NEW IMPROVEMENT OF HEINZ INEQUALITIES FOR MATRICES

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Abstract. In this paper we mainly discuss Heinz inequalities involving unitarily invariant norms. By using the Hermite-Hadamard inequality, we get some refinements of the Heinz inequalities, thus new improvements of Heinz Inequalities for matrices are obtained. Our results are different from those of [7, 1, 3, 6, 8].

Keywords: refinements, Heinz inequality, convex function, Hermite-Hadamard inequality, unitarily invariant norm.

1. Introduction

On a complex separable Hilbert space, let A and B are positive operations, X is an operator, then the function

$$f(v) = |||A^v XB^{1-v} + A^{1-v} XB^v||| ,$$

where $|||\cdot|||$ denotes unitarily invariant norm, is convex on the interval $[0, 1]$. $f(v)$ get its minimum at $v = \frac{1}{2}$, and gets its maximum at $v = 0$ and $v = 1$. Moreover, $f(v) = f(1 - v)$ for $0 \leq v \leq 1$. Thus , from [4] we know that the Heinz inequalities are valid

$$2\left|\left|\left|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right|\right|\right| \leq |||A^v XB^{1-v} + A^{1-v} XB^v||| \leq |||AX + XB||| . \tag{1.1}$$

The convexity of the function

$$f(v) = |||A^v XB^{1-v} + A^{1-v} XB^v|||$$

on $[0, 1]$ is obvious.

Similar to the methods of [5] we begin our main results.

2. Main results

A real-valued function f on the interval $[a,b]$ is convex, if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2},$$

where $x, y \in [a,b]$.

Let f be a convex real-valued function on the interval $[a,b]$, then the Hermite-Hadamard integral inequality [2] is

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

In [5], Kittaneh gave several refinements of the Heinz inequalities by using the Hermite-Hadamard integral inequality. Feng improved the results of [5] in the paper [3]. Wang [6] also improved the results of Kittaneh. In [8], Yan and Feng refined Kittaneh again. Abbas and Mourad [1] used a parameter to generalize the results of [3, 6, 8]. Xue [7] also generalized the results and gave a formula shown as follows,

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(t)dt \\ &\le \frac{1}{2n}\left((n-1)f(a) + 2f\left(\frac{a+b}{2}\right) + (n-1)f(b)\right) \\ &\le \frac{f(a)+f(b)}{2}, \end{aligned}$$

where $n \ge 2$ is an integer. It is easy to know that results of [7] is more general than of [1]. Other kinds of improvement of Heinz Inequalitie can be seen, for example , see [9].

In this paper, we give other refinements of Heinz inequalities for matrices which are different from [5, 3, 6, 8, 7]. We will use the following lemma to obtain several better improved Heinz inequalities.

Let's first give a lemma.

Lemma 1. *If f is a convex real-valued function on the interval $[a,b]$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{12}\left(f(a) + 10f\left(\frac{a+b}{2}\right) + f(b)\right) \\ &\le \frac{1}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right) \\ &\le \frac{f(a)+f(b)}{2}. \end{aligned}$$

Proof. From the Hermite-Hadamard integral inequality we know

$$2f\left(\frac{a+b}{2}\right) \le f(a) + f(b).$$

So

$$12f\left(\frac{a+b}{2}\right) \le f(a) + 10f\left(\frac{a+b}{2}\right) + f(b).$$

Thus

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{12}\left(f(a) + 10f\left(\frac{a+b}{2}\right) + f(b)\right).$$

Next, the following inequality will be proved.

$$\frac{1}{12}\left(f(a) + 10f\left(\frac{a+b}{2}\right) + f(b)\right) \le \frac{1}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right).$$

From

$$2f\left(\frac{a+b}{2}\right) \le f(a) + f(b),$$

we conclude that

$$f(a) + 10f\left(\frac{a+b}{2}\right) + f(b) \le 3f(a) + 6f\left(\frac{a+b}{2}\right) + 3f(b).$$

And so

$$\frac{1}{12}\left(f(a) + 10f\left(\frac{a+b}{2}\right) + f(b)\right) \le \frac{1}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right).$$

From

$$f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \le 2f(a) + 2f(b)$$

we get that

$$\frac{1}{4}\left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right) \le \frac{f(a)+f(b)}{2}.$$

So, we finish the proof. □

Set $f(v) = \left|\left|\left|A^v X B^{1-v} + A^{1-v} X B^v\right|\right|\right|$, then the following theorems can be obtained.

Theorem 1. *Let A, B are positive operators, X be an operator, then for $0 \le \mu \le 1$, we obtain*

$$\begin{aligned}
&2\left|\left|\left|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right|\right|\right| \\
&\le \frac{1}{6}\left(\left|\left|\left|A^{\mu} X B^{1-\mu} + A^{1-\mu} X B^{\mu}\right|\right|\right| + 5\left|\left|\left|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right|\right|\right|\right) \\
&\le \frac{1}{2}\left(\left|\left|\left|A^{\mu} X B^{1-\mu} + A^{1-\mu} X B^{\mu}\right|\right|\right| + 2\left|\left|\left|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right|\right|\right|\right) \\
&\le \left|\left|\left|A^{\mu} X B^{1-\mu} + A^{1-\mu} X B^{\mu}\right|\right|\right|.
\end{aligned} \tag{2.1}$$

Proof. Applying the previous lemma on $f(v)$ on the interval $[\mu, 1-\mu]$ when $0 \le \mu \le \frac{1}{2}$, then

$$\begin{aligned} f\left(\frac{1-\mu+\mu}{2}\right) &\le \frac{1}{12}\left(f(\mu)+10f\left(\frac{1-\mu+\mu}{2}\right)+f(1-\mu)\right) \\ &\le \frac{1}{4}\left(f(\mu)+2f\left(\frac{1-\mu+\mu}{2}\right)+f(1-\mu)\right) \\ &\le \frac{f(\mu)+f(1-\mu)}{2}, \end{aligned}$$

and thus

$$f\left(\frac{1}{2}\right) \le \frac{1}{6}\left(f(\mu)+5f\left(\frac{1}{2}\right)\right) \le \frac{1}{2}\left(f(\mu)+f\left(\frac{1}{2}\right)\right) \le f(\mu).$$

So,

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\le \frac{1}{6}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+5\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le \frac{1}{2}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||. \end{aligned} \tag{2.2}$$

Applying the previous lemma on $f(v)$ on the interval $[1-\mu, \mu]$ when $\frac{1}{2} \le \mu \le 1$, then

$$\begin{aligned} 2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\le \frac{1}{6}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+5\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le \frac{1}{2}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||. \end{aligned} \tag{2.3}$$

By combining (2.2) and (2.3) we obtain (2.1). □

Theorem 2. *Let A, B are positive operators, X be an operator, then for $0 \le \mu \le 1$, we have*

$$\begin{aligned} &\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right| \le \frac{1}{12}\Big(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}||| \\ &+10\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\Big) \\ &\le \frac{1}{4}\Big(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}||| \\ &+2\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\Big) \\ &\le \frac{1}{2}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right). \end{aligned} \tag{2.4}$$

Proof. Applying the lemma to the function $f(v)$ on $[\mu, \frac{1}{2}]$, where $0 \le \mu \le \frac{1}{2}$, and on the $[\frac{1}{2}, \mu]$, where $\frac{1}{2} \le \mu \le 1$, respectively, we can finish the proof. □

From (2.4) and the first part of (1.1) we obtain the refinement of first part of (1.1).

Corollary 1. *Let A, B are positive operators, X be an operator, then for $0 \leq \mu \leq 1$, we get*

$$\begin{aligned}
2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| &\leq \left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right| \\
&\leq \frac{1}{12}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right. \\
&\left.+10\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\
&\leq \frac{1}{4}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right. \\
&\left.+2\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|\right|\right|+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\
&\leq \frac{1}{2}\left(|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+2\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right|\right) \\
&\leq |||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||.
\end{aligned} \tag{2.5}$$

Applying the lemma to $f(v)$ on $[0, \mu]$, where $0 \leq \mu \leq \frac{1}{2}$, and $[\mu, 1]$, where $\frac{1}{2} \leq \mu \leq 1$, respectively, we get the following theorem.

Theorem 3. *Let A, B are positive operators, X be an operator, then*

(1) *for $0 \leq \mu \leq \frac{1}{2}$, we have*

$$\begin{aligned}
&\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}+A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right| \\
&\leq \frac{1}{12}\left(|||AX+XB|||+10\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}+A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right|\right. \\
&\left.+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right) \\
&\leq \frac{1}{4}\left(|||AX+XB|||+2\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}+A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right|\right. \\
&\left.+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right) \\
&\leq \frac{1}{2}\left(|||AX+XB|||+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right).
\end{aligned} \tag{2.6}$$

(2) for $\frac{1}{2} \leq \mu \leq 1$, we have

$$\begin{aligned}
&\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right| \\
&\leq \frac{1}{12}\left(|||AX+XB|||+10\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right|\right. \\
&\left.+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right) \\
&\leq \frac{1}{4}\left(|||AX+XB|||+2\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}+A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right|\right. \\
&\left.+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right) \\
&\leq \frac{1}{2}\left(|||AX+XB|||+|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||\right).
\end{aligned} \tag{2.7}$$

Because on the interval $[0, \frac{1}{2}]$, the function $f(v)$ is decreasing and on the interval $[\frac{1}{2}, 1]$, it is increasing, then by using the inequalities (2.6) and (2.7), we get the refinement of the second inequality in (1.1).

Corollary 2. *Let A, B are positive operators, X be an operator, then for $0 \leq \mu \leq 1$, we have*

(1) *for $0 \leq \mu \leq \frac{1}{2}$ and for every unitarily norm,*

$$\begin{aligned}
&|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}||| \\
&\leq \left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right| \\
&\leq \frac{1}{12}\Big(|||AX + XB||| + 10\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right| \\
&+ |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\Big) \\
&\leq \frac{1}{4}\Big(|||AX + XB||| + 2\left|\left|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}} + A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|\right|\right| \\
&+ |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\Big) \\
&\leq \frac{1}{2}\left(|||AX + XB||| + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right) \\
&\leq |||AX + XB|||.
\end{aligned} \tag{2.8}$$

(2) *for $\frac{1}{2} \leq \mu \leq 1$,*

$$\begin{aligned}
&|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}||| \\
&\leq \left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right| \\
&\leq \frac{1}{12}\Big(|||AX + XB||| + 10\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right| \\
&+ |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\Big) \\
&\leq \frac{1}{4}\Big(|||AX + XB||| + 2\left|\left|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}} + A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|\right|\right| \\
&+ |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\Big) \\
&\leq \frac{1}{2}\left(|||AX + XB||| + |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||\right) \\
&\leq |||AX + XB|||.
\end{aligned} \tag{2.9}$$

3. Results and discussion

By refining the Hermite-Hadamard Integral Inequality, we get a new refined Heinz Inequalities of matrices.

Can we add a parameter to get a family of refined Heinz Inequalities of matrices as [7] and [1]? We are going to consider this case in next papers.

4. Conclusion

Xue [7], Abbas and Mouradand [1] have generalized the results of [3, 6, 8]. In a different way, this paper gives a new generalization of Heinz Inequalities for matrices.

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References

[1] H. Abbas and B. Mourad, *A family of refinements of Heinz inequalities of matrices*, Journal of Inequalities and Applications, 267 (2014).

[2] P.S. Bullen, *A dictionary of inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 97, Addison Wesley Longman Ltd., U.K., 1998.

[3] Y. Feng, *Refinements of the Heinz inequalities,* Journal of Inequalities and Applications, 18 (2012).

[4] R. Bhatia and C. Davis, *More matrix forms of the arithmeticgeometric mean inequality,* SIAM J. Matrix Anal. Appl., 14 (1993), 132-136.

[5] F. Kittaneh, *On the convexity of the Heinz means*, Integr. Equ. Oper. Theory, 68 (2010), 519-527.

[6] S. Wang, *Some new refinements of Heinz inequalities of matrices*, Journal of Inequalities and Applications, 2013, 424.

[7] J. Xue, *Some refinements of the Heinz inqualities,* Italian Journal of Pure and Applied Mathematics, 34 (2015), 291-298.

[8] Y. Yan, Y. Feng and G. Chen, *Refinements of the Heinz inequalities for matrices*, Journal of Inequalities and Applications, 50 (2014).

[9] L. Zou and Y. Jiang, *Improved Heinz inequality and its application*, Journal of Inequalities and Applications, 113 (2012).

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