ERROR ESTIMATES OF FINITE VOLUME ELEMENT METHOD FOR NONLINEAR HYPERBOLIC OPTIMAL CONTROL PROBLEMS

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Abstract. The goal of this paper is to investigote the error estimates of the fininte volume element approximation of optimal control problems governed by nonlinear hy-

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perbolic equations. By using optimize-then-discretize, variational discretization and the finite volume method to solve the distributed optimal control problems. A semi-discrete optimal system is obtained. Meanwhile, we obtain the optimal order error estimates in $L^\infty(J;L^2)$ and $L^\infty(J;H^1)$-norm.

Keywords: error estimates, variational discretization, hyperbolic optimal control problems, finite volume element method.

1. Introduction

As is known to all, optimal control problems are widely used in science and engineering. Over the past decade, a large number of numerical methods have been applied to approximate the solutions of these optimal control problems, such as finite element method, mixed finite element method, spectral method, and finite volume method, see, e.g., [25, 24, 22, 11, 9, 13, 10]. For the finite element method, some error estimates for the finite element approximation of a class of nonlinear optimal control problems can be found in [28, 29]. The error estimates of mixed finite element approximation for optimal control problems are investigated in [7, 27, 23]. Furthermore, in [26], the finite volume element method is applied to solve the distributed optimal control problems governed by hyperbolic equation, and a priori error estimates were presented. There are plenty of others studies of the numerical methods for the optimal control problems, see, e.g., [19, 30, 31, 32, 33, 34, 20].

Finite volume method, as a type of important numerical tool for solving differential equations, has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer and petroleum engineering. Perhaps the most important property of Finite volume method is that it can preserve the conservation laws (mass, momentum and heat flux) on each computational cell. This important property, combined with adequate accuracy and ease of implementation, has attracted more people to do research in this field. There have been a lot of studies of the mathematical theory for finite volume element methods, see, e.g., [4, 5, 6, 8, 14, 16] and the references cited therein.

In this paper, we use the standard notations $W^{m,p}(\Omega)$ for Sobolev spaces and their associated norms $||v||_{m,p}$ (see, e.g., [1, 3]). To simplify the notations, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and drop the index $p=2$ and Ω whenever possible, i.e., $||u||_{m,2,\Omega} = ||u||_{m,2} = ||u||_m,\ ||u||_0 = ||u||$. Let $H_0^1(\Omega) = \{v \in H^1 : v|_{\partial\Omega} = 0\}$. As usual, we use $(\cdot,\cdot)$ to denote the $L^2(\Omega)$-inner product. We denote by $L^s(J;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $||v||_{L^s(J;W^{m,p}(\Omega))} = (\int_0^T ||v||^s_{W^{m,p}(\Omega)}dt)^{1/s}$ for $s \in [1,\infty)$ and the standard modification for $s = \infty$.

Now, we consider the following optimal control problem:

$$(1.1)\quad \min_{u\in U_{ad}} \frac{1}{2}\int_0^T (||y - y_d(x,t)||^2_{L^2(\Omega)} + ||u(x,t)||^2_{L^2(\Omega)})dt,$$

$$(1.2)\quad y_{tt}(x,t) - \nabla\cdot(A\nabla y(x,t)) + \phi(y(x,t)) = Bu(x,t) + f(x,t),\ t \in J, x \in \Omega,$$

$$y(x,t) = 0, \quad t \in J, x \in \Gamma, \tag{1.3}$$

$$y(x,0) = y_0(x), \; y_t(x,0) = g(x), \quad x \in \Omega, \tag{1.4}$$

where

$$\nabla \cdot (A\nabla y) = \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right),$$

$\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain and Γ is the boundary of Ω, $f(\cdot,t)$, $y_d(\cdot,t) \in L^2(\Omega)$ or $H^1(\Omega)$, $J = (0,T]$, $A = (a_{i,j})_{2\times 2}$ is a symmetric, smooth enough and uniformly positive definite matrix in Ω, $B : L^2(J; L^2(\Omega)) \to L^2(J; L^2(\Omega))$ is a bounded continuous linear operator, $y_0(x) = 0, x \in \Gamma$, $y_0(x) \in H^3(\Omega)$, $g(x) \in H^2(\Omega)$. ϕ is of class C^2 with respect to the variable y, for any $R > 0$ the function $\phi(\cdot) \in W^{2,\infty}(-R,R)$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi'(y) \geq 0$. U_{ad} is a set defined by

$$U_{ad} = \{u : \; u \in L^2(J; L^2(\Omega)), \; u(x,t) \geq 0, \; a.e. \text{ in } \Omega, t \in J, \; a, b \in \mathbb{R}\}.$$

The rest of this paper is organized as follows. In Section 2, we present some notations. In Section 3, we apply finite volume method and variational discretization concept to the problem (1.1)-(1.4) and obtain the discretized optimal system. In Section 4, we analyze the error estimates between the exact solution and the finite volume element approximation.

2. Notations and preliminaeies

For a convex polygonal domain Ω, we consider a quasi-uniform triangulation $\mathcal{T}_h$ consisting of closed triangle elements K such that $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$. We use N_h to denote the set of all nodes or vertices of $\mathcal{T}_h$. To define the dual partition $\mathcal{T}_h^*$ of $\mathcal{T}_h$, we divide each $K \in \mathcal{T}_h$ into three quadrilaterals by connecting the barycenter C_K of K with line segments to the midpoints of edges of Kas is shown in Figure 1.

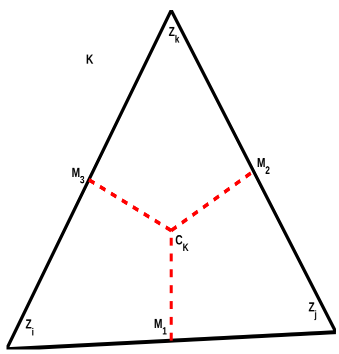

Figure 1. The dual partition of a triangular K.

The control volume V_i consists of the quadrilaterals sharing the same vertex z_i as is shown in Figure 2.

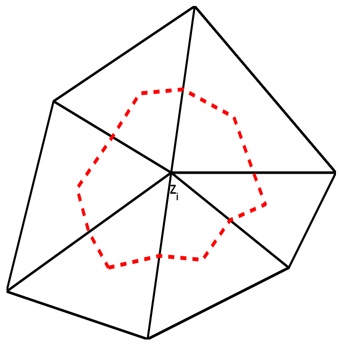

Figure 2. The control volume V_i sharing the same vertex z_i.

The dual partition $\mathcal{T}_h^*$ consists of the union of the control volume V_i. Let $h = \max\{h_K\}$, where h_K is the diameter of the triangle K. As is shown in [17], the dual partition $\mathcal{T}_h^*$ is also quasi-uniform. Throughout this paper, the constant C denotes different positive constant at each occurrence, which is independent of the mesh size h and the time step k.

We define the finite dimensional space V_h (i.e. trial space) associated with $\mathcal{T}_h$ for the trial functions by $V_h = \{v : v \in C(\Omega), v|_K \in P_1(K), \forall K \in \mathcal{T}_h, v|_\Gamma = 0\}$ and define the finite dimensional space Q_h (i.e. test space) associated with the dual partition $\mathcal{T}_h^*$ for the test functions by $Q_h = \{q : q \in L^2(\Omega), q|_V \in P_0(V), \forall V \in \mathcal{T}_h^*; q|_{V_z} = 0, z \in \Gamma\}$, where $P_l(K)$ or $P_l(V)$ consists of all the polynomials with degree less than or equal to l defined on K or V.

To connect the trial space and test space, we define a transfer operator $I_h : V_h \to Q_h$ as follows:

$$I_h v_h = \sum_{z_i \in N_h} v_h(z_i)\chi_i, \quad I_h v_h|_{V_i} = v_h(z_i), \quad \forall V_i \in \mathcal{T}_h^*,$$

where χ_i is the characteristic function of V_i. For the operator I_h, it is well known that there exists a positive constant C such that for all $v \in V_h$

$$||v - I_h v|| \leq Ch||v||_1. \tag{2.1}$$

Let $a(w, v) = \int_\Omega A\nabla w \cdot \nabla v dx$. We define the standard Ritz projection $R_h : H^2 \cap H_0^1 \to V_h$ by

$$a(R_h u, \chi) = a(u, \chi), \ \forall \chi \in V_h. \tag{2.2}$$

And let

$$a_h(\phi, I_h\psi) = -\sum_{z_i \in N_h} \psi(z_i) \int_{\partial V_i} A\nabla\phi \cdot \mathbf{n} ds,$$

where $\mathbf{n}$ is the unit outward normal vector to ∂V_i. π_h is defined as the linear interpolation on the triangulation $\mathcal{T}_h$.

3. Finite volume method for optimal control problems

In this section, we will use the optimize-then-discretize approach to obtain the finite volume element approximation for nonlinear hyperbolic optimal control problems.

It is well known (see, e.g.,[29]) that the optimal control problem (1.1)-(1.4) has a solution $(y(\cdot,t), p(\cdot,t), u(\cdot,t))$, and that if a triplet $(y(\cdot,t), p(\cdot,t), u(\cdot,t))$ is the solution of (1.1)-(1.4), then there is a co-state $p(\cdot,t) \in H_0^1(\Omega)$ such that $(y(\cdot,t), p(\cdot,t), u(\cdot,t))$ satisfies the following optimality conditions:

$$(y_{tt}, w) + (A\nabla y, \nabla w) + (\phi(y), w) = (Bu + f, w), \ \forall\ w \in H_0^1(\Omega), \quad y(x,0) = y_0(x), \quad y_t(x,0) = g(x); \tag{3.1}$$

$$(p_{tt}, q) + (A\nabla p, \nabla q) + (\phi'(y)p, q) = (y - y_d, q), \ \forall\ q \in H_0^1(\Omega), \quad p(x,T) = 0, \quad p_t(x,T) = 0; \tag{3.2}$$

$$\int_0^T (u + B^*p, v - u) d\tau \geq 0, \ \forall\ v \in U_{ad}. \tag{3.3}$$

If $y(\cdot,t) \in H_0^1(\Omega) \cap C^2(\Omega)$ and $p(\cdot,t) \in H_0^1(\Omega) \cap C^2(\Omega)$, then the optimal system (3.1)-(3.3) can be written by

$$y_{tt} - \nabla\cdot(A\nabla y) + \phi(y) = Bu + f, t \in J, x \in \Omega, y(x,t) = 0, t \in J, x \in \Gamma, \quad y(x,0) = y_0(x), \quad y_t(x,0) = g(x), \ x \in \Omega; \tag{3.4}$$

$$p_{tt} - \nabla\cdot(A\nabla p) + \phi'(y)p = y - y_d, t \in J, x \in \Omega, \ p(x,t) = 0, t \in J, x \in \Gamma, \quad p(x,T) = 0, \quad p_t(x,T) = 0, \quad x \in \Omega; \tag{3.5}$$

$$\int_0^T (u + B^*p, v - u) d\tau \geq 0, \ \forall\ v \in U_{ad}. \tag{3.6}$$

We use the finite volume method to discretized the state and costate equations directly. Then the optimal control problem (3.4)-(3.6) again has a solution $(y_h(\cdot,t), p_h(\cdot,t), u_h(\cdot,t))$, and that if a triplet $(y_h(\cdot,t), p_h(\cdot,t), u_h(\cdot,t))$ is the solution of (3.4)-(3.6), then there is a co-state $p_h(\cdot,t) \in V_h$ such that

$(y_h(\cdot,t), p_h(\cdot,t), u_h(\cdot,t))$ satisfies the following optimality conditions:

$$(3.7)\quad (y_{h,tt}, I_h w_h) + a_h(y_h, I_h w_h) + (\phi(y_h), I_h w_h) = (Bu_h + f, I_h w_h), \forall w_h \in V_h,$$
$$y_h(x,0) = \pi_h y_0(x), \quad y_{h,t}(x,0) = \pi_h g(x), \quad x \in \Omega;$$
$$(3.8)\quad (p_{h,tt}, I_h q_h) + a_h(p_h, I_h q_h) + (\phi'(y_h)p_h, I_h q_h) = (y_h - y_d, I_h q_h), \forall q_h \in V_h,$$
$$p_h(x,T) = 0, \quad p_{h,t}(x,T) = 0, \quad x \in \Omega;$$
$$(3.9)\quad \int_0^T (u_h + B^* p_h, v - u_h) d\tau \geq 0, \ \forall\ v \in U_{ad}.$$

In order to express the control in a concise form, we introduce a projection (see, e.g., [18])

$$P_{[a,b]}(f(x)) = \max\left(a, \min(b, f(x))\right),$$

we can denote the variational inequality (3.6) by

$$(3.10)\qquad u(x) = P_{[a,b]}(-B^* p(x,t)).$$

And the variational inequality (3.9) is equivalent to

$$(3.11)\qquad u_h(x) = P_{[a,b]}(-B^* p_h(x,t)).$$

Then the discrete optimality condition can be rewritten by: find $(y_h(\cdot,t), p_h(\cdot,t), u_h) \in V_h \times V_h \times U_{ad}$ such that

$$(3.12)\quad (y_{h,tt}, I_h w_h) + a_h(y_h, I_h w_h) + (\phi(y_h), I_h w_h) = (Bu_h + f, I_h w_h), \forall w_h \in V_h,$$
$$y_h(x,0) = \pi_h y_0(x), \quad y_{h,t}(x,0) = \pi_h g(x), \quad x \in \Omega;$$
$$(3.13)\quad (p_{h,tt}, I_h q_h) + a_h(p_h, I_h q_h) + (\phi'(y_h)p_h, I_h q_h) = (y_h - y_d, I_h q_h), \forall q_h \in V_h,$$
$$p_h(x,T) = 0, \quad p_{h,t}(x,T) = 0, \quad x \in \Omega;$$
$$(3.14)\quad u_h(x) = P_{[a,b]}(-B^* p_h(x,t)).$$

This is our finite volume method for the problems (1.1)-(1.4) which the variational concept is used for the variational inequality (3.6).

For $\varphi \in V_h$, we shall write

$$(3.15)\qquad \phi(\varphi) - \phi(\rho) = -\tilde{\phi}'(\varphi)(\rho - \varphi) = -\phi'(\rho)(\rho - \varphi) + \tilde{\phi}''(\varphi)(\rho - \varphi)^2,$$

where

$$\tilde{\phi}'(\varphi) = \int_0^1 \phi'(\varphi + s(\rho - \varphi)) ds,$$
$$\tilde{\phi}''(\varphi) = \int_0^1 (1 - s)\phi''(\rho + s(\varphi - \rho)) ds$$

are bounded functions in $\bar{\Omega}$, more details can be found in [12].

4. Error estimates

In this section, to begin with, we present some useful results. Then we obtain two Lemmas to deduce the error estimates. At last, we derive some error estimates for the finite volume element approximation of the problems (1.1)-(1.4).

To describe error estimates for the finite volume methods, we will give some useful results. As shown in [15, 17], for all $w_h, v_h \in V_h$, there exist positive constants C and $h_0 > 0$ such that for all $0 < h < h_0$

$$|a_h(w_h, I_h v_h) - a_h(v_h, I_h w_h)| \leq Ch||w_h||_1 \; ||v_h||_1, \tag{4.1}$$

$$a_h(v_h, I_h v_h) \geq C||v_h||_1^2, \tag{4.2}$$

$$a_h(w_h, I_h v_h) \leq C||w_h||_1 \; ||v_h||_1. \tag{4.3}$$

Let $\varepsilon_a(\varphi, \chi) = a(\varphi, \chi) - a_h(\varphi, I_h\chi)$, we have (see, e.g., [21])

$$|\varepsilon_a(\varphi, \chi)| \leq Ch \; ||\varphi||_1 \; ||\chi||_1, \quad \varphi, \chi \in V_h. \tag{4.4}$$

Then, we present two auxiliary problems to deduce the error estimates. Let $y_h(u)$ be the solution of

$$\begin{aligned} &(y_{h,tt}(u), I_h w_h) + a_h(y_h(u), I_h w_h) + (\phi(y_h(u)), I_h w_h) = (Bu + f, I_h w_h), \\ &y_h(u)(x, 0) = \pi_h y_0, \quad y_{h,t}(u)(x, 0) = \pi_h g, \quad x \in \Omega, \end{aligned} \tag{4.5}$$

and $p_h(y)$ be the solution of

$$\begin{aligned} &(p_{h,tt}(y), I_h q_h) + a_h(p_h(y), I_h q_h) + (\phi'(y_h(u))p_h(u), I_h q_h) = (y - y_d, I_h q_h), \\ &p_h(y)(x, T) = 0, \quad p_{h,t}(y)(x, T) = 0, \quad x \in \Omega, \end{aligned} \tag{4.6}$$

where $w_h, q_h \in V_h$, and note that $y_h = y_h(u_h)$, $p_h = p_h(y_h)$. We have the following lemmas for $y_h(u)$, $p_h(y)$.

Lemma 4.1. *Assume that $y_h(u), p_h(y)$ are the solutions of (4.5) and (4.6), respectively. Then there exists an $h_0 > 0$ such that for all $0 < h \leq h_0$*

$$||y_h(u) - y_h||_{L^\infty(J;H^1)} \leq C||u - u_h||_{L^2(J;L^2(\Omega))}, \tag{4.7}$$

$$||p_h(y) - p_h||_{L^\infty(J;H^1)} \leq C||y - y_h||_{L^2(J;L^2(\Omega))}. \tag{4.8}$$

Proof. Subtracting (3.7) from (4.5), we have

$$\begin{aligned} &(y_{h,tt}(u) - y_{h,tt}, I_h w_h) + a_h(y_h(u) - y_h, I_h w_h) + (\phi(y_h(u)) - \phi(y_h), I_h w_h) \\ &\qquad = (B(u - u_h), I_h w_h), \; \forall w_h \in V_h. \end{aligned}$$

A direct calculation using (3.15) shows

$$\begin{aligned} &(y_{h,tt}(u) - y_{h,tt}, I_h w_h) + a_h(y_h(u) - y_h, I_h w_h) + (\tilde{\phi}'(y_h(u))(y_h(u) - y_h), I_h w_h) \\ &\qquad = (B(u - u_h), I_h w_h). \end{aligned}$$

For convenience, let $\theta = y_h(u) - y_h$. We get $(\theta_{tt}, I_h w_h) + a(\theta, w_h) = \varepsilon_a(\theta, w_h) + (B(u-u_h), I_h w_h) - (\tilde{\phi}'(y_h(u))\theta, I_h w_h),\ \forall w_h \in V_h$. Choosing $w_h = \theta_t$, we obtain

$$\frac{1}{2}\frac{d}{dt}[(\theta_t, I_h\theta_t) + a(\theta,\theta)] = \varepsilon_a(\theta,\theta_t) + (B(u-u_h), I_h\theta_t) - (\tilde{\phi}'(y_h(u))\theta, I_h\theta_t).$$

Integrating both sides from 0 to t and noticing that $\theta(x,0)=0$ and $\theta_t(x,0)=0$, we have

$$\begin{aligned}(\theta_t, I_h\theta_t) + a(\theta,\theta) = & 2\int_0^t \varepsilon_a(\theta,\theta_t)d\tau + 2\int_0^t (B(u-u_h), I_h\theta_t)d\tau \\ & - 2\int_0^t (\tilde{\phi}'(y_h(u))\theta, I_h\theta_t)d\tau.\end{aligned}$$

The coercive property of $a(\cdot,\cdot)$ implies

$$\begin{aligned}(\theta_t, I_h\theta_t) + ||\theta||_1^2 \le & C\int_0^t \varepsilon_a(\theta,\theta_t)d\tau \\ & + C\int_0^t (B(u-u_h), I_h\theta_t)d\tau - C\int_0^t (\tilde{\phi}'(y_h(u))\theta, I_h\theta_t)d\tau.\end{aligned} \tag{4.9}$$

Using (4.4) and the inverse estimate, we derive

$$\begin{aligned}\int_0^t \varepsilon_a(\theta,\theta_t)d\tau \le & \int_0^t Ch||\theta||_1\ ||\theta_t||_1 d\tau \\ \le & \int_0^t C||\theta||_1\ ||\theta_t|| d\tau \\ \le & C\int_0^t ||\theta||_1^2 d\tau + C\delta\int_0^t ||\theta_t||^2 d\tau.\end{aligned} \tag{4.10}$$

Using linear bound properties of B and I_h, we can write the inequality as

$$\begin{aligned}\int_0^t (B(u-u_h), I_h\theta_t)d\tau \le & \int_0^t C||u-u_h||\ ||\theta_t|| d\tau \\ \le & C\int_0^t ||u-u_h||^2 d\tau + C\delta\int_0^t ||\theta_t||^2 d\tau.\end{aligned} \tag{4.11}$$

Note that

$$\begin{aligned}\int_0^t (\tilde{\phi}'(y_h(u))\theta, I_h\theta_t)d\tau \le & \int_0^t C\|\theta\|\cdot\|\theta_t\| d\tau \\ \le & C\int_0^t \|\theta\|^2 d\tau + C\delta\int_0^t \|\theta_t\|^2 d\tau \\ \le & C\int_0^t \|\theta\|_1^2 d\tau + C\delta\int_0^t \|\theta_t\|^2 d\tau.\end{aligned} \tag{4.12}$$

Using (4.10)-(4.12) and for δ sufficiently small, note that $(\theta_t, I_h\theta_t)$ is equivalent to (θ_t, θ_t)(see, e.g., [17]) , we can obtain from (4.9) that

$$||\theta_t||^2 + ||\theta||_1^2 \leq C\int_0^t ||\theta||_1^2 d\tau + C\int_0^t ||u - u_h||^2 d\tau.$$

The Gronwall's lemma implies that

$$||\theta_t||^2 + ||\theta||_1^2 \leq C\int_0^T ||u - u_h||^2 d\tau = C||u - u_h||^2_{L^2(J;L^2)},$$

which completes the proof of (4.7). In a similar way, (4.8) can be verified easily. □

We consider the following problem

$$\begin{cases} w_{tt}(x,t) - \nabla\cdot(A\nabla w(x,t)) = f(x,t), \quad t\in J, x\in\Omega, \\ w(x,t) = 0, \quad t\in J, x\in\Gamma, \\ w(x,T) = w_0(x), \quad w_t(x,T) = w_1(x), \ x\in\Omega, \end{cases} \tag{4.13}$$

where A, J, Ω are as described as in (1.1)-(1.4). The finite volume method for the problem (4.13) is to find $w_h(\cdot,t)\in V_h$ such that

$$\begin{cases} (w_{h,tt}, I_h\chi) + a_h(w_h, I_h\chi) = (f, I_h\chi), \ \forall\chi\in V_h, \\ w_h(x,T) = u_0(x), \quad w_{h,t}(x,T) = u_1(x). \end{cases} \tag{4.14}$$

For the finite volume method, we have the following results.

Lemma 4.2. *Let w_h, w be the solutions of* (4.14) *and* (4.13) *respectively. Assume that $f_t, f_{tt}\in L^2(J;L^2(\Omega))$, $f\in L^2(J;H^1(\Omega))$, If $u_h(0) = R_hu_0$ and $u_{h.t}(0) = R_hu_1$.Then there exists a constant C independent of h such that for all* $0 < h < h_0$

$$||w_h(t) - w(t)|| \leq Ch^2, \tag{4.15}$$

$$||w_h(t) - w(t)||_1 \leq Ch. \tag{4.16}$$

Proof. The proofs of (4.15) and (4.16) are similar to the proof of Theorem 1 in [7] and Theorem 2 in [7], respectively. □

Let $(p(y), y(u))$ and $(p_h(y), y_h(u))$ be the solutions of (3.7)-(3.8) and (3.12)-(3.13), respectively. Let $J(\cdot) : U_{ad}\to\mathbb{R}$ be a G-differential convex functional near the solution u which satisfies the following form:

$$J(u) = \frac{1}{2}(||y(u) - y_d||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}).$$

Then we have a sequence of convex functional $J_h : U_{ad} \to \mathbb{R}$:

$$J_h(u) = \frac{1}{2}(||y_h(u) - y_d||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}),$$
$$J_h(u_h) = \frac{1}{2}(||y_h(u_h) - y_d||^2_{L^2(\Omega)} + ||u_h||^2_{L^2(\Omega)}).$$

It can be shown that

$$(J'(u), v) = (u + B^*p, v),$$
$$(J'_h(u), v) = (u + B^*p_h(y), v),$$
$$(J'_h(u_h), v) = (u_h + B^*p_h, v).$$

In the following we estimate $\|u - u_h\|_{L^2(J;L^2)}$. We assume that the cost function J is strictly convex near the solution u, i.e., for the solution u there exists a neighborhood of u in L^2 such that J is convex in the sense that there is a constant $c > 0$ satisfying:

$$(J'(u) - J'(v), u - v) \geq c\|u - v\|^2, \tag{4.17}$$

for all v in this neighborhood of u. The convexity of $J(\cdot)$ is closely related to the second order sufficient optimality conditions of optimal control problems, which are assumed in many studies on numerical methods of the problem. For instance, in many references, the authors assume the following second order sufficiently optimality condition (see [16, 28]): there is $c > 0$ such that $J''(u)v^2 \geq c\|v\|_0^2$.

From the assumption (4.17), by the proof contained in [2], there exists a constant $c > 0$ satisfying

$$(J'_h(v) - J'_h(u), v - u) \geq c\|v - u\|^2, \quad \forall v \in U_{ad}. \tag{4.18}$$

Theorem 4.1. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of problems* (3.1)-(3.3) *and* (3.7)-(3.9)*, respectively. Assume that* $f_t, f_{tt}, y_{d,t}, y_{d,tt} \in L^2(J; L^2(\Omega))$, $f, y_d \in L^2(J; H^1(\Omega))$, $y_0(x) \in H^3(\Omega)$, $g(x) \in H^2(\Omega)$*. Then there exists an* $h_0 > 0$ *such that for all* $0 < h \leq h_0$

$$||u - u_h||_{L^2(J;L^2)} \leq Ch^2. \tag{4.19}$$

Proof. Let $v = u_h$ in (3.6) and $v = u$ in (3.9), then we have

$$\int_0^T (u + B^*p, u_h - u)d\tau \geq 0, \tag{4.20}$$

$$\int_0^T (u_h + B^*p_h, u - u_h)d\tau \geq 0. \tag{4.21}$$

From (4.20) and (4.21), it is easy to see that

$$\int_0^T (u - u_h, u - u_h)d\tau \leq \int_0^T (B^*(p - p_h), u_h - u)d\tau. \tag{4.22}$$

By using (4.18) and (4.22), we obtain

$$\begin{aligned}
c\|u-u_h\|^2_{L^2(J;L^2)} \leq & \int_0^T (u,u-u_h)d\tau - \int_0^T (u_h,u-u_h)d\tau \\
= & \int_0^T (u+B^*p_h(y),u-u_h)d\tau - \int_0^T (u_h+B^*p_h,u-u_h)d\tau \\
= & \int_0^T (u-u_h,u-u_h)d\tau + \int_0^T (B^*(p_h(y)-p_h),u-u_h)d\tau \\
\leq & \int_0^T (B^*(p_h(y)-p_h,u-u_h)d\tau - \int_0^T (B^*(p-p_h),u-u_h)d\tau \\
= & \int_0^T (B^*(p_h(y)-p),u-u_h)d\tau. \qquad (4.23)
\end{aligned}$$

Now, we estimate all terms at the right side of (4.23). From the continuity of the operator B and Cauchy inequality, we deduce that

$$\begin{aligned}
\int_0^T (B^*(p_h(y)-p),u-u_h)d\tau \leq & C\|p_h(y)-p\|_{L^2(J;L^2)}\cdot\|u-u_h\|_{L^2(J;L^2)} \\
\leq & Ch^2\|u-u_h\|_{L^2(J;L^2)} \\
\leq & Ch^4+\delta\|u-u_h\|_{L^2(J;L^2)}, \qquad (4.24)
\end{aligned}$$

where $\|p-p_h(y)\|_{0,2}\leq Ch^2$(see the Theorem 2 in [7]). Combining (4.23) and (4.24), we conclude the result (4.19). □

Theorem 4.2. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of problems* (3.1)-(3.3) *and* (3.7)-(3.9)*, respectively. Assume that* $f_t,f_{tt},y_{d,t},y_{d,tt}\in L^2(J;L^2(\Omega))$, $f,y_d\in L^2(J;H^1(\Omega))$, $y_0(x)\in H^3(\Omega)$, $g(x)\in H^2(\Omega)$. *Then there exists an* $h_0>0$ *such that for all* $0<h\leq h_0$

$$||y-y_h||_{L^\infty(J;L^2)}+||p-p_h||_{L^\infty(J;L^2)}\leq Ch^2. \qquad (4.25)$$

Assume that $f,y_d,f_t,f_{tt},y_{d,t},y_{d,tt}\in L^2(J;L^2(\Omega)),y_0(x)\in H^3(\Omega),g(x)\in H^2(\Omega)$. *Then there exists an* $h_0>0$ *such that for all* $0<h\leq h_0$

$$||y-y_h||_{L^\infty(J;H^1)}+||p-p_h||_{L^\infty(J;H^1)}\leq Ch. \qquad (4.26)$$

Proof. Using the triangle inequality, we have that

$$\begin{aligned}
||y-y_h||_{L^\infty(J;L^2)} \leq ||y-y_h(u)||_{L^\infty(J;L^2)}+||y_h(u)-y_h||_{L^\infty(J;L^2)}, \\
||p-p_h||_{L^\infty(J;L^2)} \leq ||p-p_h(y)||_{L^\infty(J;L^2)}+||p_h(y)-p_h||_{L^\infty(J;L^2)}.
\end{aligned}$$

Lemma 4.1 implies that

$$||y-y_h||_{L^\infty(J;L^2)}\leq ||y-y_h(u)||_{L^\infty(J;L^2)}+C||u-u_h||_{L^2(J;L^2)}, \qquad (4.27)$$

$$||p-p_h||_{L^\infty(J;L^2)}\leq ||p-p_h(y)||_{L^\infty(J;L^2)}+C||y-y_h||_{L^\infty(J;L^2)}. \qquad (4.28)$$

Using Lemma 4.1, Lemma 4.2, (4.27)-(4.28), and Theorem 2 of [7], we can easily obtain (4.25) from Theorem 4.1.

In a similar way, (4.26) can be proved easily. □

Theorem 4.3. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of problems* (3.1)-(3.3) *and* (3.7)-(3.9), *respectively. Assume that* $f_t, f_{tt}, y_{d,t}, y_{d,tt} \in L^2(J; L^2(\Omega))$, $f, y_d \in L^2(J; H^1(\Omega))$, $y_0(x) \in H^3(\Omega)$, $g(x) \in H^2(\Omega)$. *Then there exists an* $h_0 > 0$ *such that for all* $0 < h \leq h_0$

$$||u - u_h||_{L^\infty(J;L^2)} \leq Ch^2. \tag{4.29}$$

Proof. Using the definition of $P_{[a,b]}(\cdot)$ and (3.10)-(3.11), we have that

$$\begin{aligned} |u - u_h| &= |P_{[a,b]}(-B^*p) - P_{[a,b]}(-B^*p_h)| \\ &\leq C|p - p_h|. \end{aligned} \tag{4.30}$$

Which implies that

$$\|u - u_h\| \leq C\|p - p_h\|. \tag{4.31}$$

Then we can get

$$\|u - u_h\|_{L^\infty(J;L^2)} \leq C\|p - p_h\|_{L^\infty(J;L^2)}. \tag{4.32}$$

Finally, we can obtain (4.29) from (4.25) and (4.32). □

5. Conclusion and future works

In this paper, we consider a priori error estimates for the finite volume element approximation of nonlinear hyperbolic optimal control problem. Then we use finite volume method to discretize the state and adjoint equation of the system. Under some reasonable assumptions, we obtain some optimal order error estimates. To our best knowledge in the context of optimal control problems, these priori error estimates of finite volume method for general nonlinear hyperbolic optimal control problem is new.

In future, we shall consider a posteriori error estimates and superconvergence of the finite volume element solutions for hyperbolic optimal control problems.

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