# THE NUMERICAL RANGE OF AN ELEMENT OF A CLASS OF TOPOLOGICAL ALGEBRAS

**E. Ansari-Piri**
*Department of Mathematics*
*Faculty of Science*
*University of Guilan*
*Rasht*
*Iran*
*eansaripiri@gmail.com*

**M. Sabet***
*Department of Mathematics*
*University of Guilan*
*Rasht*
*Iran*
*and*
*Department of Mathematics*
*Payame Noor University*
*Iran*
*sabet.majid@gmail.com*

**Abstract.** Kinani, Oubbi and Oudadess (1998) show that every unital and commutative locally convex algebra with a jointly continuous product is $\beta$ - subadditive, for which $\beta$ is the boundedness radius. In this paper we obtain some results on numerical range of an element in $\beta$ - subadditive algebras. To do this, at first, we study the dual space of topological algebras for which the boundedness radius is finite. Furthermore, we prove some new results for linear and multiplicative linear functionals on a class of topological algebras.
**Keywords:** multiplicative linear functional, numerical range, boundedness radius, $\beta$-subadditive algebras.

## 1. Introduction

Allan [1] provides the definition of the radius of boundedness $\beta$ to develop the spectral theory for locally convex topological algebras. After that, the radius of boundedness $\beta$ is extended for general topological algebras (see for example [11]). T.Husain [9] introduces the concepts of strongly sequential and infrasequential topological algebras and the first author [3] introduces the concept of fundamental topological algebra to generalize the famous Cohen factorization theorem. Kinani, Oubbi and Oudadess [10] show that every unital and

∗. Corresponding author

commutative locally convex algebra with a jointly continuous product is $\beta$ - subadditive. Also Oubbi [12] extended $\beta$ - subadditive algebras. In this note, at first we study the dual space of topological algebras for which the boundedness radius is finite; and provide a norm on a subspace of the algebraic dual space of topological algebras. Next, we try to study numerical range in $\beta$ - subadditive algebras. Here, we have supposed all algebras are complex unital complete metrizable topological algebra.

## 2. Definitions and related results

At first we begin with the previous definitions and related results.

**Definition 2.1.** Let $x$ be an element of a topological algebra $(A, \tau)$. We will say that $x$ is bounded if there exists some $r > 0$ such that the sequence $(\frac{x^n}{r^n})_n$ converges to zero. The radius of boundedness of $x$ with respect to $(A, \tau)$ is denoted by $\beta(x)$ and defined by

$$\beta(x) = \inf\{r > 0 \ : \ (\frac{x^n}{r^n}) \to 0\} \tag{2.1}$$

with the convention : $\inf \phi = +\infty$. We also say $A$ is a $\beta-$ finite topological algebra if all elements of $A$ are bounded.

**Definition 2.2.** Let $A$ be a topological algebra.

(i) $A$ is said to be strongly sequential if there exists a neighborhood $U$ of 0 such that for all $x \in U$, $x^n \to 0$ as $n \to \infty$ .

(ii) $A$ is said to be infrasequential if for each bounded set $B \subseteq A$ there exists $\lambda > 0$ such that for all $x \in B, (\lambda x)^n \to 0$ as $n \to \infty$.

**Proposition 2.3** ([9])**.** *With reference to the above definitions,* $(i)$ *implies* $(ii)$*, and* $(ii)$ *implies that* $\beta$ *is finite.*

**Definition 2.4** ([3])**.** A topological algebra $A$ is said to be fundamental if there exists $b > 1$ such that for every sequence $(a_n)$ of $A$, the convergence of $b^n(a_{n+1} - a_n)$ to zero in $A$ implies that $(a_n)$ is a Cauchy sequence.

**Proposition 2.5** ([5])**.** *Let* $A$ *be a strongly sequential topological algebra that* $\rho \leq \beta$*. Then* $A$ *is a* $Q-algebra$*.*

**Corollary 2.6** ([8])**.** *Let* $A$ *be a* $Q-algebra$*. Then every multiplicative linear functional is automatically continuous.*

## 3. A norm on a subspace of the algebraic dual space of a $\beta$-finite topological algebra

Let A be a topological algebra, $A'$ be the space of all linear functionals on A, and $f \in A'$. Define $v(f) = sup \{|f(x)| : \beta(x) < 1\}$. Then we have the following theorem.

**Theorem 3.1.** *Suppose $\beta$ is finite. Taking the above notations, one has:*

$(i)$ $v(f) = sup\ \{|f(x)| : \beta(x) \leq 1\} = sup\ \{|f(x)| : x \in \beta^{-1}\{0,1\}\}$.

$(ii)$ *If $\beta(x) \neq 0$ then we have*

$$|f(x)| \leq v(f)\ \beta(x) \tag{3.1}$$

*and if $\beta(x) = 0$, then we have*

$$|f(x)| \leq v(f). \tag{3.2}$$

$(iii)$ $f = 0$ *if and only if* $v(f) = 0$.

$(iv)$ *If $f, g \in A'$, $\alpha \in \mathbb{C}$ then $v(f+g) \leq v(f) + v(g)$, $v(\alpha f) = |\alpha|\ v(f)$.*

$(v)$ *If $\hat{A} = \{f \in A' : v(f) < \infty\}$ then $v(.)$ is a norm on $\hat{A}$ and $(\hat{A}, v(.))$ is a Banach space.*

$(vi)$ *If $f \in \hat{A}$, $\beta(x) = 0$, then, $f(x) = 0$ and therefore we have:*

$$f \in \hat{A} \Rightarrow (|f(x)| \leq v(f)\ \beta(x)\ for\ all\ x \in A). \tag{3.3}$$

$(vii)$ *If $A$ is an infrasequential algebra, then $\hat{A} \subseteq A^*$ where $A^*$ is the set of all continuous linear functionals on $A$.*

**Proof.** (i) Let $r = sup\{|f(x)| : \beta(x) < 1\}$, $r' = sup\{|f(x)| : \beta(x) \leq 1\}$. Then obviously, we have $r \leq r'$. If $n \in \mathbb{N}$, $\beta(x) \leq 1$ then $\beta((1 - \frac{1}{n})x) < 1$, that implies $|f((1 - \frac{1}{n})x)| \leq r$, i.e $(1 - \frac{1}{n})|f(x)| \leq r$. Since $n$ is arbitrary, it follows $r' \leq r$. On the other hand, if we suppose $r'' = sup\ \{|f(x)| : x \in \beta^{-1}\{0,1\}\}$, obviously $r'' \leq r'$. If suppose $0 \neq \beta(x) \leq 1$, We have $|f(\frac{x}{\beta(x)})| \leq r''$ that implies $r' \leq r''$.

(ii) Let $x \in A$, $\beta(x) \neq 0$, then $\beta(\frac{x}{\beta(x)}) = 1$, that implies $|f(\frac{x}{\beta(x)})| \leq v(f)$ as desired.

(iii) By (3.1)and (3.2) the result is clear.

(iv) It is clear.

(v) It suffices to show that $(\hat{A}, v(.))$ is complete. Let$(f_n)_n$ be a Cauchy sequence in $(\hat{A}, v(.))$ and $x \in A$. By (3.1) and (3.2) , the sequence $(f_n(x))_n$ is a Cauchy sequence of $\mathbb{C}$ and so, there exists a function $f : A \to \mathbb{C}$ such that $f_n(x) \to f(x)$ for all $x \in A$. Let $\varepsilon > 0$, $x \in A$ for which $\beta(x) < 1$. There exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies that $|f_n(x) - f_m(x)| \leq v(f_n - f_m) < \frac{\varepsilon}{2}$. Now, we let m tends to infinity, then we have $|f_n(x) - f(x)| < \varepsilon$; therefore, if $n \geq N$ then $v(f_n - f) < \varepsilon$.

(vi) Let $n \in \mathbb{N}$. By (3.2), $|f(nx)| \leq v(f)$ that implies $|f(x)| \leq \frac{v(f)}{n}$ for all $n \in \mathbb{N}$; i.e $f(x) = 0$.

(vii) Let $f \in \hat{A}$ and $E$ be a bounded subset of $A$. Since $A$ is an infrasequential algebra, then there exists $M > 0$ such that $\beta(x) \leq M\ for\ all\ x \in E$. By (3.3) we get $f(E)$ is bounded; that implies $f \in A^*$. □

We define $\hat{weak}$-topology on $\hat{A}$, the same as the definition of $weak^*$-topology on $A^*$.

**Definition 3.2.** We call the $A$-topology of $\hat{A}$ the $\hat{weak}$-topology on $\hat{A}$. (pronunciation: weak hat topology).

According to this definition, the next theorem is clear.

**Theorem 3.3.** *Let $A$ be $\beta$-finite. We have:*

*$(i)$ The $\hat{weak}$-topology is a locally convex vector topology on $\hat{A}$.*

*$(ii)$ Every linear functional $\varphi$ on $\hat{A}$ is $\hat{weak}$-continuous if and only if there exists $a \in A$ such that $\varphi(\Lambda) = \Lambda a$ for every $\Lambda \in \hat{A}$.*

**Theorem 3.4.** *(Banach- Alaoglu theorem for $\hat{weak}$-topology) We suppose $A$ is $\beta$-finite and $D(A) = \{x \in A : \beta(x) < 1\}$ if*

$$K = \{\Lambda \in \hat{A} : \ |\Lambda x| \leq 1 \text{ on } D(A)\} \tag{3.4}$$

*then $K = \{\Lambda \in \hat{A} : \ v(\Lambda) \leq 1\}$ and $K$ is $\hat{weak}$-compact.*

**Proof.** The proof is similar to ([13], theorem 3.15). □

**Proposition 3.5.** *Let $A$ be a strongly sequential algebra for which $\rho \leq \beta$ and $\Phi_A$ be the carrier space of $A$. We have*

*$(i)$ If $f$ is multiplicative linear functional, then $v(f) = 1$ and therefore $\Phi_A \subseteq \ \hat{A} \ \subseteq A^*$.*

*$(ii)$ The carrier space $\Phi_A$ is weak$^*$-compact.*

*$(iii)$ Suppose $A$ is a Banach algebra, then for every continuous linear functional $f$, $\|f\| \leq v(f)$.*

**Proof.** (i) Let $x \in A$, $\beta(x) < 1$; then there exists $b > 1$ such that $b^n \, x^n \to 0$ and by propsition 2.5 and corollary 2.6, since $f$ is continuous, we have $b^n(f(x))^n = f(b^n x^n) \to 0$. Therefore $|f(x)| < 1$; and since $\beta(1) = 1$, we get the result.

(ii) By (i) and (3.4), it is sufficient to observe that $\Phi_A^\infty = \Phi_A \bigcup \{0\}$ is a $\hat{weak}$-closed subset of $\hat{A}$ (see [7] chapter2, section17, proposition 2).

(iii) It is sufficient to note that, $\beta(x) \leq \|x\|$. □

## 4. Numerical range on $\beta$- subadditive algebras

In this section, moreover, we suppose that every algebra A is $\beta$- subadditive and $\beta$-finite, that is, $\beta(x + y) \leq \ \beta(x) + \beta(y), \ \beta(x) < \infty$ for all $x, y \in A$.

**Lemma 4.1.** *Suppose $A$ is an algebra and $x_0 \in A$. Then there exists $\Lambda \in \hat{A}$ such that $\Lambda(x_0) = \beta(x_0)$ and $|\Lambda(x)| \leq \beta(x)$ for all $x \in A$.*

**Proof.** We apply the Hahn- Banach theorems where $\beta$ is indicated seminorm on $A$. □

In [7 section 10] a suitable discussion is given on the numerical range of elements of normed algebras. Here we use the similar notations and extend the ideas for topological algebras. Suppose $E(a,r) = \{x \in X : \|x-a\| \leq r\}, X_1 = \{x \in X : \|x\| \leq 1\}$ and $S(X) = \{x \in X : \|x\| = 1\}$ for which $X$ is a normed vector space. We define the following concepts.

**Definition 4.2.** We define sets $D_\beta(A;1), V_\beta(A;a)$ by
$D_\beta(A;1) = \{f \in S(\hat{A}) : f(1) = 1\}$,
$V_\beta(A;a) = \{f(a) : f \in D_\beta(A;1)\}(a \in A)$.

The elements of $D_\beta(A;a)$ are called $\beta$- normalized states on A, $V_\beta(A;a)$ is called the $\beta$- numerical range of a. We write $D(1), V(a)$ for $D_\beta(A;1)$ , $V_\beta(A;a)$ when no confusion can occur.

**Proposition 4.3.** *Let $B$ be a subalgebra of $A$ such that $1 \in B, b \in B$. Then $V_\beta(B;b) = V_\beta(A;b)$.*

**Proof.** Let $\lambda \in V_\beta(A;b)$; therefore, there exists $f \in D(A;1), \lambda = f(b)$. If $g$ is the restriction mapping $f$ to $B$, Then $g \in D_\beta(B;1), \lambda = g(b)$ that implies $\lambda \in V_\beta(B;b)$. On the other hand, we suppose $\lambda \in V_\beta(B;b)$, therefore, there exists $f \in B' : |f(x)| \leq \beta(x)\ (x \in B), \lambda = f(b), f(1) = 1$; and by the Hahn- Banach theorems, there exists $F \in A'$ *such that* $|F(x)| \leq \beta(x)$, $(x \in A)$, *and* $F = f$ *on* $B$, which implies that, $F \in D(A;1), \lambda = F(b)$; and we get the result. □

**Lemma 4.4.** *$D(1)$ is non-void weak$\hat{k}$-compact convex subset of $\hat{A}$ .*

**Proof.** Since $D(1) = (\hat{A})_1 \bigcap 1^{-1}\{1\}$ , if, in lemma 4.1, we put $x_0 = 1$, we get the result. □

**Proposition 4.5.** $(i)$ *$V(a)$ is non-void compact convex subset of $\mathbb{C}$ .*

$(ii)$ $V(\gamma + \lambda b) = \gamma + \lambda V(b),\ V(a+b) \subseteq V(a) + V(b)\ (a, b \in A,\ \lambda, \gamma \in \mathbb{C})$.

$(iii)$ $|z| \leq\ \beta(a)\ (z \in V(a))$.

**Proof.** (i) By the lemma 4.4, it is clear.

(ii) It is clear.

(iii) Let $z \in V(a)$. Then, there exists $f \in D(1)$ such that $z = f(a)$ , and by 3.3 we have, $|z| = |f(a)| \leq v(f)\beta(a) \leq \beta(a)$. □

**Lemma 4.6.** $V(a) = \bigcap_{z\in\mathbb{C}} E(z, \beta(z-a))$.

**Proof.** If $\lambda \in V(a)$, then $\lambda = f(a)$ for some $f \in D(1)$, and for all $z \in \mathbb{C}$ we have

$$|\lambda - z| = |f(z-a)| \leq \beta(z-a), \tag{4.1}$$

i.e.

$$\lambda \in E(z, \beta(z-a))\ (z \in \mathbb{C}). \tag{4.2}$$

Suppose on the other hand that $\lambda \in \bigcap_{z\in\mathbb{C}} E(z, \beta(z-a))$. If $a = z_0 1_A$, then $V(a) = \{f(a) : f \in D(1)\} = \{f(z_0 1_A) : f \in D(1)\} = \{z_0\}$; also $|\lambda - z_0| \leq \beta(z_0 - z_0) = 0$. Therefore $\lambda = z_0 \in V(a)$. Suppose then that $1, a$ are linearly independent, and define $f_0$ on their linear span by

$$f_0(w + w'a) = w + w'\lambda \ (w, w' \in \mathbb{C}). \tag{4.3}$$

Since $\lambda$ satisfies (4.2), we have $|f_0(w + w'a)| \leq \beta(w + w'a)$, $v(f_0) \leq 1$. By the Hahn-Banach theorem, $f_0$ can be extended to $f \in \hat{A}$ with $v(f) \leq 1$. Then $f \in D(1)$ and $f(a) = f_0(a) = \lambda$. □

We remember by [2] corollary 2.2 in every fundamental algebra, we have $\rho \leq \beta$.

**Proposition 4.7.** *If $\rho \leq \beta$, in particular, in fundamental algebras, we have $Sp(a) \subseteq V(a)$.*

**Proof.** Let $\lambda \in \mathbb{C} \setminus V(a)$. Then by lemma 4.6, there exists $z \in \mathbb{C}$ such that $|z - \lambda| > \beta(z - a)$. Therefore $\beta((z-\lambda)^{-1}(z-a)) < 1$, and so

$$1 - (z-\lambda)^{-1}(z-a) \in Inv(A). \tag{4.4}$$

It follows that $\lambda - a \in Inv(A)$, and as a result $\lambda \in \mathbb{C} \setminus Sp(a)$. □

## References

[1] G. R. Allan, *A spectral theory for locally convex algebras*, Proc. London Math. Soc., 115 (1965), 399-421.

[2] E. Anjidani, *On Gelfand-Mazur theorem on a class of $F$-algebras*, Topol. Algebra Appl., 2 (2014), 19-23.

[3] E. Ansari-Piri, *A class of factorable topological algebras*, Proc. Edinbourgh Math. Soc., (2) 33 (1990), 53-59.

[4] E. Ansari-Piri, *The linear functionals on fundamental locally multiplicative topological algebras*, Turk. J. Math., 34 (2010), 385 391.

[5] E. Ansari-Piri, M. Sabet, S. Sharifi, *Two famous concepts in F-algebras*, Gen. Math. Notes., 34 (2016), 1-6.

[6] F.F. Bonsall, J. Duncan, *Numerical ranges of operator on normed spaces and of normed algebras*, Cambridge at the university press, 1971.

[7] F.F. Bonsall, J. Duncan, *Complete normed algebras*, Springer-Verlag, Berlin Heidelberg-New York, 1973.

[8] T.G. Honary, M. Najfi Tavani, *Upper semicontinuity of the spectrum function and automatic continuity in topological $Q$-algebras,* Note di Matematica Note Mat., 28 (2008), 57-62.

[9] T. Husain, *Infrasequential topological algebra*, Canadian Math. Bull., 22(1979), 413-418.

[10] A. El Kinani, L. Oubbi, M. Oudadess, *Spectral and boundedness radii in locally convex algebras*, Georgian Math. J., 5 (1997), 233-241.

[11] L. Oubbi, *Further radii in topological algebras*, Bull. Belg. Math. Soc. Simon Stevin., 9 (2002), 279-292.

[12] L. Oubbi, *A Kaplansky-Meyer theorem for subalgebras*, Bull. Belg. Math. Soc. Simon Stevin, 16 (2009), 305-312.

[13] W. Rudin, *Functional Analysis*, McGraw-Hill Publishing Company LTD, 1985.

Accepted: 22.06.2016