A SHORT NOTE ON IDEMPOTENT RINGS

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Abstract. In this paper we introduce a new class of rings that we say idempotent rings. We call a ring R is idempotent, if every ideal of R is generated by an idempotent element. In this paper we prove some properties of this rings, where one of the important results is the following:

Let $t \geq 2$ be an integer number. Then the ring $\mathbb{Z}_t$ is an idempotent ring if and only if $t = p_1 p_2 \dots p_n$, where all of the p_i are distinct prime numbers.

Keywords: idempotent, artinian ring, noetherian ring.

1. Introduction

Throughout this paper, all rings are commutative rings with identity and all modules are unital. Let M be a submodule of the R-module L. We say that L is an essential extension of M precisely when $B \cap M \neq 0$ for every non-zero submodule B of L. We say that L is an injective envelope (or injective hull) of M precisely when L is an injective R-module which is also an essential extension of M. We denote by $E(M)$ the injective envelope of M. For any unexplained notation and terminology we refer the reader to [2] and [3].

2. Main results

Definition 2.1. Let R be a ring. We say that R is idempotent if every ideal of R is generated by an idempotent element.

Lemma 2.2. *Every idempotent ring is Artinian ring.*

Proof. Suppose that $\mathfrak{m} \in \operatorname{Max}(R)$. Then there exist an element $e \in R$ such that $\mathfrak{m} = \langle e \rangle$ and $e^2 = e$. Now in local Noetherian ring $R_{\mathfrak{m}}$, we have $(\mathfrak{m}R_{\mathfrak{m}})^2 = \mathfrak{m}R_{\mathfrak{m}}$, and so by *Nakayama's* lemma, $\mathfrak{m}R_{\mathfrak{m}} = 0$. Hence $\dim R_{\mathfrak{m}} = 0$. Sinc $\mathfrak{m}$ is an arbitrary maximal ideal, it follows that $\dim R = 0$ and so R is Artinian ring. □

Lemma 2.3. *In idempotent ring, the Jacobson radical is zero.*

Proof. We denote the Jacobson radical of R by $J(R)$, so we show that $J(R) = 0$. There exists an element $e \in R$ such that $J(R) = \langle e \rangle$ and $e^2 = e$. Therefore $J(R) = J(R) \cdot J(R)$ and by *Nakayama's* lemma, $J(R) = 0$. $\square$

Theorem 2.4. *Let R be an idempotent ring and $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be all of the maximal ideals of R. Then $R \approx \frac{R}{\mathfrak{m}_1} \oplus \dots \oplus \frac{R}{\mathfrak{m}_n}$.*

Proof. Let $\operatorname{Max} R = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$. By induction on n we prove the theorem.

If $n = 1$, then $J(R) = \mathfrak{m}_1 = 0$ and so $R \approx \frac{R}{\mathfrak{m}_1}$.

Now we suppose that $n \geq 0$ and consider the following exact sequence,

$$0 \longrightarrow \frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2} \longrightarrow \frac{R}{\mathfrak{m}_1} \oplus \frac{R}{\mathfrak{m}_2} \longrightarrow \frac{R}{\mathfrak{m}_1 + \mathfrak{m}_2} \longrightarrow 0$$

Since $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, it follow from the above exact sequence that $\frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2} \approx \frac{R}{\mathfrak{m}_1} \oplus \frac{R}{\mathfrak{m}_2}$. Now consider the following exact sequence

$$0 \longrightarrow \frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3} \longrightarrow \frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2} \oplus \frac{R}{\mathfrak{m}_3} \longrightarrow \frac{R}{(\mathfrak{m}_1 \cap \mathfrak{m}_2) + \mathfrak{m}_3} \longrightarrow 0.$$

Again similar the above argument, since $(\mathfrak{m}_1 \cap \mathfrak{m}_2) + \mathfrak{m}_2 = R$, it follows that

$$\frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3} \approx \frac{R}{\mathfrak{m}_1 \cap \mathfrak{m}_2} \oplus \frac{R}{\mathfrak{m}_3} \approx \frac{R}{\mathfrak{m}_1} \oplus \frac{R}{\mathfrak{m}_2} \oplus \frac{R}{\mathfrak{m}_3}.$$

By reapiting this argument we have

$$R \approx \frac{R}{(0)} \approx \frac{R}{J(R)} = \frac{R}{\cap_{i=1}^{n} \mathfrak{m}_i} \approx \oplus_{i=1}^{n} \frac{R}{\mathfrak{m}_i}.$$

$\square$

Theorem 2.5. *Every idempotent ring R as an R-module is injective.*

Proof. Let I be an ideal of R and consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{\ i\ } & R \\ & & \downarrow{\scriptstyle f} & & \\ & & R & & \end{array}$$

There exists an element $e \in R$ such that $I = \langle e \rangle$ and $e^2 = e$. Let $f(e) = x$. We define the function $g : R \to R$ with $g(r) = rx$. Then we have the following relations for all $r \in R$.

$$goi(re) = g(re) = rex = ref(e) = rf(e^2) = rf(e) = f(re),$$

and so R is an injective R-module. $\square$

Corollary 2.6. *If R is an idempotent ring, then every simple R-module is injective.*

Proof. Let N be a simple R-module. Then there exists $\mathfrak{m} \in \mathrm{Max}(R)$, such that $N \approx \frac{R}{\mathfrak{m}}$. Therefore N is a direct summand of an injective R-module R and so N is injective. □

Corollary 2.7. *Let R be an idempotent ring. Then for every $\mathfrak{m} \in \mathrm{Max}(R)$, $E_R\left(\frac{R}{\mathfrak{m}}\right) \approx \frac{R}{\mathfrak{m}}$.*

Proof. Follows from the above corollary. □

Lemma 2.8. *Let R be an idempotent ring and M be a finitely generated R-module. Then M is injective.*

Proof. Since R is Artinian ring, it follows that $l(M) < \infty$. We prove the assersion by induction on $l(M)$. If $l(M) = 1$, then M is a simple and so the assersion follows from Corollary 2.6. Now suppose that $l(M) = n \geq 2$ and the assertion holds for $n - 1$.

Since M is Artinian R-module, it follows that M has a simple submodule such as N. Consider the following exact sequence.

$$0 \longrightarrow N \longrightarrow M \longrightarrow \frac{M}{N} \longrightarrow 0$$

$l(N) = 1$ and so N is injective. $l\left(\frac{M}{N}\right) = n - 1$ then $\frac{M}{N}$ is injective. Therefore M is also injective. □

Lemma 2.9. *Let R be a Noetherian ring and $\{E_i\}_{i\in A}$ be a family of injective R-modules. Then $\varinjlim_{i\in A} E_i$ is injective.*

Proof. Is simple. □

Theorem 2.10. *Let R be an idempotent ring. Then every R-module is injective.*

Proof. Let T be an R-module. Then T is a direct limit of its finitely generated submodules. □

Theorem 2.11. *Let R be an idempotent ring. Then every R-module is projective and so is flat.*

Proof. By the above theorem every R-module is injective. Let T be an R-module. Then T is injective. By Matlis theorem

$$T = \oplus_{p\in \mathrm{Spec}(R)} E\left(\frac{R}{p}\right).$$

On the other hand $\operatorname{Max}(R) = \operatorname{Spec}(R)$. Therefore,

$$T = \oplus_{\mathfrak{m} \in \operatorname{Max}(R)} \frac{R}{\mathfrak{m}}.$$

Also for any $\mathfrak{m} \in \operatorname{Max}(R)$, $\frac{R}{\mathfrak{m}}$ is a direct summand of R and so is projective. Consequently T is projective. □

Theorem 2.12. *Let $t \geq 2$ be an integer number. Then the ring $\mathbb{Z}_t$ is an idempotent ring if and only if $t = p_1 p_2 \dots p_n$, where all of the p_i are distinct prime numbers.*

Proof. Let $\mathbb{Z}_t$ be an idempotent ring. Suppose on the contrary that there exists a prime number p such that $p^2 \mid t$.

In this case, we set $J = \langle \bar{p} \rangle$. Then $J^2 = \langle \bar{p^2} \rangle \neq J$ and so J is not an idempotent ideal of $\mathbb{Z}_t$ and therefore $\mathbb{Z}_t$ is not idempotent ring which is a contradiction.

Conversely, let $t = p_1 \dots p_n$, where p_i are distinct prime numbers. If $n = 1$, then all of the ideals of $\mathbb{Z}_t$ are $I = \langle 0 \rangle$ and $J = \mathbb{Z}_{p_1} = Z_t = \langle \bar{1} \rangle$ and so $\mathbb{Z}_t$ is idempotent ring.

Now let $n \geq 2$ and we set $\mathfrak{m}_i = \langle \bar{p_i} \rangle$. Then $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are all the maximal ideals of R. We claim that each $\mathfrak{m}_i$ is generated by an idempotent element. It is enough to show that for $\mathfrak{m}_1$. Since $(p_1, p_2 p_3 \dots p_n) = 1$, then there exist $r, s \in \mathbb{Z}$, such that $rp_1 + sp_2 \dots p_n = 1$. Therefore $p_2 \dots p_n \mid rp_1 - 1$ and $p_1 \mid rp_1$. Hence $p_1 \dots p_n \mid rp_1(rp_1 - 1) = (rp_1)^2 - rp_1$ and so in the ring $\mathbb{Z}_t$, we have $\overline{rp_1}^2 = \overline{rp_1}$. Set $e = \overline{rp_1}$ and we claim that $\mathfrak{m}_1 = \langle e \rangle$. $rp_1 + sp_2 \dots p_n = 1$ implies that $rp_1^2 + sp_1 p_2 \dots p_n = p_1$. Hence in the ring $\mathbb{Z}_t$, $\overline{rp_1^2} = \overline{p_1}$ and so $\overline{p_1 e} = \overline{p_1}$ and $\mathfrak{m}_1 = \langle \overline{p_1} \rangle = \langle \overline{p_1 e} \rangle \subseteq \langle e \rangle \subseteq \mathfrak{m}_1$. Therefore $\mathfrak{m}_1 = \langle e \rangle$ and every maximal ideal of $\mathbb{Z}_t$ is generated by an idempotent element. Now let I be an arbitrary ideal of $\mathbb{Z}_t$. Then there exists an element $p_{i_1} p_{i_2} \dots p_{i_k}$ such that $I = \langle p_{i_1} p_{i_2} \dots p_{i_k} \rangle$, where $p_{i_1}, \dots, p_{i_k}$ are different elements of the set $\{p_1, \dots, p_n\}$. Also we have,

$$I = \mathfrak{m}_{i_1} \mathfrak{m}_{i_2} \dots \mathfrak{m}_{i_k} = \langle e_{i_1} \rangle \dots \langle e_{i_k} \rangle = \langle e_{i_1} \dots e_{i_k} \rangle$$

where all of the e_{i_j} are idempotent and so the element $e_{i_1} e_{i_2} \dots e_{i_k}$ is also idempotent and the assertion follows. □

Remark 2.13. It is well known, in a Noetherian ring R, for any ideal I of R and any injective R-module E, $0 :_E (0 :_R I) = IE$.

Theorem 2.14. *Let R be a Noetherian ring and every R-module be an injective R-module. Then R is idempotent.*

Proof. Let I be an ideal of R. Then I is injective R-module and by Remark 2.13, we have

$$I \subseteq 0 :_I (0 :_R I) = II = I^2 \subseteq I$$

Hence $I = I^2$ and by [1, Corollary 2.5], there exists $a \in I$ such that $(1 - a)I = 0$ and so $I = \langle a \rangle$ and $a^2 = a$. □

Corollary 2.15. *The Noetherian ring R is idempotent iff every R-module is an injective R-module.*

Corollary 2.16. *If $p_1, \ldots, p_n$ are distinct prime numbers and $R = \mathbb{Z}_{p_1 \ldots p_n}$. Then every R-module is injective and projective.*

Corollary 2.17. *Let R be an idempotent ring and M be an R-module. Then the following are equivalent:*

i) There exists an exact sequence $0 \longrightarrow R \longrightarrow M$

ii) Ann $M = 0$

Proof. $i \to ii)$ is clear.

$ii \to i)$ Since M is injective, it follows by Matlis theorem $M = \oplus_{\gamma \in A} E\left(\frac{R}{\mathfrak{m}_\gamma}\right) = \oplus_{\gamma \in A} \frac{R}{\mathfrak{m}_\gamma}$. Set $T = \{\mathfrak{m}_\gamma \mid \gamma \in A\}$ and we prove that $T = \mathrm{Max}(R)$. Suppose on the contrary that $T \neq \mathrm{Max}(R)$. Let $\mathfrak{m} \in \mathrm{Max}(R) \backslash T$.

$$0 = \mathrm{Ann}\, M = \bigcap_{\gamma \in A} \mathfrak{m}_\gamma \Rightarrow \bigcap_{\gamma \in A} \mathfrak{m}_\gamma = 0 \subseteq \mathfrak{m}$$

and so there exist $\gamma \in A$ such that $\mathfrak{m}_\gamma \subseteq \mathfrak{m}$ which implies that $\mathfrak{m} = \mathfrak{m}_\gamma \in T$, which is a contradiction. Therefore $M = \oplus_{\mathfrak{m}_\gamma \in \mathrm{Max}\, R} \frac{R}{\mathfrak{m}_\gamma} \approx R$ and so the sequence $0 \longrightarrow R \longrightarrow M$ is exact. □

References

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