THE FORM OF AUTOMORPHISMS OF AN ABELIAN GROUP HAVING THE WEAK EXTENSION PROPERTY

S. Abdelalim
Laboratory of Topology, Algebra, Geometry and Discrete Mathematics
Departement of Mathematical and Computer Sciences
Faculty of Sciences Ain Chok
University of Hassan II Casablanca
BP 5366 Maarif, Casablanca
Morocco
seddikabd@hotmail.com

M. Zeriouh*
Department of Mathematics
CRMEF Fez-Meknes, BP. 49, Fez
Morocco
zeriouhmostafa@gmail.com

M. Ziane
Department of Mathematical and Computer
University of Mohamed first
BP.717 60000, Oujda
Morocco
ziane12001@yahoo.fr

Abstract. Let A be an abelian group and let α be an automorphism of A. In this paper we show that if the restriction of α to any p-component A_p of A is of the form: $\alpha_{\restriction A_p} = \pi id_{A_p} + \rho$, where p is a prime number, π a p-adic invertible number and $\rho \in Hom(A_p, A^1)$ with A^1 is the first subgroup Ulm of the group A. Then α satisfies the weak extension property.

Keywords: abelian groups, p-groups, torsion groups, automorphism group.

1. Introduction

The study of the characterization of automorphisms having the property of extension had begun by P. E Schupp showed, in [12], that the extension property in the category of groups, characterizes the inner automorphisms. M. R. Pettet gives in, [10], a simpler proof of Schupp's result and shows that the inner automorphisms of a group are also characterized by the lifting property in the category of groups. In [8] M. Dugas and R. Gobel gave another simpler proof of Schupp's result, using only the elementary theory of groups. In [2] L. Ben Yakoub shows that the result of Schupp is not valid in general for algebras on a

*. Corresponding author

commutative ring. It is not yet known whether this result holds true for algebras (of finite dimensions) on a commutative field. In [3] L. Ben Yakoub and M. P. Malliavin show that the property of extension also characterizes derivations in associative algebras for some algebras quantum properties. In this article, we will define the property of the weak extension by:

An automorphism α of an abelian group A has the weak extension property if for all abelian group B for all monomorphism $\lambda : A \longrightarrow B$ and if there exists an element $m \in \mathbb{N}^*$ such that the restriction of λ to mA is an isomorphism from mA to mB, then there exists $\tilde{\alpha} \in Aut(B)$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \stackrel{\lambda}{\longrightarrow} & B \\ \alpha \downarrow & & \downarrow \tilde{\alpha} \\ A & \stackrel{\lambda}{\longrightarrow} & B \end{array}$$

In other words: $\tilde{\alpha}\lambda = \lambda\alpha$. By way of example, any automorphism of an abelian group without torsion possesses the property of the weak extension (see [1], [16]).

2. Main result

Theorem 2.1. *Let A be an abelian group and let α be an automorphism of A.*

If the restriction of α to any p-component A_p of A is of the form: $\alpha_{\restriction A_p} = \pi id_{A_p} + \rho$, where p is a prime number, π a p-adic invertible number and $\rho \in Hom(A_p, A^1)$ with A^1 is the first subgroup Ulm of A. Then α satisfies the weak extension property.

Before giving proof of this theorem, we will need certain results.

Lemma 2.2. *Let T_A be the torsion part of A. If α_1 is the restriction of α to T_A then α_1 is an automorphism of T_A. Moreover, α_1 satisfies the weak extension property.*

Proof. Let $x \in T_A$. There exists $n \in \mathbb{N}^*$ such that $nx = 0$, whence $n\alpha(x) = \alpha(nx) = 0$, we deduce that $\alpha_1(T_A) \subseteq T_A$.

On the other hand, $\forall y \in T_A$, there exists $x \in T_A$ such that $y = \alpha(x)$; If $0 = my = m\alpha(x)$ for some $m \in \mathbb{N}^*$, then $\alpha(mx) = 0$ implies $mx = 0$, then $x \in T_A$. We conclude that $\alpha(T_A) = T_A$, consequently $\alpha_1 = \alpha_{\restriction T_A}$ is an automorphism of T_A. Let $(T_A)_p$ be the p-component of the torsion group T_A. From the above assumptions, $\alpha_{1\restriction (T_A)_p} = \pi\, id_{(T_A)_p} + \rho$ where π is an invertible p-adic number and $\rho \in Hom((T_A)_p, (T_A)_p^1)$ with $(T_A)_p^1$ the first subgroup Ulm of the group $(T_A)_p$. Therefore, according to the characterization of the automorphisms possessing the weak extension property in the category of torsion abelian groups see, [15], we deduce that $\alpha_1 = \alpha_{\restriction T_A}$ satisfies the weak extension property.

Proposition 2.3. *Let $\lambda : A \to A'$ be a monomorphism of abelian groups and let $\lambda_{\restriction m_0 A}$ be the restriction of λ to $m_0 A$ such that $\lambda_{\restriction m_0 A} \in Isom(m_0 A; m_0 A')$*

where $m_0 \in \mathbb{N}^*$. If T_A and $T_{A'}$ are respectively the torsion parts of A and A'. So:

(i) $\lambda_1 : T_A \to T_{A'}$ is a monomorphism.

(ii) There exists an automorphism α' of $T_{A'}$ which makes switch the following diagram:

$$\begin{array}{ccc} T_A & \xrightarrow{\lambda_1} & T_{A'} \\ \alpha_1 \downarrow & & \downarrow \alpha' \\ T_A & \xrightarrow{\lambda_1} & T_{A'} \end{array}$$

Proof. (i) It suffices to prove that: $\lambda(T_A) \subseteq T_{A'}$.

Let $a \in T_A$; There exists $m \in \mathbb{N}^*$ such that $ma = 0$, hence $m\lambda(a) = 0$, we deduce that $\lambda(T_A) \subseteq T_{A'}$.

(ii) From the assumptions we have: $m_0 A \simeq m_0 A'$ where $m_0 \in \mathbb{N}^*$.

The proposition 8.37 (see [18], p: 295) shows that for $m_0 \in \mathbb{N}^*$: $T(m_0 A) \simeq T(m_0 A')$, hence $m_0 T_A \simeq m_0 T_{A'}$, so for some $m_0 \in \mathbb{N}^*$: $\lambda_{1 \restriction m_0 T_A} \in Isom(m_0 T_A; m_0 T_{A'})$ and since the automorphism $\alpha_1 \in Aut(T_A)$ satisfies the weak extension property, then there exists an automorphism $\alpha' \in T_{A'}$ such that $\alpha' \lambda_1 = \lambda_1 \alpha_1$.

Lemma 2.4. *Let A' be an abelian group and $\lambda : A \to A'$ an monomorphism. If $\lambda(A) = A_1$ and if $\alpha_2 = \lambda\alpha\lambda^{-1}$, then, α_2 is an automorphism of A_1 which makes switch the following diagram:*

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A_1 \\ \alpha \downarrow & & \downarrow \alpha_2 \\ A & \xrightarrow{\lambda} & A_1 \end{array}$$

Proof. Since $\lambda : A \to A'$ is a monomorphism and $\lambda(A) = A_1$, then, $\lambda : A \to A_1$ is an isomorphism, consequently, $\alpha_2 = \lambda\alpha\lambda^{-1}$ is an automorphism of A_1 and we have: $\alpha_2 \lambda = \lambda\alpha$.

Proposition 2.5. *Let $\lambda : A \to A'$ be a monomorphism of abelian groups and let $\lambda_{\restriction m_0 A}$ be the restriction of λ to $m_0 A$ such that $\lambda_{\restriction m_0 A} \in Isom(m_0 A; m_0 A')$ where $m_0 \in \mathbb{N}$. If T_A and $T_{A'}$ are respectively the torsion parts of A and A'; So:*

1. *$A' = A_1 + T_{A'}$;*

2. *$A_1 \cap T_{A'} = T_{A_1}$;*

3. *$\lambda(T_A) = T_{A_1}$.*

Proof. 1) Since $\lambda : A \to A'$ is a monomorphism and $\lambda(A) = A_1$, then $\lambda : A \to A_1$ is an isomorphism.

Hence $\lambda(m_0 A) = m_0 \lambda(A) = m_0 A_1$. And since $\lambda_{\restriction m_0 A} \in Isom(m_0 A; m_0 A')$. So $\lambda(m_0 A) = m_0 A' = m_0 A_1$. Let $x \in A'$, hence $m_0 x \in m_0 A' = m_0 A_1$ which implies that there exists $a_1 \in A_1 \subset A'$ such that $m_0 x = m_0 a_1$. Therefore

$m_0(x - a_1) = 0$ and consequently $x - a_1 \in T_{A'}$. And since $x = a_1 + x - a_1$ with $a_1 \in A_1$ and $x - a_1 \in T_{A'}$ therefore $A' \subset A_1 + T_{A'}$. On the other hand $A_1 \subset A'$ and $T_{A'} \subset A'$ therefore $A_1 + T_{A'} \subset A'$. We conclude that $A' = A_1 + T_{A'}$.

2) Since $A_1 \subset A'$, then $T_{A_1} \subset T_{A'}$; of plus $T_{A_1} \subset A_1$, we deduce that $T_{A_1} \subset A_1 \cap T_{A'}$. On the other hand, for all $x \in A_1 \cap T_{A'}$, then, $x \in T_{A'}$; There exists $n \in \mathbb{N}^*$ such that $nx = 0$. Since $x \in A_1$, therefore $x \in T_{A_1}$, consequently, $A_1 \cap T_{A'} = T_{A_1}$.

3) Let $x \in T_A$; There exists $m \in \mathbb{N}^*$ such that $mx = 0$, whence $0 = \lambda(mx) = m\lambda(x)$, so $\lambda(x) \in T_{A_1}$, we deduce that $\lambda(T_A) \subset T_{A_1}$. Now either $x \in T_{A_1}$; There exists $m \in \mathbb{N}^*$ such that $mx = 0$. Since $T_{A_1} \subset A_1 = \lambda(A)$, then $x = \lambda(a)$ where $a \in A$. So $0 = mx = m\lambda(a) = \lambda(ma)$. Thus $ma = 0$, hence $a \in T_A$ which implies that $x \in \lambda(T_A)$. It is concluded that $\lambda(T_A) = T_{A_1}$.

3. The proof of theorem 2.1

Let A be an abelian group and let α be an automorphism of A.

Let A' be an abelian group and $\lambda : A \to A'$ a monomorphism.

T_A, T_{A_1} and $T_{A'}$ are Respectively the torsion parts of A, A_1 and A'.

We define the endomorphism α_3 of the group A' by: $(\alpha_3)_{\restriction_{A_1}} = \alpha_2$ and $(\alpha_3)_{\restriction_{T_{A'}}} = \alpha'$.

The endomorphism α_3 is well defined. Indeed, if $a_1 + b_1 = a_2 + b_2$ where $a_1, a_2 \in A_1$ and $b_1, b_2 \in T_{A'}$ then $a_1 - a_2 = b_2 - b_1 \in A_1 \cap T_{A'} = T_{A_1} = \lambda(T_A)$. This implies that there exists $a \in T_A$ such that $a_1 - a_2 = b_2 - b_1 = \lambda(a)$. Hence,

$$\begin{cases} \alpha_2(a_1 - a_2) = \alpha_2\lambda(a) = \lambda\alpha(a), \\ \alpha'(b_2 - b_1) = \alpha'\lambda(a) = \lambda\alpha_1(a) = \lambda\alpha(a). \end{cases}$$

Therefore, $\alpha_2(a_1 - a_2) = \alpha'(b_2 - b_1)$, i.e, $\alpha_2(a_1) + \alpha'(b_1) = \alpha_2(a_2) + \alpha'(b_2)$. Hence $\alpha_3(a_1 + b_1) = \alpha_3(a_2 + b_2)$. Moreover α_3 is an automorphism of A'. Indeed:

Injection. Let $a' = a_1 + b_1 \in A'$ where $a_1 \in A_1$ and $b_1 \in T_{A'}$ such that $a' \in Ker(\alpha_3)$.

First case. If we have $a_1 \in T_{A'}$. Then $a_1 + b_1 \in T_{A'}$, consequently $0 = \alpha_3(a_1 + b_1) = \alpha'(a_1 + b_1)$. And since $\alpha' \in Aut(T_{A'})$ then $a' = a_1 + b_1 = 0$.

Second case. If we have $a_1 \notin T_{A'}$. Then since $b_1 \in T_{A'}$, Whence there exists $m \in \mathbb{N}^*$ such that $mb_1 = 0$, so $0 = m\alpha_3(a') = \alpha_3(ma') = \alpha_3(ma_1 + mb_1) = \alpha_3(ma_1) = \alpha_2(ma_1)$ which implies that $ma_1 = 0$ because $\alpha_2 \in Aut(A_1)$, consequently $a_1 \in T_{A_1} \subset T_{A'}$ which is absurd.

Surjection. Let $a' = a_2 + b_2 \in A'$ where $a_2 \in A_1$ and $b_2 \in T_{A'}$, since $\alpha_2 \in Aut(A_1)$ and $\alpha' \in Aut(T_{A'})$, then

$$\begin{cases} \exists a_1 \in A_1 : \alpha_2^{-1}(a_1) = a_2, \\ \exists b_1 \in T_{A'} : \alpha'^{-1}(b_1) = b_2. \end{cases}$$

So $\alpha_3(a_2+b_2)=\alpha_2(a_2)+\alpha'(b_2)=\alpha_2\alpha_2^{-1}(a_1)+\alpha'\alpha'^{-1}(b_1)=a_1+b_1$. Therefore $\alpha_3 \in Aut(A')$. In addition the automorphism α_3 of group A' commutes the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A' \\ \alpha\downarrow & & \downarrow\alpha_3 \\ A & \xrightarrow{\lambda} & A' \end{array}$$

Indeed, according to proposition 0.3 and lemma 0.4 we have: $\alpha_2\lambda=\lambda\alpha$ and $\alpha'\lambda=\lambda\alpha$, so

$$\begin{aligned} \alpha_3\lambda(a) &= \alpha_3\lambda(a)=(\alpha_2+\alpha')\lambda(a)=\alpha_2\lambda(a)+\alpha'\lambda(a)=\lambda\alpha_2(a)+\lambda\alpha'(a) \\ &= \lambda(\alpha_2+\alpha')(a)=\lambda\alpha_3(a). \end{aligned}$$

We conclude that α satisfies the weak extension property.

Acknowledgments

The research supported by the University of Mohamed first, Oujda, Morocco. The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper. We also thank Professor Abdelhakim Chillali who works on elliptic curves [17].

References

[1] S. Abdelalim, H. Essannouni, M. Zeriouh and M. Ziane, *Construction of the α_3-automorphism*, International Journal of Mathematical Models and Methods in Applied Sciences, 8 (2014), 364-367.

[2] L. Ben Yakoub, *Sur un théorème de Schupp*, Portugaliae. Math., 51 (1994), 231-233.

[3] L. Ben Yakoub et M.P. Malliavin, *Caractérisation des dérivation intérieures de l'algèbre de Weyl et de l'algèbre d'Heisenberg quantique*, Comm. Alg., 24 (1996), 3131-3148.

[4] I. Kaplansky, *Infinite abelian groups*, the University of Michigan Press, Ann. Arbor., 1954.

[5] Rotman, J.J, *An introduction to the theory of groups*, Fourth edition, Graduate Texts in Mathematics, 148, Springer-Verlag, New York, 1995.

[6] L. Fuchs, *Infinite abelian groups*, Academic press New York, 1970.

[7] L. Fuchs, *Abelian groups*, Springer Monographs in Mathematics, 2015.

[8] M. Dugas and R. Gobel, *Applications of abelian groups and model theory to algebraic structures*, Infinite Groups, 1994, Proceedings of the International Conference, hed in Ravello, Italy, (1994), 42-62.

[9] M. I. Kargapolov and J. I. Merzljakov, *Fundaments of the group theory*, Nauka, Moscow, in russian, 1982.

[10] M. R. Pettet, *On inner automorphisms of finite groups*, Proceeding of A. M. S., 106 (1989), 87-90.

[11] P.A. Krylov, A.V. Mikhalev and A.A. Tuganbaev, *Endomorphism rings of abelian groups*, (Technological University), Moscow, Russia, 2003.

[12] P. E. Schupp, *A Characterizing of inner automorphisms*, Proc. of A.M.S., 101 (1987), 226-228.

[13] S. Abdelalim et H. Essannouni, *Characterization of the automorphisms of an abelian group having the extension property*, Portugaliae Mathematica, 59 (2002), 325-333.

[14] M. Zeriouh, M. Ziane, S. Abdelalim, H. Essanouni, *Weakly extension*, Recent Advances in Mathematics, Statistics and Economics, Proceedings of the 2014 International Conference on Pur Mathematics-Applied Mathematics (PM-AM'14) Venice, Italy (2014), 172-174.

[15] M. Zeriouh, M. Ziane, S. Abdelalim, H. Essanouni, *Characterization of the automorphisms of an abelian torsion group having the weakly extension property*, Journal of Taibah University for Science, 9 (2015), 357-360.

[16] S. Abdelalim, M. Zeriouh and M. Ziane, *Construction of the* α_k-*automorphism*, JP Journal of Algebra, Number Theory and Applications, in press.

[17] A. Tadmori, A. Chillali and M. Ziane, *Elliptic curve over ring* $A_4 = \mathbb{F}_{2^d}[\varepsilon]; \varepsilon^4 = 0$, Applied Mathematical Sciences, 9 (2015), 1721-1733.

[18] J. Calais, *Eléments de théorie des groupes,* Editeur: Presses Universitaires de France-PUF, Edition: 3e éd, 1998.

Accepted: 24.03.2017