = \chi_2$. We now identify the last three elementary characters $\lambda_{13}(1)(x)^{G_4(2)}$, $\lambda_{14}(1)(x)^{G_4(2)}$ and $\lambda_{24}(1)(x)^{G_4(2)}$.

We consider G_{13}. The structure of G_{13} is known from the relation 2. We use GAP [5] to construct G_{13} as a subgroup of $G_4(2)$. We have the following conjugacy classes of G_{13} computed using GAP [5] in Table 14.

classes of $G_{13}(2)$	$1a$	$2a$	$2b$	$2c$	$4a$	$2d$	$2e$	$2f$	$2g$	$4b$	$2h$	$2i$	$4c$	$2j$
$C_{G_{13}}(x)$	32	8	32	16	8	32	32	16	16	8	16	16	8	16
$[cl(x)]$	1	4	1	2	4	1	1	2	2	4	2	2	4	2

Table 14: The Conjugacy Classes of G_{13}

Using GAP [5], we get the fusions of the conjugacy classes of G_{13} to the conjugacy classes of $G_4(2)$ as in Table 15 below.

classes of G_{13}	classes of $G_4(2)$	$\lambda_{13}^{G_4(2)}(y)$
$1a$	$1A$	2
$2a$	$2H$	2
$2b$ $2e$	$2D$	2
$2c$ $2i$	$2F$	0
$2d$	$2A$	2
$2f$ $2j$	$2G$	0
$2g$	$2B$	-2
$2h$	$2E$	-2
$4a$ $4c$	$4E$	0
$4b$	$4C$	-2

Table 15: The Fusion of Classes of G_{13} to $G_4(2)$

With the information about fusions as in Table 15 above, we calculate the permutation character $(1_{G_{13}})^{G_4(2)} = \chi_1 + \chi_6$. Since $\lambda_{13}(1)(x) = 1$ or $\lambda_{13}(1)(x) = -1$ according as $x_{13} = 0$ or $x_{13} = 1$, we can induce the character $\lambda_{13}(1)(x)$ by the induction formula to obtain the values as in Table 15 above on the classes of $G_4(2)$. Note that for any $c \in G_4(2)$ not listed in Table 15, we have that $\lambda_{13}(1)^{G_4(2)}(c) = 0$. Using the character table of $G_4(2)$, we then identify $\lambda_{13}(1)^{G_4(2)} = \chi_{13}$. The same analysis applies in identifying $\lambda_{14}(1)^{G_4(2)} = \chi_{11}$ and $\lambda_{24}(1)^{G_4(2)} = \chi_5$. Therefore the six elementary characters of $G_4(2)$ are $\lambda_{12}(1)(x)^{G_4(2)} = \chi_6$, $\lambda_{23}(1)(x)^{G_4(2)} = \chi_3$, $\lambda_{34}(1)(x)^{G_4(2)} = \chi_2$, $\lambda_{13}(1)^{G_4(2)} = \chi_{13}$, $\lambda_{14}(1)^{G_4(2)} = \chi_{11}$ and $\lambda_{24}(1)^{G_4(2)} = \chi_5$.

5.3 Basic characters of $G_4(2)$

To calculate the basic characters of $G_4(2)$, we first identify the subsets of $\Phi(4)$ which are basic. Since

$$\Phi(4) = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\},$$

there are 64 subsets of $\Phi(4)$. The following subsets are basic $D_1 = \emptyset$, $D_2 = \{(1,2)\}$, $D_3 = \{(1,3)\}$, $D_4 = \{(1,4)\}$, $D_5 = \{(2,3)\}$, $D_6 = \{(2,4)\}$, $D_7 = \{(3,4)\}$. In addition, we have 7 basic 2-subsets of $\Phi(4)$, namely $D_8 = \{(1,2),(2,3)\}$, $D_9 = \{(1,2),(2,4)\}$, $D_{10} = \{(1,2),(3,4)\}$, $D_{11} = \{(1,3),(2,4)\}$, $D_{12} = \{(1,3),(3,4)\}$, $D_{13} = \{(1,4),(2,3)\}$, $D_{14} = \{(2,3),(3,4)\}$ and only one 3-subset, namely $D_{15} = \{(1,2),(2,3),(3,4)\}$. Note that none of the 4-subsets and 5-subsets is basic.

By definition $\xi_\emptyset(\varphi) = \chi_1$ is the trivial character of $G_4(2)$. In this way, we have the basic characters of $G_4(2)$, namely

$$\begin{aligned}
\xi_{D_1}(\varphi) &= \chi_1,\\
\xi_{D_2}(\varphi) &= \xi_{12}(\varphi(1,2)) = \xi_{12}(1) = \lambda_{12}(1)^{G_4} = \chi_6,\\
\xi_{D_3}(\varphi) &= \xi_{13}(\varphi(1,3)) = \xi_{13}(1) = \lambda_{13}(1)^{G_4} = \chi_{13},\\
\xi_{D_4}(\varphi) &= \xi_{14}(\varphi(1,4)) = \xi_{14}(1) = \lambda_{14}(1)^{G_4} = \chi_{11},\\
\xi_{D_5}(\varphi) &= \xi_{23}(\varphi(2,3)) = \xi_{23}(1) = \lambda_{23}(1)^{G_4} = \chi_3,\\
\xi_{D_6}(\varphi) &= \xi_{24}(\varphi(2,4)) = \xi_{24}(1) = \lambda_{24}(1)^{G_4} = \chi_5,\\
\xi_{D_7}(\varphi) &= \xi_{12}(\varphi(3,4)) = \xi_{34}(1) = \lambda_{34}(1)^{G_4} = \chi_2.\\
\xi_{D_8}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) = \lambda_{12}(1)^{G_4} \times \lambda_{23}(1)^{G_4} = \chi_6 \times \chi_3,\\
\xi_{D_9}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{24}(\phi(2,4)) = \lambda_{12}(1)^{G_4} \times \lambda_{24}(1)^{G_4} = \chi_6 \times \chi_5,\\
\xi_{D_{10}}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{34}(\varphi(3,4)) = \lambda_{12}(1)^{G_4} \times \lambda_{34}(1)^{G_4} = \chi_6 \times \chi_2,\\
\xi_{D_{11}}(\varphi) &= \xi_{13}(\varphi(1,3)) \times \xi_{24}(\varphi(2,4)) = \lambda_{13}(1)^{G_4} \times \lambda_{24}(1)^{G_4} = \chi_{13} \times \chi_5,\\
\xi_{D_{12}}(\varphi) &= \xi_{13}(\varphi(1,3)) \times \xi_{34}(\phi(3,4)) = \lambda_{13}(1)^{G_4} \times \lambda_{34}(1)^{G_4} = \chi_{13} \times \chi_2,\\
\xi_{D_{13}}(\varphi) &= \xi_{14}(\varphi(1,4)) \times \xi_{23}(\varphi(2,3)) = \lambda_{14}(1)^{G_4} \times \lambda_{23}(1)^{G_4} = \chi_{11} \times \chi_3,\\
\xi_{D_{14}}(\varphi) &= \xi_{23}(\varphi(2,3)) \times \xi_{34}(\varphi(3,4)) = \lambda_{23}(1)^{G_4} \times \lambda_{34}(1)^{G_4} = \chi_3 \times \chi_2,\\
\xi_{D_{15}}(\varphi) &= \xi_{12}(\varphi(1,2)) \times \xi_{23}(\varphi(2,3)) \times \xi_{34}(\varphi 1‘(3,4))\\
&= \lambda_{12}(1)^{G_4} \times \lambda_{23}(1)^{G_4} \times \lambda_{34}(1)^{G_4} = \chi_6 \times \chi_3 \times \chi_2.
\end{aligned}$$

Table 16 below gives a summary of the basic characters of $G_4(2)$ decomposed in terms of its irreducible characters. We observe from Table 16 that except $\xi_{D_{11}}(\phi)$, all the other basic characters are its irreducible characters.

basic characters	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
$\xi_{\emptyset}(\varphi)$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\xi_{D_2}(\varphi)$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$\xi_{D_3}(\varphi)$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
$\xi_{D_4}(\varphi)$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$\xi_{D_5}(\varphi)$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\xi_{D_6}(\varphi)$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$\xi_{D_7}(\varphi)$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\xi_{D_8}(\varphi)$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$\xi_{D_9}(\varphi)$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$\xi_{D_{10}}(\varphi)$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$\xi_{D_{11}}(\varphi)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
$\xi_{D_{12}}(\varphi)$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$\xi_{D_{13}}(\varphi)$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\xi_{D_{14}}(\varphi)$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\xi_{D_{15}}(\varphi)$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

Table 16: The Constituents of The Basic Characters of $G_4(2)$

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