SOME NEW PROPERTIES ON λ-COMMUTING OPERATORS

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Abstract. In this paper, we study the operator equation $AB = \lambda BA$ for a bounded linear operators A, B on a complex Hilbert space. We focus on algebraic relations between different operators that include normal, M-hyponormal, quasi $*$-paranormal and other classes.
Keywords: Hilbert space, λ-commute, binormal, M-hyponormal, isometry, k-paranormal, quasi $*$-paranormal.

1. Introduction

Throughout, we will denote by $\mathcal{B}(\mathcal{H})$ the complex Banach algebra of all bounded linear operators on a infinite dimensional complex Hilbert space $\mathcal{H}$. We denote the range and the kernel of $A \in \mathcal{B}(\mathcal{H})$ by $R(A)$ and $N(A)$ respectively.
Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be:

- positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$
- self-adjoint if $A = A^*$

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- isometry if $A^*A = I$, which equivalent to the condition $\|Ax\| = \|x\|$ for all $x \in \mathcal{H}$
- normal if $A^*A = AA^*$
- unitary $A^*A = AA^* = I$ (i.e. A is an onto isometry)
- quasinormal if $A(A^*A) = (A^*A)A$
- binormal if $(A^*A)(AA^*) = (AA^*)(A^*A)$ [3]
- subnormal if A has a normal extension
- hyponormal if $A^*A \geq AA^*$, which equivalent to the condition $\|A^*x\| \leq \|Ax\|$ for all $x \in \mathcal{H}$ [15]
- M-hyponormal if $A^*A \geq MAA^*$, where $M \in \mathbb{R}$ and $M \geq 1$ which equivalent to the condition $\|A^*x\| \leq M\|Ax\|$ for all $x \in \mathcal{H}$ [20]
- p-hyponormal if $(A^*A)^p \leq (AA^*)^p$, where $0 < p \leq 1$ [1]
- class $\mathcal{A}$ if $|A|^2 \leq |A^2|$, where $|A| = (A^*A)^{\frac{1}{2}}$
- paranormal if $\|Ax\|^2 \leq \|A^2x\|\|x\|$ for all $x \in \mathcal{H}$ [4]
- k-paranormal if $\|Ax\|^k \leq \|A^kx\|\|x\|^{k-1}$ for all $x \in \mathcal{H}$ and $k \geq 2$
- $*$-paranormal if $\|A^*x\|^2 \leq \|A^2x\|\|x\|$ for all $x \in \mathcal{H}$ [10]
- quasi $*$-paranormal if $\|A^*Ax\|^2 \leq \|A^3x\|\|Ax\|$ for all $x \in \mathcal{H}$ [12]
- log-hyponormal if A invertible and satisfies $\log(A^*A) \geq \log(AA^*)$ [16]
- p-quasihyponormal if $A^*[(A^*A)^p - (AA^*)^p]A \geq 0$, where $0 < p \leq 1$ [2]
- normoloid if $\|A\| = r(A)$
- quasinilpotent if $r(A) = 0$, where $r(A) = \lim \|A^n\|^{\frac{1}{n}}$.

We can notice that A is hyponormal if A is p-hyponormal with $p = 1$. By Löwner-Heinz inquality p-hyponormal is q-hyponormal for every $0 < q \leq p \leq 1$ [14]. Also we can notice that A is paranormal if A is k-paranormal with $k = 2$. It known that invertible p-hyponormal is log-hyponormal. We can consider log-hyponormal operator as 0-hyponormal [16]. It is well known that for any operators A, B and C we have

$$A^*A - 2\lambda B^*B + \lambda^2 C^*C \geq 0 \,\forall\, \lambda > 0 \;\Leftrightarrow\; ||Bx||^2 \leq ||Ax||||Cx|| \,\text{for all}\, x \in H.$$

Thus we have

- A is quasi $*$-paranormal if and only if $A^*[(A^*)^2A^2 - 2\lambda AA^* + \lambda^2]A \geq 0$ for all $\lambda > 0$.

- A is $*$-paranormal if and only if $(A^*)^2A^2 - 2\lambda AA^* + \lambda^2 \geq 0$ for all $\lambda > 0$.

We have also the following inclusions:

- quasinormal $\subseteq$ binormal

- class $\mathcal{A}$ $\subseteq$ paranormal

- hyponormal $\subseteq$ *-paranormal $\subseteq$ quasi $*$-paranormal

- invertible p-hyponormal $\subseteq$ log-hponormal $\subseteq$ paranormal.

- self-adjoint $\subseteq$ normal $\subseteq$ quasinormal $\subseteq$ subnormal $\subseteq$ hyponormal

- hyponormal $\subseteq$ p-hyponormal $\subseteq$ p-quasihyponormal $\subseteq$ class $\mathcal{A}$.

For a scalar λ, two operators A and B in $\mathcal{B}(\mathcal{H})$ are said be λ-commute if $AB = \lambda BA$. Recently many authors have studied this equation for several classes of operators, for example:

- In [11] the authors have proved that if an operator in $\mathcal{B}(\mathcal{H})$ λ-commutes with a compact, then this operator has a non-trivial hyperinvariant subspace.

- In [8] Conway and Prajitura characterized the closure and the interior of the set of operators that λ-commute with a compact operator.

- In [19] Zhang ,Ohawada and Cho have studied the properties of an operator λ-commutes with a paranormal.

- In [5] Brooke, Busch and Pearson showed that if AB is not quasinilpotent, then $|\lambda| = 1$, and if A or B is self-adjoint then $\lambda \in \mathbb{R}$.

- In [18] Yang and Du gave a simple proofs and generalizations of this results, particulary if AB is bounded below if and only if both A and B are bounded below.

- In [14] Schmeger generalized this results to hermitian or normal elements of a complex Banach algebra.

- In [6] Cho, Duggal, Harte and Ota generalized some Schmeger's results.

The aim of this paper is to study the situation for binormal, M-hyponormal, quasi $*$-paranormal operators. Again other related results are also given.

2. Main results

We begin with the following result.

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be quasi $*$-paranormal. If A is quasinilpotent, then $A = 0$.*

Proof. Let $A \in \mathcal{B}(\mathcal{H})$ be quasi $*$-paranormal, then we have

$$\|A^*Ax\| = \|A^3x\|^2\|Ax\|^2 \text{ for all } x \in \mathcal{H}.$$

Therefore $\|Ax\|^4 = \langle A^*Ax, x\rangle^2 \leq \|A^*Ax\|^2\|x\|^2 \leq \|A^3x\|\|Ax\|\|x\|^2$.

Thus $\|Ax\|^3 \leq \|A^3x\|\|x\|^2$ for all $x \in \mathcal{H}$, whence A is 3-paranormal. By [17, Lemma 1], then every k-paranormal is normaloid. Thus we conclude that A is normaloid and hence $r(A) = \|A\|$. On the other hand A is quasinilpotent, then we obtain $\|A\| = r(A) = 0$. Therefore $A = 0$. □

Corollary 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be $*$-paranormal.*
If A is quasinilpotent, then $A = 0$.

Proof. By Lemma 2.1 and since every $*$-paranormal is also quasi $*$-paranormal. □

Theorem 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$, A is quasinormal and B is normal. If $|\lambda| = 1$, then AB is quasinormal.*

Proof. Assume that $AB = \lambda BA \neq 0$, then $B^*A^* = \bar{\lambda}A^*B^*$. Since B and λB are normal operators and by Fuglede-Putnam Theorem, then $BA^* = \lambda A^*B$ and $AB^* = \bar{\lambda}B^*A$. Moreover we have

$$\begin{aligned}
AB[(AB)^*AB] &= [AB][B^*A^*AB] \\
&= [\lambda BA]B^*A^*AB \\
&= \lambda B[AB^*]A^*AB \\
&= \lambda B[\bar{\lambda}B^*A]A^*AB \\
&= |\lambda|^2[BB^*][AA^*A]B \\
&= [B^*B][A^*AA]B \\
&= B^*[BA^*]AAB \\
&= B^*[\lambda A^*B]AAB \\
&= B^*A^*[\lambda BA]AB \\
&= B^*A^*[AB]AB \\
&= [(AB)^*AB]AB.
\end{aligned}$$

Therefore AB is quasinormal. □

Theorem 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$, A is binormal and B is normal. If $|\lambda| = 1$, then AB is binormal.*

Proof. Since B and λB are normal operators and by Fuglede-Putnam Theorem, then we have $BA^* = \lambda A^* B$ and $AB^* = \bar{\lambda} B^* A$. Therefore we obtain

$$\begin{aligned}
AB(AB)^*(AB)^*AB &= A[BB^*]A^*B^*A^*AB \\
&= A[B^*B]A^*B^*A^*AB \\
&= [AB^*]BA^*[B^*A^*]AB \\
&= [\bar{\lambda}B^*A]BA^*[\bar{\lambda}A^*B^*]AB \\
&= (\bar{\lambda})^2 B^*[AB]A^*A^*[B^*A]B \\
&= (\bar{\lambda})^2 B^*[\lambda BA]A^*A^*[\frac{1}{\bar{\lambda}}AB^*]B \\
&= |\lambda|^2 B^*B[AA^*A^*A]B^*B \\
&= B^*B[A^*AAA^*]B^*B \\
&= B^*[BA^*]AA[A^*B^*]B \\
&= B^*[\lambda A^*B]AA[\frac{1}{\bar{\lambda}}B^*A^*]B \\
&= \frac{\lambda}{\bar{\lambda}}B^*A^*[BA]AB^*[A^*B] \\
&= \lambda^2 B^*A^*[\frac{1}{\lambda}AB]AB^*[\frac{1}{\lambda}BA^*] \\
&= B^*A^*ABA[B^*B]A^* \\
&= B^*A^*ABA[BB^*]A^* \\
&= (AB)^*ABAB(AB)^*,
\end{aligned}$$

then AB is binormal. □

Theorem 2.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$.*

Suppose that A is k-paranormal and B is isometry, then the following statements are equivalent:

1. *AB is k-paranormal*
2. *$\sigma(AB) \neq \{0\}$*
3. *$|\lambda| = 1$.*

Proof. Suppose that A is k-paranormal and B is isometry with $AB = \lambda BA \neq 0$.

We first show that (1) ⇒ (2). Suppose that AB is k-paranormal.

If AB is quasinilpotent ($\sigma(AB) = \{0\}$). Since every k-paranormal is isometry, then we obtain $\|AB\| = r(AB) = 0$ and hence $AB = 0$ and this is a contradiction with $AB \neq 0$. Therefore AB is not quasinilpotent and hence $\sigma(AB) \neq \{0\}$.

We prove that (2) ⇒ (3). Suppose that $\sigma(AB) \neq \{0\}$, then

$$r(AB) \neq 0. \tag{1}$$

Since $AB = \lambda BA \neq 0$ and by [5, Proposition 1], then $\sigma(AB) = \sigma(BA) = \lambda\sigma(AB)$. Hence

$$r(AB) = |\lambda| r(AB). \tag{2}$$

Therefore by (1) and (2) we obtain $|\lambda| = 1$. Finally we show that (3) $\Rightarrow$ (1). Suppose that $|\lambda| = 1$, for any unit vector $x \in \mathcal{H}$ we have

$$\begin{aligned} \|(AB)x\|^k &= \|A(Bx)\|^k \\ &\leq \|A^k(Bx)\|\|Bx\|^{k-1} \quad (A\,\text{is}\ k-\text{paranormal}) \\ &\leq \|A^kBx\| \qquad (B\,\text{is isometry}). \end{aligned}$$

Hence

$$\|(AB)x\|^k \leq \|A^kBx\|. \tag{3}$$

On the other hand by induction we show that $(AB)^k = \lambda^{\frac{k(k-1)}{2}} B^{k-1}A^kB$ for every $k \in \mathbb{N}^*$. For $k = 1$ we have $(AB)^1 = \lambda^{\frac{1(1-1)}{2}} B^{1-1}A^1B$. Assume that $(AB)^k = \lambda^{\frac{k(k-1)}{2}} B^{k-1}A^kB$ for $k \geq 2$. Finally we have

$$\begin{aligned} (AB)^{k+1} = AB(AB)^k &= (\lambda BA)(\lambda^{\frac{k(k-1)}{2}} B^{k-1}A^kB) \\ &= \lambda^{\frac{k(k-1)}{2}+1} BAB^{k-1}A^kB \\ &= \lambda^{\frac{k(k-1)}{2}+1} B(AB)B^{k-2}A^kB \\ &= \lambda^{\frac{k(k-1)}{2}+1} B(\lambda BA)B^{k-2}A^kB \\ &= \lambda^{\frac{k(k-1)}{2}+2} B^2AB^{k-2}A^kB \\ &\vdots \\ &\vdots \\ &\vdots \\ &= \lambda^{\frac{k(k-1)}{2}+k} B^kAB^{k-k}A^kB \\ &= \lambda^{\frac{(k+1)k}{2}} B^kA^{k+1}B. \end{aligned}$$

We conclude that $(AB)^k = \lambda^{\frac{k(k-1)}{2}} B^{k-1}A^kB$, for every $k \in \mathbb{N}^*$. Then for every unit vector $x \in \mathcal{H}$ we obtain

$$\begin{aligned} \|(AB)^kx\| &= \|\lambda^{\frac{k(k-1)}{2}} B^{k-1}A^kBx\| \\ &= |\lambda|^{\frac{k(k-1)}{2}} \|B^{k-1}A^kBx\| \\ &= \|A^kBx\| \qquad (B^{k-1}\ \text{is isometry and}\ |\lambda| = 1). \end{aligned}$$

Hence

$$\|(AB)^kx\| = \|A^kBx\| \ \text{for any unit vector}\ x. \tag{4}$$

Finally by (3) and (4) we conclude that $\|(AB)x\|^k \leq \|A^kBx\| = \|(AB)^kx\|$, for any unit vector x. Therefore AB is k-paranormal. □

Theorem 2.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$. Then*

1. *if A^* is M_1-hyponormal and B is M_2-hyponormal, then $|\lambda| \leq (M_1M_2)^{\frac{1}{2}}$*
2. *if A is M_1-hyponormal and B^* is M_2-hyponormal, then $|\lambda| \geq (M_1M_2)^{-\frac{1}{2}}$.*

Proof. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$.

1. Since we have

$$\begin{aligned}|\lambda|\|BA\| &= \|\lambda BA\| \\ &= \|AB\| \\ &= \|B^*A^*AB\|^{\frac{1}{2}} \quad (\|T\| = \|TT^*\|^{\frac{1}{2}} \\ &\leq M_1^{\frac{1}{2}}\|B^*AA^*B\|^{\frac{1}{2}} \ (A^* \text{ is } M_1-\text{hyponormal}: \ A^*A \leq M_1AA^*) \\ &\leq M_1^{\frac{1}{2}}\|A^*B\| \ (\|T^*T\|^{\frac{1}{2}} = \|T\|) \\ &\leq M_1^{\frac{1}{2}}\|A^*BB^*A\|^{\frac{1}{2}} \quad (\|T\| = \|TT^*\|^{\frac{1}{2}}) \\ &\leq (M_1M_2)^{\frac{1}{2}}\|A^*B^*BA\|^{\frac{1}{2}} \ (B \text{ is } M_2-\text{hyponormal: } BB^* \leq M_2B^*B) \\ &\leq (M_1M_2)^{\frac{1}{2}}\|BA\| \quad (\|T^*T\|^{\frac{1}{2}} = \|T\|).\end{aligned}$$

Therefore $|\lambda|\|BA\| \leq (M_1M_2)^{\frac{1}{2}}\|BA\|$ Hence $|\lambda| \leq (M_1M_2)^{\frac{1}{2}}$.

2. Since $AB = \lambda BA$ and $\lambda \neq 0$, then $BA = \lambda^{-1}AB$ and by first implication we obtain $|\lambda^{-1}| \leq (M_2M_1)^{\frac{1}{2}}$ and hence $|\lambda| \geq (M_2M_1)^{-\frac{1}{2}}$.

□

Corollary 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$. Then*

1. *if A^* and B are hyponormal, then $|\lambda| \leq 1$*
2. *if A and B^* are hyponormal, then $|\lambda| \geq 1$.*

Proof. By Theorem 2.4 and we take $M_1 = M_2 = 1$. □

Theorem 2.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$.*
If A^* is M_1-hyponormal and B is M_2-hyponormal, then A^*B and BA^* are $M_1M_2|\lambda|^2$-hyponormal.

Proof. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$. Then

$$\begin{aligned}(A^*B)^*A^*B &= B^*AA^*B \\ &\geq M_1B^*A^*AB \\ &\geq M_1\bar{\lambda}A^*B^*\lambda BA \\ &\geq M_1|\lambda|^2A^*B^*BA \\ &\geq M_1|\lambda|^2A^*M_2BB^*A \\ &\geq M_1M_2|\lambda|^2(B^*A)^*B^*A.\end{aligned}$$

Therefore A^*B is $M_1M_2|\lambda|^2$-hyponormal.

In the same way we obtain BA^* is $M_1M_2|\lambda|^2$-hyponormal. □

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