LIE IDEALS WITH SYMMETRIC LEFT BI-DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a prime ring and U be a nonzero lie ideal of R. A symmetric bi-additive mapping $D(.,.) : R \times R \to R$ is called a symmetric bi-derivation and d is a trace of D. In this paper we shall show that $U \subseteq Z(R)$ such that R admitting the trace d satisfying the several conditions of symmetric left bi-derivation.

Keywords: prime ring, symmetric mapping, trace, derivation, symmetric bi-derivation, symmetric bi-additive mapping, symmetric left bi-derivation.

1. Introduction

The concept of a symmetric bi-derivation has been introduced by Maksa.Gy in [5, 6]. A classical result in the theory of centralizing mappings is a theorem first proved by E. Posner [8] which stated that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. Vukman.J [9, 10] has studied some results concerning symmetric bi-derivations on prime and semi prime rings. In [1] Argac, Yenigul and in [7] Muthana obtained the similar type of results on lie ideals of R. In this paper we proved some results in symmetric left bi-derivations in prime rings.

Throughout this paper R will be associative. We shall denote by $Z(R)$ the center of a ring R. Recall that a ring R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$. We shall write $[x, y]$ for $xy - yx$. The symbol $x \circ y$ stands for anti commutator $xy + yx$. An additive map $d : R \to R$ is called

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derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x,\ y \in R$.A mapping $B(.,\ .) : R \times R \to R$ is said to be symmetric if $B(x,\ y) = B(y,\ x)$ holds for all pairs $x,\ y \in R$. A mapping map $f : R \to R$ defined by $f(x) = B(x,\ x)$, where $B(.,\ .) : R \times R \to R$ is a symmetric mapping, is called a trace of B. It is obvious that, in case $B(.,\ .) : R \times R \to R$ is symmetric mapping which is also bi-additive (i. e. additive in both arguments) the trace of B satisfies the relation $f(x+y) = f(x) + f(y) + 2B(x,\ y)$ for all $x,\ y \in R$. We shall use also the fact that the trace of a symmetric bi-additive mapping is an even function. A symmetric bi-additive mapping $D(.,\ .) : R \times R \to R$ is called a symmetric bi-derivation if $D(xy,\ z) = D(x,\ z)y + xD(y,\ z)$ is fulfilled for all $x,\ y,\ z \in R$.Obviously, in this case also the relation $D(x,\ yz) = D(x,\ y)z + yD(x,\ z)$ $x,\ y,\ z \in R$. A symmetric bi-additive mapping $D(.,\ .) : R \times R \to R$ is called a symmetric left bi-derivation if $D(xy,\ z) = xD(y,\ z) + yD(x, z)$ for all $x,\ y,\ z \in R$. Obviously, in this case also the relation $D(x,\ yz) = yD(x,\ z) + zD(x, y)$ for all $x,\ y,\ z \in R$. A mapping $f : R \to R$ is said to be commuting on R if $[f(x),\ x] = 0$ holds for all $x \in R$. A mapping $f : R \to R$ is said to be centralizing on R if $[f(x),\ x] \in Z(R)$ is fulfilled for all $x \in R$. A ring R is said to be n-torsion free if whenever $na = 0$ with $a \in R$ then $a = 0$, where n is nonzero integer.

We shall frequently use the following identities and several well known facts about the semiprime rings without specific mention.

$$
\begin{aligned}
&[xy, z] = x[y, z] + [x, z]y;\\
&[x,\ yz] = y[x,\ z] + [x,\ y]z;\\
&x \circ yz = (x \circ y)z - y[x,\ z] = y(x \circ z) + [x,\ y]z;\\
&xy \circ z = x(y \circ z) - [x,\ z]y = (x \circ z)y + x[y, z].
\end{aligned}
$$

Remark 1. Let U be a square closed lie ideal of R. Notice that $xy + yx = (x+y)^2 - x^2 - y^2$, for all $x,\ y \in U$. Since $x^2 \in U$, for all $x \in U$ $xy + yx \in U$ for all $x,\ y \in U$. Hence we find that $2xy \in U$ for all $x \in U$. Therefore, for all $r \in R$, we get $2r[x,\ y] = 2[x,\ ry] - 2[x,\ r]y \in U$ and $2[x,\ y]r = 2[x,\ ry] - 2[y,\ r]y \in U$ so that $2R[U, U] \subseteq U$ and $2[U, U]R \subseteq U$.

This remark will be freely used in the whole paper without specific reference.

Lemma 1 (4, Corollary 2.1)**.** *Let R be a 2-torsion free semiprime ring, U a Lie ideal of R such that $U \nsubseteq Z(R)$ and $a, b \in U$.*

(i) if $aUa = \{0\}$, then $a = 0$;

(ii) if $aUa = \{0\}$ $(Ua = \{0\})$, then $a = 0$;

(iii) if U is a square closed Lie ideal and $aUb = \{0\}$, then $ab = 0$ and $ba = 0$.

Lemma 2 (1, Theorem 3)**.** *Let R be 2-torsion free prime ring and U be a nonzero Lie ideal of R. Let $B : R \times R \to R$ be a symmetric bi-derivation and f be the trace of B be such that:*

(i) $f(U) = 0$, then $U \subseteq Z(R)$ or $f = 0$;

(ii) $f(U) \subseteq Z(R)$ *and* U *be a square closed Lie ideal, then* $U \subseteq Z(R)$ *or* $f = 0$.

Lemma 3 (3, Lemma 1)**.** *Let* R *be a 2-torsion free semiprime ring and* U *be a Lie ideal of* R. *Suppose that* $[U, U] \subseteq Z(R)$, *then* $U \subseteq Z(R)$.

Lemma 4 (2, Lemma 4)**.** *Let* R *be a 2-torsion free prime ring and* $U \nsubseteq Z(R)$ *be a Lie ideal of* R *and* $a, b \in R$, *if* $aUb = \{0\}$ *then* $a = 0$ *and* $b = 0$.

Lemma 5. *Let* R *be a 2-torsion free prime ring and* U *be a square closed lie ideal of* R. *Suppose that* $D : R \times R \to R$ *is a symmetric left bi-derivation and* d *the trace of* D *such that* $[d(x), y] \in Z(R)$, *for all* $x, y \in U$, *then either* $U \subseteq Z(R)$ *or* $d = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$

(1) We have $[d(x), y] \in Z(R)$, for all $x, y \in U$.

We replace y by $2yz$ in (1), we get

$$[d(x), 2yz] \in Z(R),$$
$$2y[d(x), z] + 2[d(x), y]z \in Z(R),$$
$$y[d(x), z] + [d(x), y]z \in Z(R), \text{ for all } x, y, z \in U.$$

This implies that $[[d(x), y]z + y[d(x), z], r] = 0$, for all $x, y, z \in U$ and $r \in R$

(2) $[d(x), y][z, r] + [y, r][d(x), z] = 0$, for all $x, y, z \in U$ and $r \in R$.

We replacing r by $2yt$ in (2), we get

(3) $[y, z][d(x), z] = 0$ for all $x, y, z \in U$.

We replacing y by $2yt$ in (3), we get

$$[2yt, z][d(x), z] = 0,$$
$$2[y, z]t[d(x), z] + 2y[t, z][d(x), z] = 0,$$
$$[y, z]t[d(x), z] = 0, \text{ for all } x, y, z, t \in U,$$
$$[y, z]U[d(x), z] = 0, \text{ for all } x, y, z, t \in U.$$

Thus in view of Lemma 4 we find that for each pair of $x, y, z \in U$ either $[y, z] = 0$ or $[d(x), z] = 0$ For each $z \in U$, let $A^1 = \{y \in U / [y, z] = 0\}$ and $B^1 = \{x \in U / [f(x), z] = 0\}$. Hence A^1 and B^1 are the additive subgroups of U whose unionis U. By Brauer's trick, we have either $U = A^1$ or $U = B^1$. If $U = A^1$, then $[y, z] = 0$ for all $y, z \in U$ and have $U \subseteq Z(R)$ a contradiction. On the other hand if $U = B^1$ then $[d(x), z] = 0$, for all $x, z \in U$ and hence $f(U) \subseteq C_R(U) = Z(R)$ then by Lemma 2, we get $d = 0$. This completes the proof of the lemma. □

Theorem 6. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D. If $[d(x), x] = 0$, for all $x \in U$, then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$

(4) Since we have given that $[d(x), x] = 0$, for all $x, y \in U$

We replacing x by $x + y$ in (4), we get $d(x + y), x + y] = 0$, $[d(x) + d(y) + 2D(x, y), x + y] = 0$, $[d(x), x] + [d(x), y] + [d(y), x] + [d(y), y] + 2[D(x, y), x] + 2[D(x, y), y] = 0$. By using (4), in the above equation we get

$$[d(x), y] + [d(y), x] + 2[D(x, y), x] + 2[D(x, y), y] = 0 \text{ for all } x, y \in U \tag{5}$$

We replacing x by $-x$ in (5), we get

$$\begin{aligned} &[d(-x), y] + [d(y), -x] + 2[D(-x, y), -x] + 2[D(-x, y), y] = 0, \\ &[d(x), y] - [d(y), x] + 2[D(x, y), x] - 2[D(x, y), y] = 0, \text{ for all } x, y \in U. \end{aligned} \tag{6}$$

By adding (5) and (6), we get

$$[d(x), y] + 2[D(x, y), x] = 0, \text{ for all } x, y \in U. \tag{7}$$

We replacing y by $2yz$ in (7), we get

$$\begin{aligned} &[d(x), 2yz] + 2[D(x, 2yz), x] = 0, \\ &2y[d(x), z] + 2[d(x), y]z + 4[yD(x, z) + zD(x, y), x] = 0, \\ &2y[d(x), z] + 2[d(x), y]z + 4[yD(x, z), x] + 4[zD(x, y), x] = 0, \\ &2y[d(x), z] + 2[d(x), y]z + 4[y, x]D(x, z) + 4y[D(x, z), x] \\ &+ 4[z, x]D(x, y) + 4z[D(x, y), x] = 0, \\ &2y[d(x), z] + 2z[d(x), y] + 4[y, x]D(x, z) + 4y[D(x, z), x] \\ &+ 4[z, x]D(x, y) + 4z[D(x, y), x] = 0, \\ &2y([d(x), z] + 2[D(x, z), x]) + 2z([d(x), y] + 2[D(x, y), x]) \\ &+ 4[y, x]D(x, z) + 4[z, x]D(x, y) = 0. \end{aligned}$$

By using (4) in the above equation we get

$$\begin{aligned} &4[y, x]D(x, z) + 4[z, x]D(x, y) = 0, \\ &[y, x]D(x, z) + [z, x]D(x, y) = 0, \quad \text{for all } x, y, z \in U. \end{aligned} \tag{8}$$

We replace z by x in (8) we get

$$\begin{aligned} &[y, x]D(x, x) + [x, x]D(x, y) = 0, \\ &[y, x]D(x, x) = 0, \text{ for all } x, y \in U. \end{aligned} \tag{9}$$

We replacing y by $2yz$ in (9), we get $[2yz,\ x]\,D\,(x,\ x) = 0$, $2\,[y,\ x]\,zD\,(x,\ x) + 2y\,[z,\ x]\,D\,(x,\ x) = 0$.

By using (9) in the above equation we get $2[y,x]zD(x,x) = 0$, $[y,x]zD(x,x) = 0$, for all $x, y, z \in U$, this gives $[y,x]UD(x,x) = 0$, for all $x, y \in U$. By Lemma 4 for each $x \in U$ either $[y,x] = 0$ or $D(x,x) = 0$, for all $x, y \in U$. In the first case it fallows that by Lemma 3, $x \in Z(R)$ for all $x \in U$. Thus if $x \notin Z(R)$ then $D(x,x) = 0$. Let $x, z \in U$ such that $x \in Z(R)$ and $z \notin Z(R)$. Hence $x+z \notin Z(R)$ and $x - z \notin Z(R)$. Thus $D(x+z, x+z) = 0$ and $D(x-z, x-z) = 0$. Adding the above two relations, we get $2D(x,x) = 0$, since R is 2-torsion free ring, we get $D(x,x) = 0$. Thus for all $x \in U$, $D(x,x) = 0$ and by Lemma 2, $d = 0$. □

Theorem 7. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D such that $d\,([x,\ y]) - [d\,(x)\,,\ y] \in Z(R)$, for all $x,\ y \in U$. Then either $U \subseteq Z(R)$ or $d\ = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have

$$d\,([x,\ y]) - [d\,(x)\,,\ y] \in Z(R),\ \text{for all}\ x,\ y \in U. \tag{10}$$

We replace y by $y + z$ in (10), we get $d([x, y + z]) - [d(x), y + z] \in Z(R)$, $d([x,y]+[x,z])-[d(x),y]-[d(x),z] \in Z(R)$, $d([x,y])+d([x,z])+2D([x,y],[x,z])-[d(x),y] - [d(x),z] \in Z(R)$. By using (10) in the above equation we get

$$D\,([x,\ y]\,,\ [x,\ z]) \in Z(R),\ \text{for all}\ x, y, z \in U \tag{11}$$

We replace z by y in (11), we get $D\,([x,\ y]\,,\ [x,\ y]) \in Z(RD\,([x,\ y]\,,\ [x,\ y]) \in Z(R)$, for all $x,\ y \in U$

$$d\,([x,\ y]) \in Z\,(R)\,,\ \text{for all}\ x,\ y \in U \tag{12}$$

By subtracting (10) from (12) we get $[d\,(x)\,,\ y] \in Z(R)$, for all $x,\ y \in U$ By using Lemma 5, we get the required result. □

Theorem 8. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D such that $d\,(x \circ y) - [d\,(x)\,,\ y] \in Z\,(R)$, for all $x,\ y \in U$. Then either $U \subseteq Z\,(R)$ or $d\ =\ 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z\,(R)$. We have

$$d\,(x \circ y) - [d\,(x)\,,\ y] \in Z(R),\ \text{for all}\ x,\ y \in U. \tag{13}$$

We replace y by $y + z$ in (13), we get, $d\,(x \circ y + z) - [d\,(x)\,,\ y + z] \in Z(R)$, $d\,(x \circ y) + d\,(x \circ z) + 2D\,(x \circ y,\ x \circ z) - [d\,(x)\,,\ y] - [\mathrm{d\ (x)\ ,\ z}] \in Z(R)$. By using (13) in the above equation we get $2D\,(x \circ y,\ x \circ z) \in Z(R)$

$$D\,(x \circ y,\ x \circ z) \in Z(R),\ \ \text{for all}\ x, y, z \in U \tag{14}$$

We replace z by y in (14), we get $D(x \circ y,\ x \circ y) \in Z(R)$, for all $x,\ y \in U$

$$d(x \circ y) \in Z(R), \text{ for all } x,\ y \in U. \tag{15}$$

By subtracting (13) from (15), we get $[d(x),\ y] \in Z(R)$, for all $x,\ y \in U$. By using Lemma 5, we get the required result. □

Theorem 9. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D such that $d(x) \circ y - [d(x),\ y] \in Z(R)$, for all $x,\ y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \not\subseteq Z(R)$. We have

$$d(x) \circ y - [d(x),\ y] \in Z(R), \text{ for all } x,\ y \in U. \tag{16}$$

$d(x)y + yd(x) - d(x)y + yd(x) \in Z(R)$, $2yd(x) \in Z(R)$, $yd(x) \in Z(R)$, for all $x,\ y \in U$, $[yd(x),\ r] = 0$, for all $x,\ y \in U$ and $r \in R$.

$$y[d(x),\ r] + [y,\ r]d(x) = 0, \text{ for all } x,\ y \in U \text{ and } r \in R. \tag{17}$$

We replace y by $2yt$ in (17), we get $2ty[d(x),\ r]+[2ty,\ r]d(x) = 0$, $2ty[d(x),\ r]+ 2t[y,\ r]d(x) + 2[t,\ r]yd(x) = 0$. By using (17) in the above equation we get $[t,\ r]yd(x) = 0$, for all $x,\ y,\ t \in U$ and $r \in R$, $[t,\ r]Ud(x) = 0$, for all $x,\ t \in U$ and $r \in R$. By using Lemma 4 we get either $[t,\ r] = 0$ or $d(x) = 0$, for all $x,\ t \in U$ and $r \in R$. If $[t,\ r] = 0$ then $U \subseteq Z(R)$ a contradiction. Hence if $d(x) = 0$ for all $x \in U$, then by Lemma 2, we get $d = 0$. □

Theorem 10. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R\times R \to R$ is a symmetric left bi-derivation and d the trace of D and $g : R \to R$ is any mapping such that $[d(x),\ y] - [x,\ g(x)] \in Z(R)$, for all $x,\ y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \not\subseteq Z(R)$. We have

$$[d(x),\ y] - [x,\ g(y)] \in Z(R), \text{ for all } x,\ y \in U. \tag{18}$$

We replace x by $x + z$ in (18), we get $[d(x+z),\ y] - [x+z,\ g(y)] \in Z(R)$

$$[d(x),\ y] + [d(z),\ y] + 2[D(x,\ z),\ y] - [x,\ g(y)] - [z,\ g(y)] \in Z(R)$$

By using (18) in the above equation we get $2[D(x,\ z),\ y] \in Z(R)$

$$[D(x,\ z),\ y] \in Z(R) \text{ for all } x, y, z \in U \tag{19}$$

We replace z by x in (19), we get $[D(x,\ x),\ y] \in Z(R)$, for all $x,\ y \in U$, $[d(x),\ y] \in Z(R)$, for all $x,\ y \in U$. Hence by Lemma 3, we get the required result. □

Theorem 11. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D and $g : R \to R$ is any mapping such that $d(x) \circ d(y) - [d(x), y] \in Z(R)$, for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have

$$d(x) \circ d(y) - [d(x), y] \in Z(R), \text{ for all } x, y \in U. \tag{20}$$

We replace y by $y + z$ in (20), we get $d(x) \circ d(y + z) - [d(x), y + z] \in Z(R)$, $(x) \circ d(y) + d(x) \circ d(z) + 2d(x) \circ D(y, z) - [d(x), y] - [d(x), z] \in Z(R)$, for all $x, y, z \in U$. By using (20) in the above equation we get $2d(x) \circ D(y, z) \in Z(R)$

$$d(x) \circ D(y, z) \in Z(R) \tag{21}$$

We replace z by y in (21), we get $d(x) \circ D(y, y) \in Z(R)$

$$d(x) \circ d(y) \in Z(R), \text{ for all } x, y \in U \tag{22}$$

By subtracting (20) from (22), we get $[d(x), y] \in Z(R)$, for all $x, y \in Z(R)$. Thus by using Lemma 1, we get the required result. □

Theorem 12. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D and $g : R \to R$ be any mapping such that $d(x) y - xg(y) \in Z(R)$, for all $x, y \in U$, then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have

$$d(x) y - xg(y) \in Z(R), \text{ for all } x, y \in U \tag{23}$$

We replace x by $x + z$ (23), we get $d(x + z) y - (x + z) g(y) \in Z(R)$

$$d(x) y + d(z) y + 2D(x, z) y - xg(y) - zg(y) \in Z(R) \text{ for all } x, y, z \in U. \tag{24}$$

By using (23) in (24), we get $2D(x, z) y \in Z(R)$

$$D(x, z) y \in Z(R), \text{ for all } x, y, z \in U. \tag{25}$$

We replace z by x in (25), we get $D(x, x) y \in Z(R)$, $d(x) y \in Z(R)$, for all $x, y \in U$

$$[d(x) y, r] = 0, \text{ for all } x, y \in U \text{ and } r \in R \tag{26}$$

We replace y by $2yt$ in (26), we get $[d(x)2yt, r] = 0$, $2[d(x)y, r]t + 2d(x)y[t, r] = 0$. By using (26) in the above equation, we get $2d(x)y[t, r] = 0$, $2d(x)y[t, r] = 0$, for all $x, y \in U$ and $r \in R$, $2d(x)U[t, r] = 0$, for all $x, y, t \in U$ and $r \in R$, $d(x)U[t, r] = 0$, for all $x, t \in U$ and $r \in R$.

By using Lemma 4, we get either $[t, r] = 0$ or $d(x) = 0$ for all $x, t \in U$ and $r \in R$.

If $[t, r] = 0$ then $U \subseteq Z(R)$ a contradiction. Hence if $d(x) = 0$, for all $x \in U$, then by Lemma 2, we get $d = 0$. □

Theorem 13. *Let R be a 2-torsion free prime ring and U be a square closed lie ideal of R. Suppose that $D : R \times R \to R$ is a symmetric left bi-derivation and d the trace of D such that $d(x)y - xg(y) \in Z(R)$, for all $x, y \in U$, then either $U \subseteq Z(R)$ or $d = 0$.*

Proof. Suppose on the contrary that $U \not\subseteq Z(R)$. We have

$$d(xy) - d(x)y - xd(y) \in Z(R). \tag{27}$$

We replace x by $x+z$ in (27), we get $d((x+z)y) - d(x+z)y - (x+z)d(y) \in Z(R)$, $d(xy+zy) - d(x+z)y - (x+z)d(y) \in Z(R)$, $d(xy) + d(zy) + 2D(xy, zy) - d(x)y - d(z)y - 2D(x, z)y - xd(y) - zd(y) \in Z(R)$, for all $x, y, z \in U$.

By using (27) in the above equation, we get $2D(xy, zy) - 2D(x, z)y \in Z(R)$,

$$D(xy, zy) - D(x, z)y \in Z(R), \text{ for all } x, y, z \in U. \tag{28}$$

We replace z by x in (28), we get $D(xy, xy) - D(x, x)y \in Z(R)$,

$$d(xy) - d(x)y \in Z(R), \text{ for all } x, y \in U. \tag{29}$$

We replace y by $y+z$ in (29), we get $d(x(y+z)) - d(x)(y+z) \in Z(R)$, $d(xy+xz) - d(x)(y+z) \in Z(R)$, $d(xy) + d(xz) + 2B(xy, xz) - d(x)y - d(x)z \in Z(R)$, for all $x, y, z \in U$.

By using (29) in the above equation, we get

$$B(xy, xz) \in Z(R), \text{ for all } x, y, z \in U. \tag{30}$$

We replace z by y in (30), we get $D(xy, xy) \in Z(R)$,

$$d(xy) \in Z(R), \text{ for all } x, y \in U. \tag{31}$$

By subtracting (29), from (31), we get $d(x)y \in Z(R)$, for all $x, y \in Z(R)$, $[d(x)y, r] = 0$, for all $x, y, z \in U$ and $r \in R$

$$[d(x), r]y + d(x)[y, r] = 0, \text{ for all } x, y, z \in U \text{ and } r \in R. \tag{32}$$

We replace r by $d(x)$ in (32), we get $[d(x), d(x)]y + d(x)[y, d(x)] = 0$,

$$d(x)[y, d(x)] = 0. \tag{33}$$

We replace y by $2yz$ in (33), we get $d(x)[2yz, d(x)] = 0$, $2d(x)[y, d(x)] + 2d(x)y[z, d(x)] = 0$. By using (33) in the above equation we get $2d(x)y[z, d(x)] = 0$.

$$d(x)y[z, d(x)] = 0, \text{ for all } x, y, z \in U. \tag{34}$$

Multiplying (34) left by z we get

$$zd(x)y[z, d(x)] = 0 \text{ for all } x, y, z \in U. \tag{35}$$

We replace y by $2zy$ in (34), we get $d(x)\,2zy\,[z,\ d(x)] = 0$

$$d(x)\,zy\,[z,\ d(x)] = 0, \text{ for all } x,\ y,\ z \in U. \tag{36}$$

By combining (35) and (36), we get $[z,\ d(x)]y[z,\ d(x)] = 0$, $[z,\ d(x)]U[z,\ d(x)] = \{0\}$. By using Lemma 1, we get $[z,\ d(x)] = 0$, for all $x,\ z \in U$ and by Lemma 5, we get $d = 0$. □

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