ON THE GENERALIZATION OF $(\in, \in \vee q)$-INTUITIONISTIC FUZZY BI-IDEALS OF SEMIGROUPS

M. Sajjad Ali Khan
Asad Ali*
Fazli Amin
Khaista Rahman
Department of Mathematics
Hazara University
Mansehra, KPk
Pakistan
sajjadalimath@yahoo.com
asad_maths@hu.edu.pk
faminmaths@yahoo.com

Saleem Abdullah
Department of Mathematics
Abdul Wali Khan University
Mardan, KPk
Pakistan
saleemabdullah81@yahoo.com

Fawad Hussain
Department of Mathematics
Abbottabad University of Science and Technology
KPk
Pakistan
fawadhussain998@hotmail.com

Abstract. In this article, we introduce the notion of $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup, $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideals, $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideals and $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideals and study some of its properties. We study the related properties of the $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideals, $(1, 2)$-ideals and in particular, an $(\in, \in \vee q_k)$-fuzzy bi-ideals and $(1, 2)$-ideals in semigroups will be investigated.
Keywords: $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup, $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideals, $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideals, $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideals.

1. Introduction

The idea of a fuzzy set was first originated by Zadeh in 1965 [1]. Fuzzy set theory has been shown to be a useful tool to define conditions in which the data

*. Corresponding author

are inexact or vague. Fuzzy sets theory handle such conditions by attributing a degree to which a certain object belongs to a set. The concept of fuzzy group was first proposed by Rosenfeld [2]. The notion of fuzzy semigroups was first studied by Kuroki in his standard paper [3]. The concepts of fuzzy ideals, bi-ideals, semi-prime ideals, quasi-ideals of semigroups are initiated by Kuroki in [4, 5, 6, 7, 8, 9, 10]. A logical account of fuzzy semigroup was specified by Mordeson et.al. [11], and they have found theoretical results on fuzzy semigroups and their use in fuzzy languages, fuzzy finite state machines, and fuzzy coding. The book of Mordeson and Malik treaties with the application of fuzzy method to the notions of formal languages and automata [12]. Newly, fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. Ameri and Noari in [13] introduced fuzzy hyperalgebras and investigated some vital results. In [14], Davvaz et.al., initiated fuzzy Hv-ideals in Γ-Hv-rings. The concept of fuzzy Γ-hypernearrings was initiated by Davvaz in [15]. In [16], Davvaz originated fuzzy Krasner (m, n)-hyperrings. Sun et al., consider fuzzy hypergraphs on fuzzy relations in [17]. The notion of "belong to" relation $(\in)$ was intitiated by Pu and Lia in [18]. The concept of a fuzzy point belonging to a fuzzy subset under natural equivalence on fuzzy subset was proposed by Morali in [19]. Bhakat and Das in [20], initiated the ideas of (α, β)-fuzzy subgroups by using the "belong to" relation $(\in)$ and "quasi-coincident with" relation (q) concerning with a fuzzy point and a fuzzy subgroup, and defined an $(\in, \in \vee q)$-fuzzy subgroup of a group. Kazanci and Yamak in [21], studied generalized types fuzzy bi-ideals of semigroups and defined $(\in, \in \vee q)$-fuzzy bi-ideals of semigroups. The concept of generalized fuzzy interior ideals of semigroups was studied by Jun and Song in [22]. Shabir et. al. in [23], characterized regular semigroups by the properties of (α, β)-fuzzy ideals, bi-ideals and quasi-ideals. S In [24], Shabir et. al. originated the notion of $(\in, \in \vee q_k)$-fuzzy ideals of semigroups and characterized regular semigroups by these ideals. Shabir and Mehmood in [25], initiated the notion of $(\in, \in \vee q_k)$-fuzzy h-ideals of hemirings and characterized different classes of hemirings by the using the concept of $(\in, \in \vee q_k)$-fuzzy h-ideals. Aslam et al. in [26], originated the notion of (α, β)-fuzzy Γ-ideals of Γ-LA-semigroups and given some characterization of Γ-LA-semigroups by (α, β)-fuzzy Γ-ideals:

In 1986, the notion of intuitionistic fuzzy set (IFS) was premised by Atanassov in [27]. An Atanassov intuitionistic fuzzy set is considered as a generalization of fuzzy set [1]. In the sense of Atanassov an IFS is characterized by a pair of functions valued in $[0, 1]$: the membership function and the non-membership function. The evaluation degrees of membership and non-membership are independent. Thus, an Atanassov intuitionistic fuzzy set is most substantial and brief to designate the spirit of fuzziness, and Atanassov intuitionistic fuzzy set theory may be more appropriate than fuzzy set theory for dealing with imperfect knowledge in many problems. Biswas in [28], use the idea of intuitionistic fuzzy set and initiated the the notion of intuitionistic fuzzy subgroup of a group. Kim and Jun in [29], originated intuitionistic fuzzy ideals of semigroups. In [30], Kim and Lee initiated the notion of intuitionistic fuzzy bi-ideals of semigroups. The concepts of intuitionistic fuzzy interior ideals of semigroups was initiated by Kim and Jun in [31]. The concept of intuitionistic fuzzy point was initiated by Coker and Demirci in [32]. Jun in [33], introduced the concept of (Φ, Ψ)-intuitionistic fuzzy subgroups. Aslam and Abdullah in [34], initiated the concept of (Φ, Ψ)-intuitionistic fuzzy ideals of semigroups. Abdullah et.al., in [35], initiated the concept of (α, β)-intuitionistic fuzzy ideals of hemirings by using the "belong to" relation $(\in)$ and "quasi-coincident with" relation (q) between an intuitionistic fuzzy point and an intuitionistics fuzzy set, and they defined prime (semi-prime) (α, β)-intuitionistic fuzzy ideals of hemirings. In [36], Khan et. al. initiated the notion of $(\in, \in q_k)$-intuitionistic fuzzy bi-ideals in ordered semigroups.

In this article, we introduce the notion of $(\in, \in q_k)$-intuitionistic fuzzy bi-ideal, $(\in, \in q_k)$-intuitionistic fuzzy $(1,2)$-ideal of semigroup, and studied related properties. We also prove that in regular semigroup, every $(\in, \in q_k)$-intuitionistic fuzzy $(1,2)$- ideal of semigroup S is an $(\in, \in q_k)$-intuitionistic fuzzy bi-ideal of semigroup S.

2. Preliminaries

In this section we give some basic definitions and results which are use in this note. Throughout in this article S will denote semigroup unless otherwise stated.

An algebraic system $(S, \cdot)$ consisting of a non-empty set S together with an associative binary operation " $\cdot$ " is called a semigroup. A subsemigroup of S is a non-empty set A such that $A^2 \subseteq A$. A left (resp. right) ideal of S is a non-empty set if $SA \subseteq A(AS \subseteq A)$. It is called two sided ideal of S if it is both left and right ideal of S. A quasi-ideal Q of S is a non-empty subset of S if $QS \cap SQ \subseteq Q$. A bi-ideal of S is a subsemigroup B of S if $BSB \subseteq B$. A generalized bi-ideal B of S is a non-empty subset of S if $BSB \subseteq B$. An interior ideal A of S is a subsemigroup of S if $SAS \subseteq A$. An element "x" of S is called a regular element if there exists an element $a \in S$ such that $x = xax$. "S" is called regular if every element of S is regular.

Definition 2.1 ([27]). Suppose X is a non-empty set. An intuitionistic fuzzy set (briefly, IFS) F is object having the form

$$F = \{\langle x, \lambda_F(x), \mu_F(x)\rangle : x \in X\}$$

where the functions $\lambda_F : X \to [0,1]$ and $\mu_F : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set F, respectively, and $\lambda_F(x) + \mu_F(x) \leq 1$ for all $x \in S$ for simplicity, we use the symbol $F = \langle \lambda_F, \mu_F \rangle$ for the IFS $F = \{\langle x, \lambda_F(x), \mu_F(x)\rangle : x \in X\}$.

Definition 2.2 ([28]). An intuitionistic fuzzy subsemigroup of S is an IFS $F = \langle \lambda_F, \mu_F \rangle$ in S if the satisfy the following conditions:

(IF1) $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y)$,

(IF2) $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y)$,
$\forall\ x, y \in S$.

Definition 2.3 ([28])**.** An intuitionistic fuzzy left (resp. right) ideal of S is an IFS $F = \langle \lambda_F, \mu_F \rangle$ in S if it satisfy $\lambda_F(xy) \geq \lambda_F(y)\,(\lambda_F(xy) \geq \lambda_F(x))$ and $\mu_F(xy) \leq \mu_F(y)\,(\mu_F(xy) \leq \mu_F(x))$ for all $x, y \in S$.

Definition 2.4 ([29])**.** An intuitionistic fuzzy subsemigroup of S is an IFS $F = \langle \lambda_F, \mu_F \rangle$ in S if the following conditions hold:

(IF1) $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y), \lambda_F(xyz) \geq \lambda_F(x) \wedge \lambda_F(z)$
(IF2) $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y), \mu_F(xyz) \leq \mu_F(x) \vee \mu_F(z)$
$\forall\ x, y, z \in S$.

Definition 2.5 ([36])**.** Let c be a point in a non-empty set X. If $t_1, t_2 \in (0, 1]$ are two real numbers such that $0 \leq t_1 + t_2 \leq 1$, then the IFS $\langle x; (t_1, t_2) \rangle = \langle a, x_{t_1}, 1 - x_{1-t_2} \rangle$ is said to be an intuitionistic fuzzy point (IFP for short) in X, where t_1 (resp, t_2) is the degree of membership (resp, non-membership) of $\langle x; (t_1, t_2) \rangle$ and $x \in X$ is the support of $\langle x; (t_1, t_2) \rangle$. Let $\langle x; (t_1, t_2) \rangle$ be an IFP in X and let $F = \langle \lambda_F, \mu_F \rangle$ be an IFS in X. Then, $\langle x; (t_1, t_2) \rangle$ is said to belong to F, written $\langle x; (t_1, t_2) \rangle \in F$, if $\lambda_F(x) \geq t$ and $\mu_F(x) \leq t_2$. We say that $\langle x; (t_1, t_2) \rangle$ is quasi-coincident with F, written $\langle x; (t_1, t_2) \rangle\, q_k F$, if $\lambda_F\,(x) + t_1 + k > 1$ and $\mu_F(x) + t_2 + k < 1$. To say that $\langle x; (t_1, t_2) \rangle \in \vee q_k F$ (resp, $\langle x; (t_1, t_2) \rangle \in \wedge q_k F$) means that $\langle x; (t_1, t_2) \rangle \in F$ or $\langle x; (t_1, t_2) \rangle\, q_k F$ (resp, $\langle x; (t_1, t_2) \rangle \in F$ and $\langle x; (t_1, t_2) \rangle\, q_k F$) and $\langle x; (t_1, t_2) \rangle \overline{\in \vee q_k} F$ means that $\langle x; (t_1, t_2) \rangle \in \vee q_k F$ does not hold and $t_1 \wedge t_2 = \min\{t_1, t_2\}$; $r_1 \vee r_2 = \max\{r_1, r_2\}$.

3. $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideals

In this section, we initiated the notion of $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup, $(\in, \in \vee q_k)$-intuitionistic fuzzy left(resp. right, two sided) ideal, $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal and $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideals in semigroups and investigated some of its properties.

Definition 3.1. An IFS $F = \langle \lambda_F, \mu_F \rangle$ in a semigroup S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S if satisfy the following condition:

$\langle x; (t_1, r_1) \rangle \in F$ and $\langle y; (t_2, r_2) \rangle \in F \Rightarrow \langle xy; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$, $\forall$ $x, y \in S$, $k \in [0, 1$, $t_1, t_2 \in (0, 1]$ and $r_1, r_2 \in [0, 1)$

Definition 3.2. An IFS $F = \langle \lambda_F, \mu_F \rangle$ in a semigroup S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S if satisfy the following condition:

$(IFI1)$ $\langle y; (t, r) \rangle \in F \Rightarrow \langle xy; (t, r) \rangle \in \vee q_k F$, (resp. $\langle x; (t, r) \rangle \in F \Rightarrow \langle xy; (t, r) \rangle \in \vee q_k F$), $\forall x, y \in S$, $k \in [0, 1)$, $t \in (0, 1]$ and $r \in [0, 1)$.

Definition 3.3. An IFS $F = \langle \lambda_F, \mu_F \rangle$ in a semigroup S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy ideal of S, if it is an $(\in, \in \vee q_k)$-intutionistic fuzzy left ideal and $(\in, \in \vee q_k)$-intuitionistic fuzzy right ideal of S.

Definition 3.4. An IFS $F = \langle \lambda_F, \mu_F \rangle$ in a semigroup S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S if satisfy the following conditions:

$(IFB1)$ $\langle x; (t_1, r_1) \rangle \in F$ and $\langle y; (t_2, r_2) \rangle \in F \Rightarrow \langle xy; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$.

$(IFB2)$ $\langle x; (t_1, r_1) \rangle \in F$ and $\langle z; (t_2, r_2) \rangle \in F \Rightarrow \langle xyz; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$.

$(\forall x, y, z \in S$ and $k \in [0, 1))(t_1, t_2 \in (0, 1]$ and $r_1, r_2 \in [0, 1))$

Definition 3.5. An IFS $F = \langle \lambda_F, \mu_F \rangle$ in a semigroup S is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideal of S if satisfy the following conditions:

$(\forall a, x, y, z \in S$ and $k \in [0, 1))(t_1, t_2 \in (0, 1]$ and $r_1, r_2 \in [0, 1))$

$(IF1)$ $\langle x; (t_1, r_1) \rangle \in F$ and $\langle y; (t_2, r_2) \rangle \in F \Rightarrow \langle xy; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$.

$(IF2)$ $\langle x; (t_1, r_1) \rangle \in F$ and $\langle z; (t_2, r_2) \rangle \in F \Rightarrow \langle xa\,(yz)\,; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in \vee q_k F$.

Theorem 3.6. *Let B be a left (resp. right) ideal of S and $F = \langle \lambda_F, \mu_F \rangle$ be an IFS such that*

1) $(\forall x \in S \backslash R)\,(\lambda_F(x) = 0$ and $\mu_F\,(x) = 1)\,,$

2) $(\forall x \in S \backslash R)\,\left(\lambda_F(x) \geq \frac{1-k}{2}\right.$ and $\left.\mu_F\,(x) \leq \frac{1-k}{2}\right)\,,$

Then, $F = \langle \lambda_F, \mu_F \rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S.

Proof. Let $x, y \in S$ and $t \in (0, 1]$, and $r \in [0, 1)$ be such that $\langle y; (t, r) \rangle\, qF$. Then, $\lambda_F\,(y) + t > 1$ and $\mu_F\,(y) + r < 1$. So, $y \in B$. Therefore, $xy \in B$. Thus, if $t \leq \frac{1-k}{2}$ and $r \geq \frac{1-k}{2}$, then $\lambda_F\,(xy) \geq \frac{1-k}{2} \geq t$ and $\mu_F\,(xy) \leq \frac{1-k}{2} \leq r$. Hence $\langle xy; (t, r) \rangle \in F$. If $t > \frac{1-k}{2}$ and $r < \frac{1-k}{2}$, then $\lambda_F\,(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $\mu_F\,(xy) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Thus $\langle xy; (t, r) \rangle\, q_k F$. Hence $\langle xy; (t, r) \rangle \in \vee q_k F$. Since $t + r \leq 1$, the case $t > \frac{1-k}{2}$ and $r \geq \frac{1-k}{2}$ does not occur. From the fact that $\langle y; (t, r) \rangle\, q_k F$, if implies that the case $t \leq \frac{1-k}{2}$ and $r < \frac{1-k}{2}$ does not occur. Hence, $F = \langle \lambda_F, \mu_F \rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy left ideal of S. □

Theorem 3.7. *Let B be a subsemigroup of S and $F = \langle \lambda_F, \mu_F \rangle$ be an IFS such that*

1) $(\forall x \in S \backslash B)\,(\lambda_F\,(x) = 0$ and $\mu_F\,(x) = 1)\,,$

2) $(\forall x \in S \backslash B)\,\left(\lambda_F\,(x) \geq \frac{1-k}{2}\right.$ and $\left.\mu_F\,(x) \leq \frac{1-k}{2}\right)\,.$

Then, $F = \langle \lambda_F, \mu_F \rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S.

Proof. Proof of the Theorem follows from Theorem 3.6. □

Theorem 3.8. *Let B be a bi-ideal of S and $F = \langle \lambda_F, \mu_F \rangle$ be an IFS such that*

1) $(\forall x \in S \backslash B)\left(\lambda_F(x) = 0 \text{ and } \mu_F(x) = 1\right)$,
2) $(\forall x \in S \backslash B)\left(\lambda_F(x) \geq \frac{1-k}{2} \text{ and } \mu_F(x) \leq \frac{1-k}{2}\right)$,
Then, $F = \langle \lambda_F, \mu_F \rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.

Proof. Let $x, y \in S$ and $t_1, t_2 \in (0,1]$ and $r_1, r_2 \in [0,1)$ be such that $\langle x; (t_1, r_1) \rangle \, qF$ and $\langle y; (t_2, r_2) \rangle \, qF$. Then $\lambda_F(x) + t_1 > 1$ and $\mu_F(y) + r_1 < 1$, and $\lambda_F(x) + t_2 > 1$ and $\mu_F(y) + r_2 < 1$. Hence $x, y \in B$. since B is a subsemigroup, therefore $xy \in B$ and so, $\lambda_F(xy) \geq \frac{1-k}{2}$ and $\mu_F(xy) \leq \frac{1-k}{2}$. If $t_1 \wedge t_2 > \frac{1-k}{2}$ and $r_1 \vee r_2 < \frac{1-k}{2}$, then $\lambda_F(xy) + t_1 \wedge t_2 + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $\mu_F(xy) + r_1 \vee r_2 + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Hence $\langle xy; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \, q_k F$. If $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$, then, $\lambda_F(xy) \geq t_1 \wedge t_2$ and $\mu_F(xy) \leq r_1 \vee r_2$ and so, $\langle xy; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in F$. Since $t_1 + r_1 \leq 1$ and $t_2 + r_2 \leq 1$, the case $t_1 \wedge t_2 > \frac{1-k}{2}, r_1 \vee r_2 \geq \frac{1-k}{2}$ does not hold. From the fact that $\langle x; (t_1, r_1) \rangle \, qF$ and $\langle y; (t_2, r_2) \rangle \, qF$, it implies that $t_1 \wedge t_2 \leq \frac{1-k}{2}, r_1 \vee r_2 < \frac{1-k}{2}$ does not hold. Hence, $F = \langle \lambda_F, \mu_F \rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S. Now let, $a, b, c \in S$ and $t_1, t_2 \in (0,1]$ and $r_1, r_2 \in [0,1)$ be such that $\langle x; (t_1, r_1) \rangle \, qF$ and $\langle z; (t_2, r_2) \rangle \, qF$. Then $\lambda_F(x) + t_1 > 1$ and $\mu_F(x) + r_1 < 1$, and $\lambda_F(z) + t_2 > 1$ and $\mu_F(z) + r_2 < 1$. Hence $x, z \in B$. since B is a bi-ideal. Therefore $xyz \in B$. Hence $\lambda_F(xyz) \geq \frac{1-k}{2}$ and $\mu_F(xyz) \leq \frac{1-k}{2}$. If $t_1 \wedge t_2 > \frac{1-k}{2}$ and $r_1 \vee r_2 < \frac{1-k}{2}$, then $\lambda_F(xyz) + t_1 \wedge t_2 + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $\mu_F(xyz) + r_1 \vee r_2 + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Hence, $\langle xyz; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \, q_k F$. If $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$, then, $\lambda_F(xyz) \geq t_1 \wedge t_2$ and $\mu_F(xyz) \leq r_1 \vee r_2$ implies that $\langle xyz; (t_1 \wedge t_2, r_1 \vee r_2) \rangle \in F$. Since $t_1 + r_1 \leq 1$ and $t_2 + r_2 \leq 1$, the case $t_1 \wedge t_2 > \frac{1-k}{2}, r_1 \vee r_2 \geq \frac{1-k}{2}$ does not hold. From the fact that $\langle x; (t_1, r_1) \rangle \, qF$ and $\langle y; (t_2, r_2) \rangle \, qF$, it implies that $t_1 \wedge t_2 \leq \frac{1-k}{2}, r_1 \vee r_2 < \frac{1-k}{2}$ does not hold. Hence, $F = \langle \lambda_F, \mu_F \rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. $\square$

Theorem 3.9. *Let B be a $(1,2)$-ideal of S and $F = \langle \lambda_F, \mu_F \rangle$ be an IFS such that:*
1) $(\forall x \in S \backslash B)\left(\lambda_F(x) = 0 \text{ and } \mu_F(x) = 1\right)$,
2) $(\forall x \in S \backslash B)\left(\lambda_F(x) \geq \frac{1-k}{2} \text{ and } \mu_F(x) \leq \frac{1-k}{2}\right)$.
Then, $F = \langle \lambda_F, \mu_F \rangle$ is an $(q, \in \vee q_k)$-intuitionistic fuzzy $(1,2)$-ideal of S.

Proof. Proof of the Theorem follows from Theorem 3.8. $\square$

Theorem 3.10. *Let $F = \langle \lambda_F, \mu_F \rangle$ is an intuitionistic fuzzy set in S. Then $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S if and only if the following conditions satisfied:*

1) $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$.
2) $\lambda_F(xyz) \geq \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}$ and $\mu_F(xyz) \leq \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}$.

Proof. Let $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.
1) Suppose $x, y \in S$. We consider the following two cases:
i) $\lambda_F(x) \wedge \lambda_F(y) < \frac{1-k}{2}$ and $\mu_F(x) \vee \mu_F(y) > \frac{1-k}{2}$
ii) $\lambda_F(x) \wedge \lambda_F(y) \geq \frac{1-k}{2}$ and $\mu_F(x) \vee \mu_F(y) \leq \frac{1-k}{2}$

Case i. Suppose that $\lambda_F(xy) < \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) > \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$. Then, $\lambda_F(xy) < \lambda_F(x) \wedge \lambda_F(y)$ and $\mu_F(xy) > \mu_F(x) \vee \mu_F(y)$. We choose $t \in (0,1]$ and $r \in [0,1)$ in such a way that $\lambda_F(xy) < t < \lambda_F(x) \wedge \lambda_F(y)$ and $\mu_F(xy) > r > \mu_F(x) \vee \mu_F(y)$. Then, $\langle x;(t,r)\rangle \in F$ and $\langle y;(t,r)\rangle \in F$, but $\langle xy;(r,s)\rangle \overline{\in \vee q_k} F$, which is a contradiction.

Case ii. Suppose that, $\lambda_F(xy) < \frac{1-k}{2}$ and $\mu_F(xy) > \frac{1-k}{2}$. Then, $\langle x;(\frac{1-k}{2}, \frac{1-k}{2})\rangle \in F$ and $\langle y;(\frac{1-k}{2}, \frac{1-k}{2})\rangle \in F$, but $\langle xy;(\frac{1-k}{2}, \frac{1-k}{2})\rangle \overline{\in \vee q_k} F$, which is a contradiction. Hence, $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$.

2) Now suppose $x, y, z \in S$. We consider the following two cases:

i) $\lambda_F(x) \wedge \lambda_F(z) < \frac{1-k}{2}$ and $\mu_F(x) \vee \mu_F(z) > \frac{1-k}{2}$

ii) $\lambda_F(x) \wedge \lambda_F(z) \geq \frac{1-k}{2}$ and $\mu_F(x) \vee \mu_F(z) \leq \frac{1-k}{2}$

Case i. Suppose that $\lambda_F(xyz) < \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}$ and $\mu_F(xyz) > \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}$, then $\lambda_F(xyz) < \lambda_F(a) \wedge \lambda_F(c)$ and $\mu_F(xyz) > \mu_F(x) \vee \mu_F(z)$. We choose $t \in (0,1]$ and $r \in [0,1)$ in such a way that $\lambda_F(xyz) < t < \lambda_F(x) \wedge \lambda_F(z)$ and $\mu_F(xyz) > r > \mu_F(x) \vee \mu_F(z)$. Then, $\langle x;(t,r)\rangle \in F$ and $\langle z;(t,r)\rangle \in F$, but $\langle xyz;(t,r)\rangle \overline{\in \vee q_k} F$, which is a contradiction.

Case ii. Suppose that $\lambda_F(xyz) < \frac{1-k}{2}$ and $\mu_F(xyz) > \frac{1-k}{2}$. Then, $\langle x;(\frac{1-k}{2}, \frac{1-k}{2})\rangle \in F$ and $\langle z;(\frac{1-k}{2}, \frac{1-k}{2})\rangle \in F$. But $\langle xyz;(\frac{1-k}{2}, \frac{1-k}{2})\rangle \overline{\in \vee q_k} F$, which is a contradiction. Hence $\lambda_F(xyz) \geq \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}$ and $\mu_F(xyz) \leq \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}$.

Conversely, suppose that $F = \langle \lambda_F, \mu_F \rangle$ satisfy (i) and (ii). Let $x, y \in S$, $t_1, t_2 \in (0,1]$ and $r_1, r_2 \in [0,1)$ be in a way that $\langle x;(t_1,r_1)\rangle \in F$ and $\langle y;(t_2,r_2)\rangle \in F$. Then, $\lambda_F(x) \geq t_1$ and $\mu_F(x) \leq r_1$, $\lambda_F(y) \geq t_2$ and $\mu_F(y) \leq r_2$. Now we have $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$. It Implies that $\lambda_F(xy) \geq t_1 \wedge t_2 \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq r_1 \vee r_2 \vee \frac{1-k}{2}$. Then, we have the following two cases.

i) $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$.

ii)$t_1 \wedge t_2 > \frac{1-k}{2}$ and $r_1 \vee r_2 < \frac{1-k}{2}$, the other cases does not occurs.

Case i. If $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$, then, $\lambda_F(xy) \geq t_1 \wedge t_2$ and $\mu_F(xy) \leq r_1 \vee r_2$, which implies that $(xy)(t_1 \wedge t_2, r_1 \vee r_2) \in F$.

Case ii. If $t_1 \wedge t_2 > \frac{1-k}{2}$ and $r_1 \vee r_2 < \frac{1-k}{2}$, then, $\lambda_F(xy) \geq \frac{1-k}{2}$ and $\mu_F(xy) \leq \frac{1-k}{2}$, which implies that $\lambda_F(xy) + t_1 \wedge t_2 + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $\mu_F(xy) + r_1 \vee r_2 + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Thus, $\langle xy;(t_1 \wedge t_2, r_1 \vee r_2)\rangle\, q_k F$. Hence, $\langle xy;(t_1 \wedge t_2, r_1 \vee r_2)\rangle \in \vee q_k F$.

Now, let $x, y, z \in S$ and $t_1, t_2 \in (0,1]$ and $r_1, r_2 \in [0,1)$ be in a way that $\langle x;(t_1,r_1)\rangle \in F$ and $\langle z;(t_2,r_2)\rangle \in F$. Then, $\lambda_F(x) \geq t_1$ and $\mu_F(x) \leq r_1, \lambda_F(z) \geq t_2$ and $\mu_F(z) \leq r_2$. Now we have $\lambda_F(xyz) \geq \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}$ and $\mu_F(xyz) \leq \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}$. It Implies that $\lambda_F(xyz) \geq t_1 \wedge t_2 \wedge \frac{1-k}{2}$ and $\mu_F(xyz) \leq r_1 \vee r_2 \vee \frac{1-k}{2}$. Then, we have the following two cases.

i) $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$

ii) $t_1 \wedge t_2 > \frac{1-k}{2}$ and $r_1 \vee r_2 < \frac{1-k}{2}$

Case i. If $t_1 \wedge t_2 \leq \frac{1-k}{2}$ and $r_1 \vee r_2 \geq \frac{1-k}{2}$, then $\lambda_F(xyz) \geq t_1 \wedge t_2$ and $\mu_F(xyz) \leq r_1 \vee r_2$, which implies that $(xyz)(t_1 \wedge t_2, r_1 \vee r_2) \in F$.

Case ii: If $t_1 \wedge t_2 > \frac{1-k}{2}$ and $r_1 \vee r_2 < \frac{1-k}{2}$, then $\lambda_F(xyz) \geq \frac{1-k}{2}$ and $\mu_F(xyz) \leq \frac{1-k}{2}$, which implies that $\lambda_F(xyz) + t_1 \wedge t_2 + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $\mu_F(xyz) + r_1 \vee r_2 + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Thus, $\langle xyz; (t_1 \wedge t_2, r_1 \vee r_2)\rangle\, q_k F$. Hence, $\langle xyz; (t_1 \wedge t_2, r_1 \vee r_2)\rangle \in \vee q_k F$. □

Every intuitionistic fuzzy bi-ideal and an $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of a semigroup S. But the converse is not true. For this we have the following example.

Example 3.11. Let $S = \{a, b, c, d, e\}$ be a semigroup with the following table.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	c
d	a	a	c	d	e
e	a	a	c	c	e

Let $F = \langle \lambda_F, \mu_F \rangle$ be an IFS is a semigroup S, defined by $\lambda_F(a) = \lambda_F(c) = 0.3, \lambda_F(b) = \lambda_F(e) = 0.6$, $\lambda_F(d) = 0.5$ and $\mu_F(a) = \mu_F(c) = 0.2$, $\mu_F(b) = \mu_F(e) = 0.3$, $\mu_F(d) = 0.5$. Take $\frac{1-k}{2} = 0.2$. Thus by simple calculation $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_{0.6})$-intuitionistic fuzzy bi-ideal of S. But $F = \langle \lambda_F, \mu_F \rangle$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal of S nor an fuzzy bi-ideal of S. i.e,

$$\lambda_F(d \cdot e \cdot d) = \lambda_F(c) = 0.3 \ngeq 0.5 = \lambda_F(d) \wedge \lambda_F(d) \wedge 0.5$$
$$\text{and } \lambda_F(d \cdot e \cdot d) = \lambda_F(c) = 0.3 \ngeq 0.5 = \lambda_F(d) \wedge \lambda_F(d)$$

Remark 3.12. From above example we say that an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S is a generalization of an$(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal and fuzzy bi-ideal of S.

Theorem 3.13. *Let $F = \langle \lambda_F, \mu_F \rangle$ is an intuitionistic fuzzy set in S. Then $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideal of S if and only if the following conditions satisfied:*

i) $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$.

ii) $\lambda_F(xa(yz)) \geq \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}$ and $\mu_F(xa(yz)) \leq \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}$.

Proof. Proof of the Theorem follows from Theorem 3.10. □

Lemma 3.14. *Every $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.*

The converse of the above Lemma not true. For this we have the following example.

Let $S = \{a, b, c, d\}$ be a semigroup with the following table:

$\cdot$	a	b	c	d
a	a	c	a	a
b	a	a	a	a
c	b	a	a	d
d	a	a	d	a

(a) : Let $F = \langle \lambda_F, \mu_F \rangle$ be an IFS defined by, $\lambda_F(a) = 0.6$, $\lambda_F(b) = 0.5$, $\lambda_F(c) = 0.3$, $\lambda_F(d) = 0.2$ and $\mu_F(a) = \mu_F(b) = 0.6$, $\mu_F(c) = \mu_F(d) = 0.4$. Then $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_{0.3})$-intuitionistic fuzzy bi-ideal of S, where $\frac{1-k}{2} = 0.3$. Clearly it is not an $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal nor intuitionistic fuzzy bi-ideal of S. Because, $\lambda_F(a \cdot b) = \lambda_F(c) = 0.3 \ngeq 0.5 = \lambda_F(a) \wedge \lambda_F(b) \wedge 0.5$ and $\lambda_F(a \cdot b) = \lambda_F(c) = 0.3 \ngeq 0.5 = \lambda_F(a) \wedge \lambda_F(b)$. Also $\lambda_F(c \cdot d) = \lambda_F(d) = 0.2 \ngeq 0.3 = \lambda_F(c) \wedge \frac{1-k}{2}$. Which shows that $F = \langle \lambda_F, \mu_F \rangle$ is not $(\in, \in \vee q_{0.3})$-intuitionistic fuzzy right ideal of S.

(b) : Let $F = \langle \lambda_F, \mu_F \rangle$ be an IFS defined by, $\lambda_F(a) = 0.7$, $\lambda_F(b) = 0.3$, $\lambda_F(c) = 0.4$, $\lambda_F(d) = 0.2$ and $\mu_F(a) = \mu_F(b) = 0.2$, $\mu_F(c) = \mu_F(d) = 0.32$.Then $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_{0.3})$-intuitionistic fuzzy bi-ideal of S. Clearly it is not an $(\in, \in \vee q)$-intuitionistic fuzzy bi-ideal nor intuitionistic fuzzy bi-ideal of S. Because, $\lambda_F(c \cdot a) = \lambda_F(b) = 0.3 \ngeq 0.5 = \lambda_F(c) \wedge \lambda_F(a) \wedge 0.5$ and $\lambda_F(c \cdot a) = \lambda_F(b) = 0.3 \ngeq 0.5 = \lambda_F(c) \wedge \lambda_F(a)$. Also $\lambda_F(d \cdot c) = \lambda_F(d) = 0.2 \ngeq 0.3 = \lambda_F(c) \wedge \frac{1-k}{2}$. Which shows that $F = \langle \lambda_F, \mu_F \rangle$ is not an $(\in, \in \vee q_{0.3})$-intuitionistic fuzzy left ideal of S.

Lemma 3.15. (i) *Every* $(\in \vee q_k, \in \vee q_k)$*-intuitionistic fuzzy bi-ideal of* S *is an* $(\in, \in \vee q_k)$*-intuitionistic fuzzy bi-ideal of* S.

(ii) *Every* $(\in, \in \vee q)$*-intuitionistic fuzzy bi-ideal of* S *is an* $(\in, \in \vee q_k)$*-intuitionistic fuzzy bi-ideal of* S.

(iii) *Every* $(\in, \in)$*-intuitionistic fuzzy bi-ideal of* S *is an* $(\in, \in \vee q_k)$*-intuitionistic fuzzy bi-ideal of* S.

Proof. Straightforward. □

Examples 3.11 and 3 shows that the converse of the above Lemma 3.15 is not true in general.

Lemma 3.16. *Let* $\{F_i\}_{i \in I}$ *be a family of an* $(\in, \in \vee q_k)$*-intuitionistic fuzzy bi-ideal of* S. *Then* $\bigcap_{i \in I} F_i$ *is an* $(\in, \in \vee q_k)$*-intuitionistic fuzzy bi-ideal of* S, *where* $\bigcap_{i \in I} F_i = \left\langle \bigwedge_{i \in I} \lambda_F, \bigvee_{i \in I} \mu_F \right\rangle$.

Proof. Straightforward. □

Lemma 3.17. *Let* $\{F_i\}_{i \in I}$ *be a family of an* $(\in, \in \vee q_k)$*-intuitionistic fuzzy bi-ideal of* S. *Then* $\bigcup_{i \in I} F_i$ *is not an* $(\in, \in \vee q_k)$*-intuitionistic fuzzy bi-ideal of* S, *where* $\bigcup_{i \in I} F_i = \left\langle \bigwedge_{i \in I} \lambda_{F_i}, \bigvee_{i \in I} \mu_{F_i} \right\rangle$. *For this we have the following example.*

Example 3.18. Let $S = \{a, b, c, d\}$ be a semigroup with the following table:

$\cdot$	a	b	c	d
a	a	c	a	a
b	a	a	a	a
c	b	a	a	d
d	a	a	d	a

Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ be two $IFS's$ of semigroup S defined by $\lambda_E(a) = 0.7$, $\lambda_E(b) = 0.5$, $\lambda_E(c) = \lambda_E(d) = 0.3$ and $\mu_E(a) = \mu_E(c) = 0.5$, $\mu_E(b) = \mu_E(d) = 0.2$, and $\lambda_F(a) = 0.8$, $\lambda_F(b) = \lambda_F(d) = 0.3$, $\lambda_F(c) = 0.4$ and $\mu_F(a) = \mu_F(b) = 0.5$, $\mu_F(c) = \mu_F(d) = 0.3$. Then, both $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ are an $(\in, \in \vee q_{0.4})$-intuitionistic fuzzy bi-ideals of S, where $\frac{1-k}{2} = 0.4$. But $E \cup F$ is not an $(\in, \in \vee q_{0.4})$-intuitionistic fuzzy bi-ideal of S. i.e, $(\lambda_E \vee \lambda_F)(bc) = (\lambda_E \vee \lambda_F)(d) = \lambda_E(d) \vee \lambda_F(d) = 0.3 \vee 0.3 = 0.3$ and $(\lambda_E \vee \lambda_F)(b) \wedge (\lambda_E \vee \lambda_F)(c) \wedge \frac{1-k}{2} = 0.5 \wedge 0.4 \wedge \frac{1-k}{2} = 0.4$. Hence $(\lambda_E \vee \lambda_F)(bc) = 0.3 \ngeq 0.4 = (\lambda_E \vee \lambda_F)(b) \wedge (\lambda_E \vee \lambda_F)(c) \wedge \frac{1-k}{2}$.

Theorem 3.19. *Let $\{F_i\}_{i \in I}$ be a family of an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S such that $F_i \subseteq F_j$ or $F_j \subseteq F_i$ for all $i, j \in I$. Then $\bigcup_{i \in I} F_i$ is not an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S, where $\bigcup_{i \in I} F_i = \left\langle \bigvee_{i \in I} \lambda_F, \bigwedge_{i \in I} \mu_F \right\rangle$.*

Proof. For all $x, y \in S$, we have

$$\begin{aligned} \left(\bigvee_{i \in I} \lambda_{F_i}(xy) \right) &= \bigvee_{i \in I} (\lambda_{F_i}(xy)) \geq \bigvee_{i \in I} \left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2} \right) \\ &= \bigvee_{i \in I} \lambda_{F_i}(x) \wedge \bigvee_{i \in I} \lambda_{F_i}(y) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{i \in I} \lambda_{F_i} \right)(x) \wedge \left(\bigvee_{i \in I} \lambda_{F_i} \right)(y) \wedge \frac{1-k}{2}. \end{aligned}$$

It is clear that

$$\bigvee_{i \in I} \left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2} \right) \leq \left(\bigvee_{i \in I} \lambda_{F_i} \right)(x) \wedge \left(\bigvee_{i \in I} \lambda_{F_i} \right)(y) \wedge \frac{1-k}{2}.$$

Suppose that

$$\bigvee_{i \in I} \left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2} \right) \neq \left(\bigvee_{i \in I} \lambda_{F_i} \right)(x) \wedge \left(\bigvee_{i \in I} \lambda_{F_i} \right)(y) \wedge \frac{1-k}{2}.$$

Then there exists t such that

$$\bigvee_{i \in I} \left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2} \right) < t < \left(\bigvee_{i \in I} \lambda_{F_i} \right)(x) \wedge \left(\bigvee_{i \in I} \lambda_{F_i} \right)(y) \wedge \frac{1-k}{2}.$$

Since $\lambda_{F_i} \subseteq \lambda_{F_j}$ or $\lambda_{F_j} \subseteq \lambda_{F_i}$ for all $i, j \in I$, thus there exists $k \in I$ such that $t < \lambda_{F_k}(x) \wedge \lambda_{F_k}(y) \wedge \frac{1-k}{2}$. On the other hand $t > \lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2}$ for all $i \in I$, a contradiction. Hence

$$\bigvee_{i\in I}\left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(y) \wedge \frac{1-k}{2}\right) = \left(\bigvee_{i\in I}\lambda_{F_i}\right)(x) \wedge \left(\bigvee_{i\in I}\lambda_{F_i}\right)(y) \wedge \frac{1-k}{2}$$

and

$$\begin{aligned}\left(\bigwedge_{i\in I}\mu_{F_i}(xy)\right) &= \bigwedge_{i\in I}\left(\mu_{F_i}(xy)\right) \\ &\leq \bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}\right) \\ &= \bigwedge_{i\in I}\mu_{F_i}(x) \vee \bigwedge_{i\in I}\mu_{F_i}(y) \vee \frac{1-k}{2} \\ &= \left(\bigwedge_{i\in I}\mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I}\mu_{F_i}\right)(y) \vee \frac{1-k}{2}.\end{aligned}$$

It is clear that

$$\bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}\right) \geq \left(\bigwedge_{i\in I}\mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I}\mu_{F_i}\right)(y) \vee \frac{1-k}{2}.$$

Suppose that

$$\bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}\right) \neq \left(\bigwedge_{i\in I}\mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I}\mu_{F_i}\right)(y) \vee \frac{1-k}{2}.$$

Then there exists r such that

$$\bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}\right) > r > \left(\bigwedge_{i\in I}\mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I}\mu_{F_i}\right)(y) \vee \frac{1-k}{2}.$$

Since $\mu_{F_i} \subseteq \mu_{Fj}$ or $\mu_{Fj} \subseteq \mu_{Fi}$ for all $i, j \in I$. Thus there exists $k \in I$ such that $r > \mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}$. On the other hand, $r < \mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}$ for all $i \in I$, which is a contradiction. Hence,

$$\bigwedge_{i\in I}\left(\mu_{F_i}(x) \vee \mu_{F_i}(y) \vee \frac{1-k}{2}\right) = \left(\bigwedge_{i\in I}\mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I}\mu_{F_i}\right)(y) \vee \frac{1-k}{2}.$$

Let $a, b, c \in S$, we have

$$\begin{aligned}
\left(\bigvee_{i\in I} \lambda_{F_i}(xyz)\right) &= \bigvee_{i\in I} \left(\lambda_{F_i}(xyz)\right) \geq \bigvee_{i\in I} \left(\lambda_{F_i}(x) \wedge \lambda_{F_i}(z) \wedge \frac{1-k}{2}\right) \\
&= \bigvee_{i\in I} \lambda_{F_i}(x) \wedge \bigvee_{i\in I} \lambda_{F_i}(z) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{i\in I} \lambda_{F_i}\right)(x) \wedge \left(\bigvee_{i\in I} \lambda_{F_i}\right)(z) \wedge \frac{1-k}{2}
\end{aligned}$$

and

$$\begin{aligned}
\left(\bigwedge_{i\in I} \mu_{F_i}(xyz)\right) &= \bigwedge_{i\in I} \left(\mu_{F_i}(xyz)\right) \leq \bigwedge_{i\in I} \left(\mu_{F_i}(x) \vee \mu_{F_i}(z) \vee \frac{1-k}{2}\right) \\
&= \bigwedge_{i\in I} \mu_{F_i}(x) \vee \bigwedge_{i\in I} \mu_{F_i}(z) \vee \frac{1-k}{2} \\
&= \left(\bigwedge_{i\in I} \mu_{F_i}\right)(x) \vee \left(\bigwedge_{i\in I} \mu_{F_i}\right)(z) \vee \frac{1-k}{2}.
\end{aligned}$$

Hence, $\bigcup_{i\in I} F_i = \left\langle \bigvee_{i\in I} \lambda_F, \bigwedge_{i\in I} \mu_F \right\rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. □

Definition 3.20. Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ of S. Then, the $\frac{1-k}{2}$-product of E and F is defined by:

$$\begin{aligned}
E \circ_{\frac{1-k}{2}} F &= \left\langle \lambda_E \circ_{\frac{1-k}{2}} \lambda_F, \mu_E \circ_{\frac{1-k}{2}} \mu_F \right\rangle \\
\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F\right)(a) &= \left\{ \begin{array}{ll} \bigvee_{a=xy} \left(\lambda_E(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}\right) & \text{if } a = xy \\ 0 & \text{if } a \neq xy \end{array} \right\} \\
\left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(a) &= \left\{ \begin{array}{ll} \bigwedge_{a=xy} \left(\mu_E(x) \vee \mu_F(y) \vee \frac{1-k}{2}\right) & \text{if } a = xy \\ 0 & \text{if } a \neq xy \end{array} \right\} \\
E \cap_{\frac{1-k}{2}} F &= \left\langle \lambda_E \wedge_{\frac{1-k}{2}} \lambda_F, \mu_E \vee_{\frac{1-k}{2}} \mu_F \right\rangle \\
\left(\lambda_E \wedge_{\frac{1-k}{2}} \lambda_F\right)(a) &= \lambda_E(a) \wedge \lambda_F(a) \wedge \frac{1-k}{2} \\
\left(\mu_E \vee_{\frac{1-k}{2}} \mu_F\right)(a) &= \mu_E(a) \vee \mu_F(a) \vee \frac{1-k}{2}.
\end{aligned}$$

Remark 3.21. Let E, F, G, H are $IFS's$ of S such that $E \subseteq F$ and $G \subseteq H$. Then $E \circ_{\frac{1-k}{2}} F \subseteq G \circ_{\frac{1-k}{2}} H$.

Lemma 3.22. *Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ be $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. Then $E \cap_{\frac{1-k}{2}} F$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.*

Definition 3.23. An $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal $F = \langle \lambda_F, \mu_F \rangle$ of S is said to be $\frac{1-k}{2}$-idempotent if $F \circ_{\frac{1-k}{2}} F = F$.

Lemma 3.24. *Let $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S. Then $F \circ_{\frac{1-k}{2}} F \subseteq F$.*

Lemma 3.25. *Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideals of S. Then $F \circ_{\frac{1-k}{2}} F \subseteq 1 \circ_{\frac{1-k}{2}} F$.(resp. $F \circ_{\frac{1-k}{2}} F \subseteq E \circ_{\frac{1-k}{2}} 1$).*

Theorem 3.26. *Let $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. Then, $F \circ_{\frac{1-k}{2}} S \circ_{\frac{1-k}{2}} F \subseteq F$, where $S = \langle 1, 0 \rangle$, $1(a) = 1$ and $0(a) = 0$ for all $a \in S$.*

Proof. Let $x \in S$. Then we have the following two cases:

(i) If $x \neq ab$ $\forall a, b \in S$.

(ii) If $x = ab$ for some $a, b \in S$.

Case (i). If $a \neq xy$, then

$$\left(\lambda_F \circ_{\frac{1-k}{2}} 1 \circ_{\frac{1-k}{2}} \lambda_F\right)(x) = 0 \leq \lambda_F(x) \wedge \frac{1-k}{2}$$

and

$$\left(\mu_F \circ_{\frac{1-k}{2}} 0 \circ_{\frac{1-k}{2}} \mu_F\right)(x) = 1 \geq \mu_F(x) \vee \frac{1-k}{2}.$$

Thus, $F \circ_{\frac{1-k}{2}} S \circ_{\frac{1-k}{2}} F \subseteq F$.

Case (ii). If $x = ab$ for some $x, y \in S$, then

$$\begin{aligned}
\left(\lambda_F \circ_{\frac{1-k}{2}} 1 \circ_{\frac{1-k}{2}} \lambda_F\right)(x) &= \bigvee_{x=ab} \left\{\lambda_F(a) \wedge \left(1 \circ_{\frac{1-k}{2}} \lambda_F\right)(b) \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{x=ab} \left\{\lambda_F(a) \wedge \left\{\bigvee_{b=pq} 1(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{x=ab} \bigvee_{b=pq} \left\{\lambda_F(x) \wedge 1 \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \\
&= \bigvee_{x=apq} \left\{\lambda_F(x) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\}.
\end{aligned}$$

Since $x = ab = a(pq) = apq$ and $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S, therefore we have $\lambda_F(apq) \geq \lambda_F(a) \wedge \lambda_F(q) \wedge \frac{1-k}{2}$. Hence,

$$\begin{aligned}
\bigvee_{x=apq} \left\{\lambda_F(a) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} &\leq \bigvee_{x=apq} \{\lambda_F(apq)\} = \lambda_F(x) \\
\left(\lambda_F \circ_{\frac{1-k}{2}} 1 \circ_{\frac{1-k}{2}} \lambda_F\right)(x) &\leq \lambda_F(x)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\mu_F \circ_{\frac{1-k}{2}} 0 \circ_{\frac{1-k}{2}} \mu_F\right)(x) \\
&= \bigwedge_{x=ab} \left\{\mu_F(a) \vee \left(0 \circ_{\frac{1-k}{2}} \mu_F\right)(b) \vee \frac{1-k}{2}\right\} \\
&= \bigwedge_{x=ab} \left\{\mu_F(a) \vee \left\{\bigwedge_{y=pq} \left\{1(p) \vee \mu_F(q) \vee \frac{1-k}{2}\right\}\right\} \vee \frac{1-k}{2}\right\} \\
&= \bigwedge_{x=ab} \bigwedge_{y=pq} \left\{\mu_F(a) \vee 0 \vee \mu_F(q) \vee \frac{1-k}{2}\right\} \\
&= \bigwedge_{x=apq} \left\{\mu_F(a) \vee \mu_F(q) \vee \frac{1-k}{2}\right\}.
\end{aligned}$$

Since $x = ab = a(pq) = apq$ and $F = \langle \lambda_F, \mu_F \rangle$ is an interval valued $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S, therefore we have $\mu_F(apq) \leq \mu_F(a) \vee \mu_F(q) \vee \frac{1-k}{2}$. Hence,

$$\begin{aligned}
&\bigwedge_{x=apq} \left\{\mu_F(a) \vee \mu_F(q) \vee \frac{1-k}{2}\right\} \leq \bigwedge_{a=xpq} \{\mu_F(apq)\} = \mu_F(x) \\
&\left(\mu_F \circ_{\frac{1-k}{2}} 0 \circ_{\frac{1-k}{2}} \mu_F\right)(x) \leq \mu_F(x).
\end{aligned}$$

Thus, $F \circ_{\frac{1-k}{2}} S \circ_{\frac{1-k}{2}} F \subseteq F$. □

Theorem 3.27. *Let $F = \langle \lambda_F, \mu_F \rangle$ be an IFS. Then $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S if and only if $F \circ_{\frac{1-k}{2}} F \subseteq F$.*

Theorem 3.28. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S if and only if the following condition satisfied.*

(i) $F \circ_{\frac{1-k}{2}} F \subseteq F$,

(ii) $F \circ_{\frac{1-k}{2}} S \circ_{\frac{1-k}{2}} F \subseteq F$.

Proof. Let $F = \langle \lambda_F, \mu_F \rangle$ be an interval valued $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.Then, by Lemma 3.24 and Theorem 3.26, we have $F \circ_{\frac{1-k}{2}} F \subseteq F$ and $F \circ_{\frac{1-k}{2}} S \circ_{\frac{1-k}{2}} F \subseteq F$.

Conversely, assume that condition (i) and (ii) satisfied. Let $x, y \in S$ be such that $a = xy$. Then, we have

$$\begin{aligned}
\lambda_F(xy) &= \lambda_F(a) \geq \left(\lambda_F \circ_{\frac{1-k}{2}} \lambda_F\right)(a) \\
&= \bigvee_{a=xy} \left\{\lambda_F(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2}\right\} \\
&\geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}
\end{aligned}$$

and

$$\begin{aligned}\mu_F(xy) &= \mu_F(a) \leq \left(\mu_F \circ_{\frac{1-k}{2}} \mu_F\right)(a) \\ &= \bigwedge_{a=pq}\left\{\mu_F(p) \vee \mu_F(q) \vee \frac{1-k}{2}\right\} \\ &\leq \mu_F(p) \vee \mu_F(q) \vee \frac{1-k}{2}.\end{aligned}$$

Now, let $x, y, z \in S$ such that $a = xyz$. Then, we have

$$\begin{aligned}\lambda_F(xyz) &= \lambda_F(a) \geq \left(\lambda_F \circ_{\frac{1-k}{2}} 1 \circ_{\frac{1-k}{2}} \lambda_F\right)(a) \\ &= \bigvee_{a=pq}\left\{\lambda_F(p) \wedge \left(1 \circ_{\frac{1-k}{2}} \lambda_F\right)(q) \wedge \frac{1-k}{2}\right\} \\ &= \bigvee_{a=pq}\left\{\lambda_F(p) \wedge \left(\bigvee_{q=st}\left\{1(t) \wedge \lambda_F(s) \wedge \frac{1-k}{2}\right\}\right) \wedge \frac{1-k}{2}\right\} \\ &= \bigvee_{a=pq}\left\{\lambda_F(p) \wedge \left(\bigvee_{q=st}\left\{1 \wedge \lambda_F(s) \wedge \frac{1-k}{2}\right\}\right) \wedge \frac{1-k}{2}\right\} \\ &\geq \bigvee_{a=pq}\bigvee_{q=st}\left\{\lambda_F(p) \wedge \lambda_F(s) \wedge \frac{1-k}{2}\right\} \\ &\geq \bigvee_{a=pst}\left\{\lambda_F(p) \wedge \lambda_F(s) \wedge \frac{1-k}{2}\right\} \\ &\geq \lambda_F(x) \wedge \lambda_F(z) \wedge \frac{1-k}{2}\end{aligned}$$

and

$$\begin{aligned}\mu_F(xyz) &= \mu_F(a) \leq \left(\mu_F \circ_{\frac{1-k}{2}} 0 \circ_{\frac{1-k}{2}} \mu_F\right)(a) \\ &= \bigwedge_{a=pq}\left\{\mu_F(p) \vee \left(0 \circ_{\frac{1-k}{2}} \mu_F\right)(q) \vee \frac{1-k}{2}\right\} \\ &= \bigwedge_{a=pq}\left\{\mu_F(p) \vee \left\{\bigwedge_{q=st}\left\{0(s) \vee \mu_F(t) \vee \frac{1-k}{2}\right\}\right\} \vee \frac{1-k}{2}\right\} \\ &= \bigwedge_{a=pq}\left\{\mu_F(p) \vee \left\{\bigwedge_{q=st}\left\{0 \vee \mu_F(t) \vee \frac{1-k}{2}\right\}\right\} \vee \frac{1-k}{2}\right\} \\ &\leq \bigwedge_{a=pq}\bigwedge_{q=st}\left\{\mu_F(p) \vee \mu_F(t) \vee \frac{1-k}{2}\right\} \\ &\leq \bigwedge_{a=pst}\left\{\mu_F(p) \vee \mu_F(t) \vee \frac{1-k}{2}\right\} \\ &\leq \mu_F(x) \vee \mu_F(z) \vee \frac{1-k}{2}.\end{aligned}$$

Hence, $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. $\square$

Theorem 3.29. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right, two sided) ideal of S if and only if it satisfied:*
$S \circ_{\frac{1-k}{2}} F \subseteq F \left(F \circ_{\frac{1-k}{2}} S \subseteq F,\ S \circ_{\frac{1-k}{2}} F \subseteq F \text{ and } F \circ_{\frac{1-k}{2}} S \subseteq F \right)$.

Proof. Straightforward. $\square$

Theorem 3.30. *Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ be two $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. Then $A = E \circ_{\frac{1-k}{2}} F$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S.*

Proof. Let $E = \langle \lambda_E, \mu_E \rangle$ and $F = \langle \lambda_F, \mu_F \rangle$ be two $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S and $a \in S$. Then we have two cases:

(i) If $x \neq ab$ for any $a, b \in S$. (ii) If $x = ab$ for some $a, b \in S$.

Case i. If $x \neq ab$ for any $a, b \in S$, then

$$\left(\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right) \circ_{\frac{1-k}{2}} \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right) \right)(x) = 1 \leq \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(x)$$

and

$$\left(\left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right) \circ_{\frac{1-k}{2}} \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right) \right)(x) = 0 \geq \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right)(x).$$

Thus, $A \circ_{\frac{1-k}{2}} A \subseteq A$.

Case ii. If $x = ab$ for some $a, b \in S$, then

$$\begin{aligned}
&\left(\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right) \circ_{\frac{1-k}{2}} \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right) \right)(a) \\
= &\bigvee_{a=xy} \left\{ \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(x) \wedge \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(y) \wedge \frac{1-k}{2} \right\} \\
= &\bigvee_{a=xy} \left\{ \begin{array}{c} \left\{ \bigvee_{x=pq} \left\{ \lambda_E(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2} \right\} \right\} \wedge \\ \left\{ \bigvee_{y=st} \left\{ \lambda_E(s) \wedge \lambda_F(t) \wedge \frac{1-k}{2} \right\} \right\} \end{array} \right\} \\
= &\bigvee_{a=xy} \bigvee_{x=pq} \bigvee_{y=st} \left\{ \left\{ \lambda_E(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2} \right\} \wedge \lambda_E(s) \wedge \lambda_F(t) \wedge \frac{1-k}{2} \right\} \\
= &\bigvee_{a=xy} \bigvee_{x=pq} \bigvee_{y=st} \left\{ \lambda_E(p) \wedge \lambda_E(s) \wedge \lambda_F(t) \wedge \frac{1-k}{2} \} \right\} \\
\leq &\bigvee_{a=xy} \bigvee_{x=pq} \bigvee_{y=st} \left\{ \lambda_E(p) \wedge \lambda_E(s) \wedge \frac{1-k}{2} \wedge \lambda_F(t) \} \right\}.
\end{aligned}$$

Since $a = xy$, $x = pq$ and $y = st$. So, $a = (xy)(st) = (pqs)t$ and we have

$$\begin{aligned}
&\bigvee_{a=xy} \bigvee_{x=pq} \bigvee_{y=st} \left\{ \lambda_E(p) \wedge \lambda_E(s) \wedge \frac{1-k}{2} \wedge \lambda_F(t) \} \right\} \\
\leq &\bigvee_{a=(xys)t} \left\{ \lambda_E(p) \wedge \lambda_E(s) \wedge \frac{1-k}{2} \wedge \lambda_F(t) \right\}.
\end{aligned}$$

Since $E = \langle \lambda_E, \mu_E \rangle$ is an interval valued $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S we have

$$\lambda_E(xys) \geq \lambda_E(x) \wedge \lambda_E(s) \wedge \frac{1-k}{2}.$$

So,

$$\begin{aligned}
&\bigvee_{a=(xys)t} \left\{ \lambda_E(p) \wedge \lambda_E(s) \wedge \frac{1-k}{2} \wedge \lambda_F(t) \right\} \\
&\leq \bigvee_{a=(xys)t} \left\{ \lambda_E(xys) \wedge \lambda_E(s) \wedge \frac{1-k}{2} \right\} \\
&\leq \bigvee_{a=uv} \left\{ \lambda_E(u) \wedge \lambda_E(v) \wedge \frac{1-k}{2} \right\} \\
&= \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(a).
\end{aligned}$$

Therefore, $\left(\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right) \circ_{\frac{1-k}{2}} \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right) \right)(a) \leq \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(a)$. Now,

$$\begin{aligned}
&\left(\left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right) \circ_{\frac{1-k}{2}} \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right) \right)(a) \\
&= \bigwedge_{a=xy} \left\{ \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right)(x) \vee \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right)(y) \vee \frac{1-k}{2} \right\} \\
&= \bigwedge_{a=xy} \left\{ \bigwedge_{x=pq} \left\{ \mu_E(p) \vee \mu_F(q) \vee \frac{1-k}{2} \right\} \vee \bigwedge_{y=st} \left\{ \mu_E(s) \vee \mu_F(t) \vee \frac{1-k}{2} \right\} \vee \frac{1-k}{2} \right\} \\
&= \bigwedge_{a=xy} \bigwedge_{x=pq} \bigwedge_{y=st} \left\{ \left\{ \mu_E(p) \vee \mu_F(q) \vee \frac{1-k}{2} \right\} \vee \left\{ \mu_E(s) \vee \mu_F(t) \vee \frac{1-k}{2} \right\} \right\} \\
&= \bigwedge_{a=xy} \bigwedge_{x=pq} \bigwedge_{y=st} \left\{ \mu_E(p) \vee \mu_E(s) \vee \mu_F(t) \vee \frac{1-k}{2} \} \right\} \\
&\geq \bigwedge_{a=xy} \bigwedge_{x=pq} \bigwedge_{y=st} \left\{ \mu_E(p) \vee \mu_E(s) \vee \frac{1-k}{2} \vee \mu_F(t) \} \right\}.
\end{aligned}$$

Since $a = xy$, $x = pq$ and $y = st$. So, $a = (xy)(st) = (pqs)t$ and we have

$$\begin{aligned}
&\bigwedge_{a=xy} \bigwedge_{x=pq} \bigwedge_{y=st} \left\{ \mu_E(p) \vee \mu_E(s) \vee \frac{1-k}{2} \vee \mu_F(t) \} \right\} \\
&\geq \bigwedge_{a=(xys)t} \left\{ \mu_E(p) \vee \mu_E(s) \vee \frac{1-k}{2} \vee \mu_F(t) \} \right\}.
\end{aligned}$$

Since $E = \langle \lambda_E, \mu_E \rangle$ is an interval valued $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S we have

$$\mu_E(xys) \geq \mu_E(x) \vee \mu_E(s) \vee \frac{1-k}{2}.$$

So,

$$\begin{aligned}
&\bigwedge_{a=(xys)t}\left\{\mu_E(p)\vee\mu_E(s)\vee\frac{1-k}{2}\vee\mu_F(t)\}\right\}\\
&\geq\bigwedge_{a=(xys)t}\left\{\mu_E(xys)\vee\mu_E(s)\vee\frac{1-k}{2}\}\right\}\\
&\geq\bigwedge_{a=uv}\left\{\mu_E(u)\vee\mu_F(v)\vee\frac{1-k}{2}\right\}=\left(\mu_E\circ_{\frac{1-k}{2}}\mu_F\right)(a).
\end{aligned}$$

Therefore, $\left(\left(\mu_E\circ_{\frac{1-k}{2}}\mu_F\right)\circ_{\frac{1-k}{2}}\left(\mu_E\circ_{\frac{1-k}{2}}\mu_F\right)\right)(a)\geq\left(\mu_E\circ_{\frac{1-k}{2}}\mu_F\right)(a)$ and hence $A\circ_{\frac{1-k}{2}}A\subseteq A$. Thus, $A=E\circ_{\frac{1-k}{2}}F$ is an intutionistic $(\in,\in\vee q_k)$-intuitionistic fuzzy subsemigroup of S.

Now, let $a,b,c\in S$. Then,

$$\begin{aligned}
&\left(\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)(a)\wedge\left(\lambda_E\circ_{\frac{1-k}{2}}\lambda_F\right)\right)(c)\wedge\frac{1-k}{2}\\
&=\bigvee_{a=xy}\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\frac{1-k}{2}\right\}\wedge\bigvee_{c=pq}\left\{\lambda_E(p)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}\wedge\frac{1-k}{2}\\
&=\bigvee_{a=xy}\bigvee_{c=pq}\left\{\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\frac{1-k}{2}\right\}\wedge\left\{\lambda_E(p)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}\wedge\frac{1-k}{2}\right\}\\
&\leq\bigvee_{a=xy}\bigvee_{c=pq}\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\lambda_E(p)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}\\
&\leq\bigvee_{a=xy}\bigvee_{c=pq}\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}.
\end{aligned}$$

Since $a=xy$ and $c=pq$, so $abc=(xy)\,b\,(pq)=(x\,(yb)\,p)\,q$ and we have

$$\begin{aligned}
&\bigvee_{a=xy}\bigvee_{c=pq}\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\lambda_F(q)\wedge\frac{1-k}{2}\right\}\\
&\leq\bigvee_{abc=(x(yb)p)q}\left\{\left\{\lambda_E(x)\wedge\lambda_F(y)\wedge\frac{1-k}{2}\right\}\wedge\lambda_F(q)\right\}.
\end{aligned}$$

Since $E=\langle\lambda_E,\mu_E\rangle$ is an interval valued $(\in,\in\vee q_k)$-intuitionistic fuzzy bi-ideal of S we have

$$\lambda_E(x\,(yb)\,p)\geq\lambda_E(x)\wedge\lambda_E(yb)\wedge\frac{1-k}{2}.$$

So,

$$\begin{aligned}
&\bigvee_{abc=(x(yb)p)q} \left\{ \left\{ \lambda_E(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} \right\} \wedge \lambda_F(q) \right\} \\
&\leq \bigvee_{abc=(x(yb)p)q} \left\{ \lambda_E(x(yb)p) \wedge \lambda_F(q) \wedge \frac{1-k}{2} \right\} \\
&= \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(abc).
\end{aligned}$$

Thus,

$$\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(abc) \geq \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(a) \wedge \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(c) \wedge \frac{1-k}{2}$$

and

$$\begin{aligned}
&\left(\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(a) \wedge \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right) \right)(c) \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=xy} \left\{ \lambda_E(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} \right\} \right\} \wedge \left\{ \bigvee_{c=pq} \lambda_E(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2} \right\} \wedge \frac{1-k}{2} \\
&= \bigvee_{a=xy} \bigvee_{c=pq} \left\{ \left\{ \lambda_E(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} \right\} \wedge \left\{ \lambda_E(p) \wedge \lambda_F(q) \wedge \frac{1-k}{2} \right\} \wedge \frac{1-k}{2} \right\} \\
&\leq \bigvee_{a=xy} \bigvee_{c=pq} \left\{ \{ \lambda_E(x) \wedge \lambda_F(y) \wedge \lambda_E(p) \wedge \lambda_F(q) \} \wedge \frac{1-k}{2} \right\} \\
&\leq \bigvee_{a=xy} \bigvee_{c=pq} \left\{ \lambda_E(x) \wedge \lambda_F(y) \wedge \lambda_F(q) \wedge \frac{1-k}{2} \right\}.
\end{aligned}$$

Since $a = xy$ and $c = pq$, so $abc = (xy)\, b\, (pq) = (x\,(yb)\,p)\, q$ and we have

$$\begin{aligned}
&\bigvee_{a=xy} \bigvee_{c=pq} \left\{ \lambda_E(x) \wedge \lambda_F(y) \wedge \lambda_F(q) \wedge \frac{1-k}{2} \right\} \\
&\leq \bigvee_{abc=(x(yb)p)q} \left\{ \left\{ \lambda_E(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} \right\} \wedge \lambda_F(q) \right\}.
\end{aligned}$$

Since $E = \langle \lambda_E, \mu_E \rangle$ is an interval valued $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S we have

$$\lambda_E(x\,(yb)\,p) \geq \lambda_E(x) \wedge \lambda_E(yb) \wedge \frac{1-k}{2}.$$

So,

$$\begin{aligned}
&\bigvee_{abc=(x(yb)p)q} \left\{ \left\{ \lambda_E(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} \right\} \wedge \lambda_F(q) \right\} \\
&\leq \bigvee_{abc=(x(yb)p)q} \left\{ \lambda_E(x(yb)p) \wedge \lambda_F(q) \wedge \frac{1-k}{2} \right\} \\
&= \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(abc).
\end{aligned}$$

Thus,

$$\left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(abc) \geq \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(a) \wedge \left(\lambda_E \circ_{\frac{1-k}{2}} \lambda_F \right)(c) \wedge \frac{1-k}{2}$$

and

$$\begin{aligned}
&\left(\left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right)(a) \vee \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right) \right)(c) \vee \frac{1-k}{2} \\
&= \begin{array}{c} \bigwedge_{a=xy} \left\{ \mu_E(x) \vee \mu_F(y) \vee \frac{1-k}{2} \right\} \vee \\ \bigwedge_{c=pq} \left\{ \mu_E(p) \vee \mu_F(q) \vee \frac{1-k}{2} \right\} \vee \frac{1-k}{2} \end{array} \\
&= \bigwedge_{a=xy} \bigwedge_{c=pq} \left\{ \begin{array}{c} \left\{ \mu_E(x) \vee \mu_F(y) \vee \frac{1-k}{2} \right\} \vee \\ \left\{ \mu_E(p) \vee \mu_F(q) \vee \frac{1-k}{2} \right\} \vee \frac{1-k}{2} \end{array} \right\} \\
&\leq \bigwedge_{a=xy} \bigwedge_{c=pq} \left\{ \mu_E(x) \vee \mu_F(y) \vee \mu_E(p) \vee \mu_F(q) \vee \frac{1-k}{2} \right\} \\
&\leq \bigwedge_{a=xy} \bigwedge_{c=pq} \left\{ \mu_E(x) \vee \lambda_F(y) \vee \mu_F(q) \vee \frac{1-k}{2} \right\}
\end{aligned}$$

Since $a = xy$ and $c = pq$, so $abc = (xy)b(pq) = (x(yb)p)q$ and we have

$$\begin{aligned}
&\bigwedge_{a=xy} \bigwedge_{c=pq} \left\{ \mu_E(x) \vee \mu_F(y) \vee \mu_F(q) \vee \frac{1-k}{2} \right\} \\
&\leq \bigwedge_{abc=(x(yb)p)q} \left\{ \left\{ \mu_E(x) \vee \mu_F(y) \vee \frac{1-k}{2} \right\} \vee \lambda_F(q) \right\}.
\end{aligned}$$

Since $E = \langle \lambda_E, \mu_E \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S we have

$$\mu_E(x(yb)p) \geq \mu_E(x) \vee \mu_E(yb) \vee \frac{1-k}{2}.$$

So,

$$\begin{aligned}
&\bigwedge_{abc=(x(yb)p)q} \left\{ \left\{ \mu_E(x) \vee \mu_F(y) \vee \frac{1-k}{2} \right\} \vee \mu_F(q) \right\} \\
&\leq \bigwedge_{abc=(x(yb)p)q} \left\{ \mu_E(x(yb)p) \vee \mu_F(q) \vee \frac{1-k}{2} \right\} = \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F \right)(abc).
\end{aligned}$$

Thus,

$$\left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(abc) \leq \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(a) \vee \left(\mu_E \circ_{\frac{1-k}{2}} \mu_F\right)(c) \vee \frac{1-k}{2}.$$

Hence, $A = E \circ_{\frac{1-k}{2}} F$ is an intutionistic $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. □

For any intutionstic fuzzy set $F = \langle \lambda_F, \mu_F \rangle$ in S and $t \in (0,1], r \in [0,1)$, we denote $F_{(t,r)} = \{x \in S : \langle x; (t,r) \rangle \, q_k F\}$ and $[F]_{(t,r)} = \{x \in S : \langle x; (t,r) \rangle \in \vee q_k E\}$.

Obviously, $[F]_{(t,r)} = F_{(t,r)} \cup U_{(t,r)}$, where $U_{(t,r)}$, $F_{(t,r)}$ and $[F]_{(t,r)}$ are called $\in$-level set, q_k-level set and $\in \vee q_k$-level set of $F = \langle \lambda_F, \mu_F \rangle$ respectively.

Theorem 3.31. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S if and only if for all $t \in (0,1]$ and $r \in [0,1)$, the set $U_{(t,r)} \neq \emptyset$ is a left (resp. right) ideal of S.*

Proof. Let $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy left ideal of S and $U_{(t,r)} \neq \emptyset$ for all $t \in (0,1]$ and $r \in [0,1)$. Let $y \in U_{(t,r)}$ and $x \in S$. Then, $\lambda_F(x) \geq t$ and $\lambda_F(x) \leq r$. Since

$$\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(b) \wedge \frac{1-k}{2} \geq t \wedge \frac{1-k}{2} \geq t$$

and

$$\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2} \leq r \vee \frac{1-k}{2} \leq r.$$

So, $xy \in U_{(t,r)}$. Hence $U_{(t,r)} \neq \emptyset$ is a left ideal of S.

Conversely, Let $F = \langle \lambda_F, \mu_F \rangle$ be an intutionistic fuzzy set in a way that $U_{(t,r)} \neq \emptyset$ is a left ideal of S. Assume that there exists $x, y \in S$ such that $\lambda_F(xy) < \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) > \mu_F(x) \vee \frac{1-k}{2}$. We choose $t \in (0,1]$ and $r \in [0,1)$, then $\lambda_F(xy) < t < \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) > r > \mu_F(x) \vee \frac{1-k}{2}$. Then $y \in U_{(t,r)}$, but $xy \notin U_{(t,r)}$, which is a contradiction. Hence, $\lambda_F(xy) \leq \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \geq \mu_F(x) \vee \frac{1-k}{2}$. □

Theorem 3.32. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S if and only if for all $t \in (0,1]$ and $r \in [0,1)$, the set $U_{(t,r)} \neq \emptyset$ is a bi-ideal of S.*

Proof. Proof of the Theorem follows from Theorem 3.31. □

Theorem 3.33. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1,2)$-ideal of S if and only if for all $t \in (0,1]$ and $r \in [0,1)$, the set $U_{(t,r)} \neq \emptyset$ is a $(1,2)$-ideal of S.*

Proof. Proof of the Theorem follows from Theorem 3.31. □

Theorem 3.34. *An IFS* $F = \langle \lambda_F, \mu_F \rangle$ *of* S *is an* $(\in, \in \vee q_k)$*-intuitionistic fuzzy subsemigroup of* S *if and only if for all* $t \in (0,1]$ *and* $r \in D[0,1)$, *the set* $[F]_{(t,r)} \neq \emptyset$ *is a subsemigroup of* S.

Proof. Let $x, y \in [F]_{(t,r)}$. Then, $\lambda_F(x) \geq t$ and $\mu_F(x) \leq r$ or $\lambda_F(x) + t + k > 1$ and $\mu_F(x) + r + k < 1$, and $\lambda_F(y) \geq t$ and $\mu_F(y) \leq r$ or $\lambda_F(y) + t + k > 1$ and $\mu_F(y) + r + k < 1$, thus we have the following four cases:

(i) $\lambda_F(x) \geq t$ and $\mu_F(x) \leq r$, and $\lambda_F(y) \geq t$ and $\mu_F(y) \leq r$,

(ii) $\lambda_F(x) \geq t$ and $\mu_F(x) \leq r$ and $\lambda_F(x) + t + k > 1$ and $\mu_F(y) + r + k < 1$,

(iii) $\lambda_F(x) + t + k > 1$ and $\mu_F(y) + r + k < 1$, and $\lambda_F(y) \geq t$ and $\mu_F(y) \leq r$,

(iv) $\lambda_F(x) + t + k > 1$ and $\mu_F(y) + r + k < 1$, and $\lambda_F(y) + t + k > 1$ and $\mu_F(y) + r + k < 1$.

For the first case, by Theorem 3.10 (i), it implies that

$$\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} = t \wedge \frac{1-k}{2} = \begin{cases} \frac{1-k}{2}, & \text{if } t > \frac{1-k}{2} \\ t, & \text{if } t \leq \frac{1-k}{2} \end{cases}$$

and

$$\begin{aligned} \mu_F(xy) &\leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2} \\ &= r \vee \frac{1-k}{2} = \begin{cases} \frac{1-k}{2}, & \text{if } s < \frac{1-k}{2} \\ r, & \text{if } r \leq \frac{1-k}{2} \end{cases} \end{aligned}$$

and hence, $\lambda_F(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and $\mu_F(xy) + r + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ which implies that $(xy)(t,r)\, q_k F$ or $xy \in F_{(t,r)}$. Hence $xy \in U_{(t,r)} \cup F_{(t,r)} = [F]_{(t,r)}$. For the second case we assume that $t > \frac{1-k}{2}$ and $r < \frac{1-k}{2}$. Then $1-t < \frac{1-k}{2}$ and $1-r > \frac{1-k}{2}$. If $\lambda_F(x) \geq \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(x) \leq \mu_F(y) \vee \frac{1-k}{2}$, then $\lambda_F(x) \geq \lambda_F(y) \wedge \frac{1-k}{2} > 1-t$ and $\mu_F(y) \leq \mu_F(y) \vee \frac{1-k}{2} < 1-r$ and if $\lambda_F(y) > \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(x) < \mu_F(y) \vee \frac{1-k}{2}$, then $\lambda_F(xy) \geq \lambda_F(x) \geq t$ and $\mu_F(xy) \leq \mu_F(x) \leq r$. Hence, $xy \in U_{(t,r)} \cup F_{(t,r)} = [F]_{(t,r)}$. Now suppose that $t \leq \frac{1-k}{2}$ and $r \geq \frac{1-k}{2}$. Then $1-t \geq \frac{1-k}{2}$ and $1-r \leq \frac{1-k}{2}$. If $\lambda_F(y) \geq \lambda_F(x) \wedge \frac{1-k}{2}$ and $\mu_F(y) \leq \mu_F(x) \vee \frac{1-k}{2}$, then $\lambda_F(xy) \geq \lambda_F(y) \wedge \frac{1-k}{2} \geq t$ and $\mu_F(x) \leq \mu_F(y) \vee \frac{1-k}{2} \leq r$ and if $\lambda_F(y) < \lambda_F(x) \wedge \frac{1-k}{2}$ and $\mu_F(y) > \mu_F(x) \vee \frac{1-k}{2}$, then $\lambda_F(xy) \geq \lambda_F(y) \geq 1-t$ and $\mu_F(xy) \leq \mu_F(y) \leq 1-r$. Hence, $ab \in U_{(t,r)} \cup F_{(t,r)} = [F]_{(t,r)}$. We have similar result for the case (iii). For the case four, if $t > \frac{1-k}{2}$ and $r < \frac{1-k}{2}$. Then$1-t < \frac{1-k}{2}$ and $1-r > \frac{1-k}{2}$. Hence,

$$\begin{aligned} \lambda_F(xy) &\geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} \\ &= \begin{cases} \frac{1-k}{2} > 1-t, & \text{if } \lambda_F(x) \wedge \lambda_F(y) \geq \frac{1-k}{2} \\ \lambda_F(x) \wedge \lambda_F(y) > 1-t, & \text{if } \lambda_F(x) \wedge \lambda_F(y) < \frac{1-k}{2} \end{cases} \end{aligned}$$

and

$$\begin{aligned}\mu_F(xy) &\leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2} \\ &= \begin{cases} \frac{1-k}{2} < 1-r, & \text{if } \mu_F(x) \vee \mu_F(y) \leq \frac{1-k}{2} \\ \mu_F(x) \vee \mu_F(y) > 1-r, & \text{if } \mu_F(x) \vee \mu_F(y) > \frac{1-k}{2} \end{cases}\end{aligned}$$

and hence $xy \in F_{(t,r)} \subseteq [F]_{(t,r)}$. If $t \leq \frac{1-k}{2}$ and $r \geq \frac{1-k}{2}$, then $\frac{1-k}{2} \leq 1-t$ and $\frac{1-k}{2} \geq 1-r$. Thus,

$$\begin{aligned}\lambda_F(xy) &\geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2} \\ &= \begin{cases} \frac{1-k}{2} \geq t, & \text{if } \lambda_F(x) \wedge \lambda_F(y) \geq \frac{1-k}{2} \\ \lambda_F(x) \wedge \lambda_F(y) > 1-t, & \text{if } \lambda_F(x) \wedge \lambda_F(y) < \frac{1-k}{2} \end{cases}\end{aligned}$$

and

$$\begin{aligned}\mu_F(xy) &\geq \mu_F(x) \wedge \mu_F(y) \wedge \frac{1-k}{2} \\ &= \begin{cases} \frac{1-k}{2} \leq r, & \text{if } \mu_F(x) \vee \mu_F(y) \leq \frac{1-k}{2} \\ \mu_F(x) \vee \mu_F(y) < 1-r, & \text{if } \mu_F(x) \vee \mu_F(y) > \frac{1-k}{2} \end{cases}\end{aligned}$$

which implies that $xy \in U_{(t,r)} \cup F_{(t,r)} = [F]_{(t,r)}$.

Conversely, assume that $F = \langle \lambda_F, \mu_F \rangle$ is not an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S. Then, there exists $x, y \in S$ such that $\lambda_F(xy) < \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) > \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$. Let

$$t = \frac{1}{2}\left[\lambda_F(xy) + \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}\right]$$

and

$$r = \frac{1}{2}\left[\mu_F(xy) + \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}\right].$$

Then,

$$\lambda_F(xy) < t < \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$$

and

$$\mu_F(xy) > r > \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}.$$

Which implies that $x, y \in [F]_{(t,r)}$ and $xy \in [F]_{(t,r)}$. Hence, $\lambda_F(xy) \geq t$ and $\mu_F(xy) \leq r$ or $\lambda_F(xy) + t + k > 1$ and $\mu_F(xy) + r + k < 1$, which is contradiction. Therefore, we have $\lambda_F(xy) \geq \lambda_F(x) \wedge \lambda_F(y) \wedge \frac{1-k}{2}$ and $\mu_F(xy) \leq \mu_F(x) \vee \mu_F(y) \vee \frac{1-k}{2}$. Thus, $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy subsemigroup of S. □

Theorem 3.35. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S if and only if for all $t \in (0, 1]$ and $r \in [0, 1)$, the set $[F]_{(t,r)} \neq \emptyset$ is a left (resp. right) ideal of S.*

Proof. Proof of the Theorem follows from Theorem 3.34. □

Theorem 3.36. *An IFS $F = \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S if and only if for all $t \in (0, 1]$ and $r \in [0, 1)$, the set $[F]_{(t,r)} \neq \emptyset$ is a bi-ideal of S.*

Proof. Proof of the Theorem follows from Theorem 3.34. □

Theorem 3.37. *An IFSF $= \langle \lambda_F, \mu_F \rangle$ of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideal of S if and only if for all $t \in (0, 1]$ and $r \in [0, 1)$, the set $[F]_{(t,r)} \neq \emptyset$ is a $(1, 2)$-ideal of S.*

Proof. Proof of the Theorem follows from Theorem 3.34. □

Theorem 3.38. *Every $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideal of S.*

Proof. Straightforward. □

Theorem 3.39. *In a regular semigroup every $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy left (resp. right) ideal of S.*

Proof. Suppose S is regular, then every bi-ideal of S is left (resp. right) ideal of S. Let $F = \langle \lambda_F, \mu_F \rangle$ be an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S and $x, y \in S$, xSx is a bi-ideal of S. Then aSa is a right ideal of S. Since S is regular, thus we have $ab \in (aSa)\, S \subseteq aSa$, this implies that $xy = xyx$ for some $y \in S$. Also since $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. It implies that

$$\lambda_F\,(xy) = \lambda_F\,(xyx) \geq \lambda_F\,(x) \wedge \lambda_F\,(x) \wedge \frac{1-k}{2} \geq \lambda_F\,(x) \wedge \frac{1-k}{2}$$

and

$$\mu_F\,(xy) = \mu_F\,(xyx) \leq \mu_F\,(x) \vee \mu_F\,(x) \vee \frac{1-k}{2} \leq \mu_F\,(x) \vee \frac{1-k}{2}.$$

Hence, $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy right ideal of S. □

Theorem 3.40. *In a regular semigroup every $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideal of S is an $(\in, \in \vee q_k)$-intuitionistic fuzzy left bi-ideal of S.*

Proof. Let S be a regular semigroup and $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideal of S. Let $a, b, x \in S$. Since S is regular, we have $ax \in (aSa)\, S \subseteq aSa$, this implies that $ax = asa$ for some $s \in S$. Thus

$$\lambda_F(axb) = \lambda_F((asa)b) = \lambda_F(ax(ab)) \geq \lambda_F(a) \wedge \lambda_F(a) \wedge \lambda_F(b) \wedge \frac{1-k}{2}$$

and $\mu_F(axb) = \mu_F((asa)b) = \mu_F(ax(ab)) \leq \mu_F(a) \vee \mu_F(a) \vee \mu_F(b) \vee \frac{1-k}{2}$. Hence, $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. □

Theorem 3.41. *Let $F = \langle \lambda_F, \mu_F \rangle$ is an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal of S. If S is completely regular and $\lambda_F(a) \leq \frac{1-k}{2}$, and $\mu_F(a) \geq \frac{1-k}{2}$ for all $a \in S$, then $F(a) = F(a^2)$ for all $a \in S$.*

Proof. Straightforward. □

4. Conclusion

It is recognized that semigroups are basic algebraic structures in several applied branches like automata and formal languages, coding theory, finite state machines and others. Due to these prospects of applications, semigroups are now widely studied in fuzzy setting. An intuitionistic fuzzy set is more substantial and brief to designate the essence of fuzziness. Intuitionistic fuzzy set theory is more appropriate than the fuzzy set theory for dealing with imperfect knowledge in several problems. In the structural study of semigroup, we notified that intuitionistic fuzzy ideals with superior properties continuously play an significant role. The intuitionistic fuzzy point of a semigroup S is basic tools to define the algebraic subsystems of S. So, we combined the above notions and initiated new types of intuitionistic fuzzy bi-ideals and $(1, 2)$-ideals of semigroups which are said to be an $(\in, \in \vee q_k)$-intuitionistic fuzzy bi-ideal and $(\in, \in \vee q_k)$-intuitionistic fuzzy $(1, 2)$-ideal. The results in the paper are generalizations of results about ordinary $(\in, \in \vee q)$- intuitionistic fuzzy ideals in semigroups. In future, we will focus on the following topics:

(1) Characterizations of regular semigroups by the properties of $(\in, \in \vee q_k)$-intuitionistic fuzzy ideals

(2) We will define $(\in, \in \vee q_k)$-intuitionistic fuzzy (interior, prime, generalized bi, prime bi) ideals of a semigroup and characterize different classes of semigroups by the properties of $(\in, \in \vee q_k)$-intuitionistic-fuzzy ideals. In future we will extend our study to other algebraic structures like ring theory, module theory, soft semigroups etc.

References

[1] L.A. Zadeh, *Fuzzy sets*, Inform. and Control, 8 (1965), 338-353.

[2] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., 35 (1971), 512-517.

[3] N. Kuroki, *On fuzzy semigroups*, Inform. Sci., 53 (1991), 203-236.

[4] N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Systems, 5 (1981), 203-215.

[5] N. Kuroki, *Fuzzy semiprime ideals in semigroups*, Fuzzy Sets and Systems, 8 (1982), 71-79.

[6] N. Kuroki, *On fuzzy semigroups*, Inform. Sci., 53 (1991), 203-236.

[7] N. Kuroki, *Fuzzy generalized bi-ideals in semigroups*, Inform. Sci., 66 (1992), 235-243.

[8] N. Kuroki, *On fuzzy semiprime quasi-ideals in semigroups*, Inform. Sci., 75 (1993), 201-211.

[9] N. Kuroki, *Fuzzy bi-ideals in semigroups*, Commentarii Mathematici Universitatis Sancti Pauli, 28 (1979), 17-21.

[10] N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Systems, 5 (1981), 203-215.

[11] J.N. Mordeson, D.S. Malik and N. Kuroki, *Fuzzy semigroups*, Studies in Fuzziness and Soft Computing, Springer-Verlag Berlin, 131 (2003).

[12] J.N. Mordeson and D.S. Malik, *Fuzzy automata and languages, theory and applications*, Computational Mathematics Series, Chapman and Hall/CRC, Boca Raton, 2002.

[13] R. Ameri and T. Nozeri, *Fuzzy hyperalgebras*, Comp. Math. App., 61 (2011), 149-154.

[14] B. Davvaz, J. Jhan and Y. Yin, *Fuzzy Hv-ideals in* Γ*-Hv-rings*, Comp. Math. App., 61 (2011), 690-698.

[15] B. Davvaz, J. Zhan and K.H. Kim, *Fuzzy* Γ*-hypernear-rings*, Comp. Math. App., 59 (2010), 2846-2853.

[16] B. Davvaz, *Fuzzy krasner* (m, n)*-hyperrings*, Comp. Math. App., 59 (2010), 3879-3891.

[17] K. Sun, X. Yuan and H. Li, *Fuzzy hypergraphs on fuzzy relations*, Comp. Math. App., 60 (2010), 610-622.

[18] P.M. Pu and M. Liu, *Fuzzy topology I: nieghbourhood structure of a fuzzy point and moore-smith convergence*, J. Math. Anl. Appl., 76 (1980), 571-599.

[19] V. Murali, *Fuzzy points of equivalent fuzzy subsets*, Inform. Sci., 158 (2004), 277-288.

[20] S.K. Bhakat and P. Das, $(\in, \in \vee q_k)$*-fuzzy subgroup*, Fuzzy Sets and Systems, 80 (1996), 359-368.

[21] O. Kazanci and S. Yamak, *Generalized fuzzy bi-ideal of semigroups*, Soft. Comput., 12 (2008), 1119-1124.

[22] Y.B. Jun and S.Z. Song, *Generalized fuzzy interior ideals in semigroups*, Inform. Sci., 176 (2006), 3079-3093.

[23] M. Shabir, Y.B. Jun and Y. Nawaz, *Characterizations of regular semigroups by (α,β)-fuzzy ideals*, Comp. Math. App., 59 (2010), 161-175.

[24] M. Shabir, Y.B. Jun and Y. Nawaz, *Semigroups characterized by* $(\in, \in \vee q_k)$-*fuzzy ideals,* Comp. Math. App., 60 (2010), 1473-1493.

[25] M. Shabir and T. Mehmood, *Characterizations of hemirings by* $(\in, \in \vee q_k)$-*fuzzy ideals*, Comp. Math. App., 61 (2011), 1059-1078.

[26] M. Aslam, S. Abdullah and N. Amin, *Characterization of gamma LA-semigroups by generalized fuzzy gamma ideals*, Int. J. Math. Statistics, 12 (2012), 29-50.

[27] K.T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986), 87-96.

[28] R. Biswas, *Intuitionistic fuzzy subgroups*, Mathematical Fortum, 10 (1989), 37-46.

[29] K.H. Kim and Y.B. Jun, *Intuitionistic fuzzy ideals in semigroups*, Indian J. pure Appl. Math., 33 (2002), 443-449.

[30] K.H. Kim and J.G. Lee, *On intuitionistic fuzzy bi-ideals of semigroups*, Turk. J. Math., 29 (2005), 201-210.

[31] K.H. Kim and Y.B. Jun, *Intuitionistic fuzzy interior ideals of semigroups*, Int. J. Math. Math. Anal., 27 (2001), 261-267.

[32] D. Coker and M. Demirci, *On intuitionistic fuzzy points*, Notes IVIFS, 1 (1995), 79-84.

[33] Y.B. Jun, *On* (Φ, Ψ)-*intuitionistic fuzzy subgroups*, KYUNGPOOK Math. J., 45 (2005), 87-94.

[34] S. Abdullah and M. Aslam, (Φ, Ψ)-*intuitionistic fuzzy ideals in semigroups*, Italian J. Pure Appl. Math., 32 (in press).

[35] S. Abdullah, B. Davvaz and M. Aslam, (α, β)-*intuitionistic fuzzy ideals of hemirings*, Computer and Mathematics with Applications, 62 (2011), 3077-3090.

[36] A. Khan, B. Davvaz, N. H. Sarmin and H. Ullah Khan, *Redefined intuitionistic fuzzy bi-ideals of ordered semigroups*, Journal of Inequalities and Applications 2013, 397.

Accepted: 13.03.2017