ON THE p-NILPOTENCE OF FINITE GROUPS

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Abstract. In this paper, we give a new definition–a CSS subgroup(a subgroup H of a finite group G is called a CSS-subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is SS-quasinormal in G). By this definition, we investigate the relationship between the p-nilpotence of G and the p-nilpotence of $N_G(P)$, and generalize the corresponding results to a saturated formation $\mathcal{F}$ which contains the class $\mathcal{N}_p$ of all p-nilpotent groups, where p is an odd prime factor of $|G|$, P a Sylow p-subgroup of a group G.

Keywords: CSS-subgroups, Sylow subgroups, mxiamal subgroups, p-nilpotent groups, saturated formations.

1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G(see [5]). Recall that a subgroup H of a group G is c-normal in G if there is a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the core of H in G(see [8]). Recently, Li[6] defined that a subgroup H of G is said to be an SS-quasinormal subgroup of G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. By use of these definitions, many authors investigated such structure of a group as the nilpotence, the supersolvability and so on, see [8, 1, 6, 4, 2, 5].

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Enlightened by the above concepts, we give a new definition —– a CSS-subgroup, which is a generalization of C-normality and SS-quasinormality.

Definition 1.1. A subgroup H of a group G is called a CSS-subgroup of G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is SS-quasinormal in G. In this case, K is called a normal CSS-supplement of H in G.

Remark. Obviously, if a subgroup H of G is c-normal in G, then there is a normal subgroup K_1 of G such that $G = HK_1$ and $H \cap K_1 \leq H_G$. In this case, writing $K = H_G K_1$, then we have $G = HK$ and $H \cap K = H_G$; Of course, $H \cap K$ is SS-quasinormal in G. Therefore c-normal subgroups of G also are CSS-subgroups of G. Besides, a SS-quasinormal subgroup H of G, by taking $K = G$, must be a CSS-subgroup.

However, the following examples show the above converse is not true.

Example 1. Let $G = A_5$ be the alternating group of degree five. We have $G = A_4C_5$. Then A_4 is SS-quasinormal in G, and is a CSS-subgroup of G. However, A_4 is not c-normal in G.

Example 2. Consider $G = S_4$, the symmetric group of degree four. Take $\alpha = (34)$ and $\beta = (123)$. Then $G = \langle\alpha\rangle A_4$ and $\langle\alpha\rangle \cap A_4 = 1$, hence $\langle\alpha\rangle$ is c-normal in G. Of course $\langle\alpha\rangle$ is a CSS-subgroup of G. Let B be a subgroup of G satisfying $G = \langle\alpha\rangle B$. Then B is either A_4 or G. As $\langle\alpha\rangle\langle\beta\rangle \neq \langle\beta\rangle\langle\alpha\rangle$, which indicates that $\langle\alpha\rangle$ is not SS-quasinormal in G.

In Section 2 of this paper, we give some properties of CSS-subgroups. Let p be an odd prime factor of $|G|$, P a Sylow p-subgroup of a group G. In Section 3, we investigate the relationship between p-nilpotence of G and p-nilpotence of $N_G(P)$, and generalize the corresponding results to a saturated formation $\mathcal{F}$ which contains the class $\mathcal{N}_p$ of all p-nilpotent groups.

2. Preliminaries

In this section we list some known results and some properties about CSS-subgroups which are needed in the main results.

Lemma 2.1 ([6, Lemma 2.1]). *Suppose that H is SS-quasinormal in a group G, $K \leq G$ and N a normal subgroup of G. We have:*

(1) *If $H \leq K$, then H is SS-quasinormal in K.*

(2) *HN/N is SS-quasinormal in G/N.*

(3) *If $N \leq K$ and K/N is SS-quasinormal in G/N, then K is SS-quasinormal in G.*

(4) *If K is quasinormal in G, then HK is SS-quasinormal in G.*

Lemma 2.2 ([6, Lemma 2.5])**.** *If a p-subgroup P of G is SS-quasinormal, where p is a prime. Then P permutes with every Sylow q-subgroup of G with $q \neq p$.*

Lemma 2.3. *Let H be a CSS-subgroup of a group G. We have:*

(1) *If $H \leq M \leq G$, then H is a CSS-subgroup of M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$, then H is a CSS-subgroup of G if and only if H/N is a CSS-subgroup of G/N.*

(3) *Let π be a set of some primes and N a normal π'-subgroup of G. If H is a π-subgroup of G, then HN/N is a CSS-subgroup of G/N.*

Proof. (1) Since H is a CSS-subgroup of G, there exists a subgroup $K \trianglelefteq G$ such that $G = HK$ and $H \cap K$ is SS-quasinormal in G. Notice that $H \cap K \leq H \leq M \leq G$ and $K \trianglelefteq G$, we have that $M = H(M \cap K)$ and $M \cap K \trianglelefteq M$. On the other hand, Lemma 2.1.1 shows that $H \cap K$ is SS-quasinormal in M. Also, $H \cap (M \cap K) = H \cap K$, so $H \cap (M \cap K)$ is SS-quasinormal in M. Therefore H is a CSS-subgroup of M.

(2) Let $K \trianglelefteq G$ be a CSS-supplement of H in G, then $G = HK$ and $H \cap K$ is SS-quasinormal in G. Consider the group KN/N, we have: $KN/N \trianglelefteq G$, $G/N = HK/N = (H/N) \cdot (KN/N)$, and $(H/N) \cap (KN/N) = (H \cap K)N/N$ is SS-quasinormal in G/N by Lemma 2.1(2). Thus H/N is a CSS-subgroup of G/N.

Conversely, if H/N is a CSS-subgroup of G/N, then there exists a normal subgroup K/N of G/N such that $G/N = (H/N) \cdot (K/N)$ and $(H/N) \cap (K/N)$ is SS-quasinormal in G/N. Obviously, $G = HK$ and $K \trianglelefteq G$. Also, by Lemma 2.1(3), $H \cap K$ is SS-quasinormal in G. Thus H is a CSS-subgroup of G.

(3) Let K be a normal CSS-supplement of H in G, then $G = HK$, and $H \cap K$ is SS-quasinormal in G. Since N is a normal π'-subgroup and H is a π-subgroup of G, then $N \leq K$. Consequently $G/N = (HN/N) \cdot (K/N)$ and $K/N \trianglelefteq G/N$. Also, $(HN/N) \cap (K/N) = (H \cap K)N/N$ is SS-quasinormal in G/N by Lemma 2.1.2. Therefore HN/N is a CSS-subgroup of G/N. □

Lemma 2.4. *Let P be a Sylow p-subgroup of a finite group G and N a normal subgroup of G. If every maximal subgroup of P is a CSS-subgroup of G, then every maximal subgroup of PN/N is a CSS-subgroup of G/N.*

Proof. Let S/N be a maximal subgroup of PN/N, and T a Sylow p-subgroup of S. Without loss of generality, we may assume that $T \leq P$. Obviously, $S/N = TN/N$. Notice that $p = |PN : T| = |PN : TN| = |P : T|$, it shows that T is a maximal subgroup of P. By hypothesis of the theorem, T is a CSS-subgroup of G, so there exists a normal subgroup K such that $G = TK$ and $T \cap K$ is SS-quasinormal in G. Let K_q be a Sylow q-subgroup of K, where q is a prime factor of $|K|$ and $q \neq p$. Obviously, K_q is also a Sylow q-subgroup of G, and $N \cap K_q$ is a Sylow q-subgroup of N. Set $D = \langle N \cap K_q \,|q \neq p\rangle$, then $D \leq K$ and $N = (T \cap N)D$. So $TN \cap KN = (TN \cap K)N = (TD \cap K)N = (T \cap K)DN =$

$(T \cap K)N$. By Lemma 2.1(2), we know $TN/N \cap KN/N = (T \cap K)N/N$ is SS-quasinormal in G/N, it is to say S/N is SS-quasinormal in G/N. □

Lemma 2.5. *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

Proof. Notice that $P \trianglelefteq G$, so $M \leq N_G(P \cap M)$. Also, it is clear that $P \cap M$ is not a Sylow p-subgroup of G, so $P \cap M \lneqq N_P(P \cap M)$. It follows that $M \lneqq N_G(P \cap M)$. By the maximality of M, we have that $N_G(P \cap M) = G$, which implies that $(P \cap M) \trianglelefteq G$. □

Lemma 2.6 ([3, Proposition IV.3.11]). *Let $\mathcal{F}_1 = LF(F_1)$ and $\mathcal{F}_2 = LF(F_2)$, where F_i is both an integrated and full formation function of $\mathcal{F}_i (i = 1,\ 2)$. Then the following statements are equivalent:*

(1) $\mathcal{F}_1 \subseteq \mathcal{F}_2$,

(2) $F_1(p) \subseteq F_2(p)$ *for all* $p \in P$.

3. Main results

Theorem 3.1. *Let H be a normal subgroup of a group G, p an odd prime factor of $|H|$ and P a Sylow p-subgroup of H. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is a CSS-subgroup of G, then G is p-nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. We proceed by the following steps:

Step 1. $O_{p'}(G) = 1$.

Otherwise, consider the quotient group $G/O_{p'}(G)$ and $HO_{p'}(G)/O_{p'}(G)$. For convenience, we write $D = O_{p'}(G)$. Obviously, PD/D is a Sylow p-subgroup of HD/D. Notice that $(|D|, p) = 1$, so we have $N_{G/D}(PD/D) = N_G(P)D/D$, which shows that $N_{G/D}(PD/D)$ is p-nilpotent. By applying Lemma 2.4, we have that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. The choice of G implies that $G/O_{p'}(G)$ is p-nilpotent, so is G, a contradiction.

Step 2. *If $P \leq L < G$, then L is p-nilpotent.*

As $N_L(P) \leq N_G(P)$, so L satisfies the hypotheses of the theorem by Lemma 2.3.1. The minimality of G implies that L is p-nilpotent, as desired.

Step 3. *3*.1 $H = G$;

3.2 If K be a normal subgroup of G, then G/K is p-nilpotent. So G has a unique minimal normal subgroup N and $N \nleq \Phi(G)$.

3.1 If $H < G$, by using Step 2, we have that H is p-nilpotent. Also, Step 1 implies that H is a p-group, so $H = P$. Therefore, $N_G(P) = N_G(H) = G$ is p-nilpotent, a contradiction.

3.2 It is clear that P is a Sylow p-subgroup by 3.1, therefore $N_{G/K}(PK/K) = N_G(P)K/K$ is p-nilpotent. Applying Lemma 2.4, we have that G/K satisfies the hypotheses of the theorem. So the choice of G implies that G/K is p-nilpotent.

Since the class of all p-nilpotent groups is a saturated formation, we may assume that G has the unique minimal normal subgroup, say N and $N \nleq \Phi(G)$.

Step 4. *$O_p(G) > 1$, moreover, G is p-solvable.*

Since G is not p-nilpotent, by a result of Thompson [[7], Corollary], there exists a non-trivial character subgroup T of P such that $N_G(T)$ is not p-nilpotent. Now, T *char* $P \trianglelefteq N_G(P)$, so $N_G(P) \leq N_G(T)$. Step 2 implies that $N_G(T) = G$, so $T \trianglelefteq G$, therefore $O_p(G) > 1$. By Step 3.2, we have that $G/O_p(G)$ is p-nilpotent, so G ia p-solvable.

Step 5. *$N = O_p(G)$ and $|N| = p$.*

Since $O_p(G) > 1$, by Step 3.2, we have that $N \leq O_p(G)$. Therefore there exists a maximal subgroup M such that $G = NM$ and $N \cap M = 1$. So $G = O_p(G)M$. By Lemma 2.5, we have $O_p(G) \cap M = 1$, hence $N = O_p(G)$.

Next, we affirm that $N \nleq \Phi(P)$. Otherwise, $N \leq \Phi(G)$, a contradiction to Step 3.2. Therefore there exists a maximal subgroup P_1 of P such that $N \nleq P_1$, so $P = NP_1$. Put $N_1 = N \cap P_1$, then $|N : N_1| = |N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p$. By hypotheses, there exists a normal subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1$ is SS-quasinormal in G. By the minimality of N, we have that $N \leq K_1$ and $P \cap K_1 = P_1N \cap K_1 = (P_1 \cap K_1)N$ is a Sylow p-subgroup of K_1. Let K_{1q} be a Sylow q-subgroup of K_1, where $q \neq p$. K_{1q} is also a Sylow q-subgroup of G. By Lemma 2.2, we have $(P_1 \cap K_1)K_{1q}$ is a subgroup. Since $N \leqslant K_1$, it follows that $N_1 = N \cap (P_1 \cap K_1)K_{1q} \trianglelefteq (P_1 \cap K_1)K_{1q}$. Therefore N_1 is normal in the subgroup $\langle N, (P_1 \cap K_1)K_{1q} | q \in \pi(G), q \neq p\rangle = K_1$. On the other hand, $N_1 = N \cap P_1 \trianglelefteq P_1$, then $N_1 \trianglelefteq P_1K_1 = G$. The minimality of N yields $N_1 = 1$. Consequently, N is a cyclic subgroup of order p.

Step 6. *The final contradiction.*

Notice that $N = O_p(G) \trianglelefteq G$ and $|N| = p$, we have that $N \leq Z(P)$, so $P \leq C_G(N)$. On the other hand, G is p-solvable and $O_{p'}(G) = 1$, so $C_G(O_p(G)) \leq O_p(G)$. It follows that $P = O_p(G) = N$, hence $N_G(P) = G$. Therefore, by using the hypothesis of the theorem, we have that G is p-nilpotent, a contradiction. □

Remark. In Theorem 3.1, if take $H = G$, then we have the following:

Corollary 3.2. *Let G be a group, and P a Sylow p-subgroup of G such that $N_G(P)$ is p-nilpotent, where p is an odd prime factor of $|G|$. If every maximal subgroup of P is a CSS-subgroup of G, then G is p-nilpotent.*

Corollary 3.3. *Let H be a normal subgroup of a group G and p an odd prime factor of $|H|$. Also, let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{N}_p$ of all p-nilpotent groups and $G/H \in \mathcal{F}$. If $N_G(P)$ is p-nilpotent and every maximal* *subgroup of P is a CSS-subgroup of G, then $G \in \mathcal{F}$, where P is a Sylow p-subgroup of H.*

Proof. It is clear that $N_H(P)$ is p-nilpotent and every maximal subgroup of P is a CSS-subgroup of H. By Corollary 3.2, we have that H is p-nilpotent. Now let $H_{p'}$ be the normal Hall p'-subgroup of H, then $H_{p'} \trianglelefteq G$. By using similar arguments such as in the proof of Theorem 3.1, we have that G/H satisfies the hypotheses of the corollary, so $G/H \in \mathcal{F}$ by induction. Let $F_i(i = 1, 2)$ be the full and integrated formation function such that $\mathcal{N}_p = LF(F_1)$ and $\mathcal{F} = LF(F_2)$, respectively, then $G/C_G(K_1/K_2) \in F_1(q)$ for every chief factor K_1/K_2 of G with $K_1 \leq H_{p'}$ and every prime q dividing $|K_1/K_2|$. By Lemma 2.6, we have that $G/C_G(K_1/K_2) \in F_2(q)$ for every chief factor K_1/K_2 of G with $K_1 \leq H_{p'}$ and every prime q dividing $|K_1/K_2|$. Therefore, it follows that $G \in \mathcal{F}$. Hence, we may assume that $H_{p'} = 1$ and henceforth $H = P$ is a p-group. So, by the hypotheses of the corollary, $N_G(P) = G$ is p-nilpotent and therefore $G \in \mathcal{F}$. □

Remark. In Theorem 3.1, Corollary 3.2 and Corollary 3.3, the assumption that "$N_G(P)$ is p-nilpotent" is necessaey. For example, we consider the group $G = A_5$ and $p = 5$. In this case, since every maximal subgroup of Sylow 5-subgroup of G is 1, we see that every maximal subgroup of Sylow 5-subgroup of G is a CSS-subgroup of G, but G is not 5-nilpotent.

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References

[1] M. Asaad and A. A. Heliel, *On S-quasinormally embedded subgroups of finite groups*, J. Pure Appl. Algebra, 165 (2001), 129-135.

[2] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, *Sufficient conditions for supersolubility of finite groups,* J. Pure Appl. Algebra, 127 (1998), 113-118.

[3] K. Doerk and T. Hawkes, *Finite Solvable Groups*, Walter de Gruyter, Berlin-New York, 1992.

[4] X. Y. Guo and K. P. Shum, *On c-normal maximal and minimal subgroups of Sylow p-subgroups of finite groups,* Arch. Math., 80 (2003), 561-569.

[5] O. H. Kegel, *Sylow-gruppen und subnormal endlicher gruppen,* Math. Z., 78 (1962), 205-201.

[6] S. Li and Z. Shen, *The influence of SS-quasinormality of some subgroups on the structure of finite groups,* J. Algebra, 319 (2008), 4275-4287.

[7] J. G. Thompson, *Normal p-complements for finite groups,* J. Algebra, 1 (1964), 43-46.

[8] Y. Wang, *c-normality of groups and its properties,* J. Algebra, 180 (1996), 954-965.

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