A CATASTROPHIC TEMPERATURE CHANGE IN THE HEAT EQUATION

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Abstract. In this paper we modify the heat equation with the inclusion of a delta function contribution and find the solution of such an equation. It is found that there is an abrupt rise in the temperature across the board.

1. Introduction

The heat equation [1, 2] is a type of second order partial differential equation used to study the distribution of heat, or, more precisely the variation in temperature in a specific region in accordance with time. In this paper, we examine the scenario where in a large body of water, say a lake or a flooded region, there is a sudden rise in the temperature at some instant. The temperature rises momentarily to an extreme value. This could happen if one drops a device (e.g., a bomb of a few kilotons) [3, 4]. At that instant of time, the sharp rise of temperature is beyond the usual temperature variations.

In the next section, we commence by introducing the one dimensional heat equation and modify it with a delta function contribution which represents a singularity due to it's properties. Then, we solve it by the original methodology used by Fourier himself and others [5, 6, 7] in order to find out the exact nature of the modified equation. In the third section, we discuss the opposite case of sudden and catastrophic cooling.

2. The modified heat equation

Here, we would like to observe how we can explain the phenomenon of a sudden and catastrophic temperature rise in a very small time interval, $-\epsilon < t < +\epsilon$, by

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resorting to the heat equation. It is to be borne in mind that the negative sign in the lower limit signifies the time before the instant of the blast. Let $\Theta(x,t)$ denote the temperature at a position x and instant t in a long, thin thermally conducting rod of length d that extends from $x = 0$ to $x = d$ in 1D. We assume that the sides of the rod are insulated so that heat energy neither enters nor leaves the rod through its sides. Also, we assume that heat energy is neither created nor destroyed in the interior of the rod and there are no radiative losses. Then, the temperature $\Theta(x,t)$ abides by the heat equation given as [2]

$$\alpha \frac{\partial^2 \Theta}{\partial x^2} = \frac{\partial \Theta}{\partial t}, \quad \forall\ 0 < x < d, \forall\ t \geq 0, \tag{1}$$

where $\alpha = \frac{k}{c_p \rho}$, k being the thermal conductivity, c_p being the specific heat capacity and ρ being the mass density of the material under consideration. Without loss of generality, we consider the one dimensional case: it's generalization to 3D is immediate. We would like to propose the following modified form of equation (1) by adding an extra term to the right hand side

$$\alpha \frac{\partial^2 \Theta}{\partial x^2} = \{1 - \delta(t)\} \frac{\partial \Theta}{\partial t}, \tag{2}$$

where $\delta(t)$ represents the Dirac delta as a function of time and the negative sign is due to the fact that we consider the heat equation moments before the blast and the temperature rises radically. It should be observed that in the modified equation (2) the derivative with respect to time (t) suddenly reaches a very high value because of the introduction of the delta function which in physical terms can be looked upon as a device which artificially triggers such an steep rise in the temperature.

As we stated earlier, the objective of the current work is to find out what are the new solutions due to this extra term. Using separation of variables as [5, 6]

$$\Theta(x,t) = X(x)T(t)$$

one can solve (2) in terms of x and t. Now, suppose the temperature and the boundaries of the rod are kept fixed as 0. Then, we have the following boundary conditions:

$$\Theta(0,t) = 0, \quad \forall\ t > 0,$$

$$\Theta(d,t) = 0, \quad \forall\ t > 0.$$

Now, considering the fact that

$$\Theta(x,t) = X(x)T(t)$$

is a solution for the heat equation (2), we must have

$$\alpha X(x)T'(t) = \{(1 - \delta(t))\} X(x)T'(t)$$

which implies that

$$\frac{X''}{X(x)} = \frac{\{(1-\delta(t))\}}{\alpha}\frac{T'(t)}{T(t)} = \xi, \tag{3}$$

where ξ is some constant. Now, from (3) we have the following two equations which are to be solved.

$$X''(x) - \xi X(x) = 0$$

and

$$T'(t) - \frac{\alpha\xi}{1-\delta(t)}T(t) = 0.$$

It is obvious that we will have two cases depending on ξ being zero or nonzero. We shall consider both cases below.

Case-1: $\xi = 0$.
Now, for $\xi = 0$, the solution in terms of x is simply given by

$$X(x) = a_1 + a_2 x \tag{4}$$

and for the solution in terms of t is given by

$$T(t) = a_3. \tag{5}$$

Therefore the solution to the heat equation (2) is given by

$$\Theta(x,t) = a_3(a_1 + a_2 x), \tag{6}$$

where a_1, a_2 and a_3 are integration constants. Now, let us impose the boundary conditions mentioned before. The first boundary condition, $\Theta(0,t) = 0$, is satisfied when we have

$$X(0) = 0$$

which in turn is satisfied when we have

$$a_1 = 0.$$

Again, the second boundary condition, $\Theta(d,t) = 0$, is satisfied when we have

$$X(d) = 0$$

which is satisfied only when we have

$$a_1 + da_2 = 0 \Rightarrow a_2 = 0.$$

Hence, the boundary conditions are satisfied when we have

$$a_1 = a_2 = 0$$

which implies that in general

$$X(x) = 0.$$

But, this will make the whole solution of the heat equation (2) to be trivial and of no fruitful result. Thus, we discard the case of $\xi = 0$.

Case-2: $\xi \neq 0$

In this case, we have the general solution in terms of x as

$$X(x) = a_1 e^{\sqrt{\xi} x} + a_2 e^{-\sqrt{\xi} x}.$$

Now, the first boundary condition is satisfied when we have

$$X(0) = 0$$

which in turn is satisfied when

$$a_1 + a_2 = 0 \Rightarrow a_1 = -a_2.$$

Again, the second boundary condition is satisfied when we have

$$X(d) = 0$$

which again is satisfied only when

$$a_1 e^{\sqrt{\xi} d} + a_2 e^{-\sqrt{\xi} d} = 0 \Rightarrow a_1 [e^{\sqrt{\xi} d} - e^{-\sqrt{\xi} d}] = 0$$

since, $a_1 = -a_2$. From this we obtain

$$e^{\sqrt{\xi} d} = e^{-\sqrt{\xi} d} \Rightarrow e^{2\sqrt{\xi} d} = 1$$

which implies that we have

$$\sqrt{\xi} = \frac{n\pi}{d} i \Rightarrow \xi = -\frac{n^2 \pi^2}{d^2},$$

where n is some integer. Thus, the general solution in terms of x can be written as

$$X(x) = a_1 [e^{\frac{in\pi x}{d}} - e^{-\frac{in\pi x}{d}}] = 2ia_1 \sin(\frac{n\pi x}{d}). \tag{7}$$

Now, let us delve into finding the general solution in terms of t. We have

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} - \frac{\alpha \xi}{1 - \delta} T(t) = 0.$$

From this we can write

$$\int \frac{\{1 - \delta(t)\} \mathrm{d}T(t)}{T(t)} = \alpha \xi \int \mathrm{d}t$$

which yields

$$\ln[T(t)] + \int \frac{\delta(t)\mathrm{d}T(t)}{T(t)} = \alpha\xi t + c. \tag{8}$$

Now, let us consider the second term on the left hand side. We can integrate it by parts such that we have

$$\int \frac{\delta(t)\mathrm{d}T(t)}{T(t)} = \delta(t)\int \frac{\mathrm{d}T(t)}{T(t)} - \int \frac{\mathrm{d}}{\mathrm{d}T(t)}\{\delta(t)\}\ln[T(t)]\mathrm{d}T(t)$$

which gives

$$\int \frac{\delta(t)\mathrm{d}T(t)}{T(t)} = \delta(t)\ln[T(t)] - \int \frac{\mathrm{d}\{\delta(t)\}}{\mathrm{d}T(t)}\ln[T(t)]\mathrm{d}T(t).$$

It is to be noted that the integration constant will be merged with the integration constant (c) to make it c', on the right hand side of equation (8). Now, we would like to define the following terms

$$\omega(t) = \int \frac{\mathrm{d}\{\delta(t)\}}{\mathrm{d}T(t)}\ln[T(t)]\mathrm{d}T(t)$$

and

$$\sigma(t) = \delta(t)\ln[T(t)].$$

Again, it is conspicuous that the delta function depends explicitly on the variable time (t) only and hence

$$\frac{\mathrm{d}\{\delta(t)\}}{\mathrm{d}T(t)} = \frac{\partial\delta(t)}{\partial T(t)} = 0$$

and thus, $\omega(t) = 0$. Therefore, we are left with the improper function $\sigma(t)$. Now, we know that the delta function has the following property

$$\delta(t) = 0, \quad \forall\ t \neq 0,$$

$$\delta(t) = L, \quad \text{when } t = 0,$$

where, L is arbitrarily very large. Using this property of the delta function, the function $\sigma(t)$ can be defined as follows

$$\sigma(t) = 0, \quad \forall\ t \neq 0,$$

$$\sigma(t) = R, \quad \text{when } t = 0,$$

where, R is extremely large. Thus, from equation (8) we have

$$\ln[T(t)] - \sigma(t) = \alpha\xi t + c'$$

which gives the general solution in terms of t as

$$T(t) = a_3 e^{\alpha\xi t + \sigma(t)}, \tag{9}$$

where $a_3 = e^{c'}$. Thus, the partial solutions (7) and (9) finally yield the solution for the heat equation (2) as

$$\Theta(x,t) = \chi_n \sin(\frac{n\pi x}{d}) e^{-\frac{\alpha n^2 \pi^2}{d^2} t + \sigma(t)}, \tag{10}$$

where $\chi_n = 2ia_1a_3$. Here, it is easy to see that at the moment of the blast when $t = 0$, we have

$$\Theta(x,0) = \chi_n \sin(\frac{n\pi x}{d}) e^{R} \tag{11}$$

and since R is extremely large, the value of Θ is very large, i.e., we have an abrupt and extreme rise in the temperature of the medium. Interestingly, there is a sudden change in the system due to the abrupt phenomenon of the bomb blast [3, 4]. Now, far away from the instant $t = 0$, the heat equation is the same as the original equation (1). As we already mentioned in the beginning of this section, our work encompasses a small time interval, namely, $-\epsilon < t < +\epsilon$. Now, it is to be noted that this mathematical formulation can be extended to higher dimensions too.

Now suppose, in a flooded region the spread (or length) of the water body is d and at instant $t = 0$ an explosion device (just to vaporize the water) is detonated. Then, from (10) it is obvious that the temperature $\Theta(x,t)$ will be infinite. Practically, this means that we would achieve an extremely high temperature. Also, in equation (11) we see that since the trigonometric function, $\sin(\frac{n\pi x}{d})$, is bounded, the temperature rise at the instant, $t = 0$, is independent across the position x. So, irrespective of the coordinate we have a great amount of temperature. This can be very useful to evaporate unnecessary water bodies emanating from floods or other reasons.

3. Sudden decrease

Now, let us consider a sudden temperature drop in a localized area. To take this into account, we modify equation (2) by replacing the $\{-\delta(t)\}$ with $\{+\delta(t)\}$, such that we have

$$\alpha \frac{\partial^2 \Theta}{\partial x^2} = \{1 + \delta(t)\} \frac{\partial \Theta}{\partial t}.$$

In this case, proceeding with the same methodology we would have the solution as

$$\Theta(x,t) = \chi_n \sin(\frac{n\pi x}{d}) e^{-\frac{\alpha n^2 \pi^2}{d^2} t - \sigma(t)}.$$

Thus at the instant, $t = 0$ (in the interval, $-\epsilon < t < +\epsilon$), we have

$$\Theta(x,0) = \chi_n \sin(\frac{n\pi x}{d}) e^{-R},$$

where R is again an arbitrarily large number. Therefore, there would be a sudden precipitous localized cooling. This would, for instance, represent an extreme climate scenario or even a cryogenic device. However, we can only draw a qualitative conclusion, as of now. In the case of climate,the problem is very complex, as is well known [8, 9, 10, 11, 12, 13].

4. Discussion

Now, it is known that since (10) is the solution to the modified heat equation (2), one has

$$\Theta(x,t) = \sum_{n=1}^{\infty} \chi_n \sin(\frac{n\pi x}{d}) e^{-\frac{\alpha n^2\pi^2}{d^2}t+\sigma(t)}. \tag{12}$$

Now, if we denote the solution at $t = 0$ as

$$\Theta(x,0) = f(x)$$

then the constants χ_n are given by

$$\chi_n = \frac{2}{d}\int_0^d f(x)\sin(\frac{n\pi x}{d})\mathrm{d}x.$$

However, the most important conclusion of the current paper is embodied in equation (11) of the preceding section. The instant, $t = 0$, is the key feature in the solution of the modified heat equation (2). As we have already mentioned in the preceding section, for all instants of time after the blast, i.e. for $t > 0$, the heat equation and it's solution presume their original form without the singularity.

Our work could have immediate practical results based on what we have shown and also the introduction of the delta function in the parabolic second order differential equation could inspire similar approaches in other cases too.

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