ON THE GENERALIZED DRAZIN INVERSE IN A BANACH ALGEBRA

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Abstract. We give the representations of $(a+b)^{\mathsf{d}}, (a+ab)^{\mathsf{d}}$ and $(a+ba)^{\mathsf{d}}$ in terms of a, b, a^{d} and b^{d} being elements of a Banach algebra with $a^3b = ba$ and $b^3a = ab$. We also give the representations of $(a+b)^{\mathsf{d}}$ under the assumptions $a^3b = ba$, $\|a^{\mathsf{D}}b\| < 1$, and $\|(1-aa^{\mathsf{D}})b^{\mathsf{D}}a\| < 1$.

Keywords: generalized Drazin inverse, Banach algebra.

1. Introduction

Let $\mathscr{A}$ be a complex Banach algebra with unit 1. The symbols $\mathscr{A}^{-1}, \mathscr{A}^{\mathsf{nil}}, \mathscr{A}^{\mathsf{qnil}}$ and $\mathscr{A}^{\bullet}$ stand for the sets of all invertible, nilpotent, quasinilpotent and idempotent elements in the Banach algebra $\mathscr{A}$, respectively.

For $a \in \mathscr{A}$, if there exists a unique $x \in \mathscr{A}$ such that

$$xax = x, \qquad ax = xa, \qquad a^{k+1}x = a^k \tag{1.1}$$

x is the Drazin inverse of a (denoted by a^{D}). The least nonnegative integer k for which satisfies the above equations is the Drazin index ind(a) of a. If ind$(a) \leq 1$,

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then a^{D} reduces to the group inverse of a in this case it is customary to denote $a^{\mathsf{D}} = a^{\#}$. Note that $a \in \mathscr{A}$ is invertible if and only if $\mathrm{ind}(a) = 0$.

In [12] Koliha generalized the Drazin invertibility by changing (1.1). Let $a \in \mathscr{A}$. An element $x \in \mathscr{A}$ is called a generalized Drazin inverse of a if

$$xax = x, \qquad ax = xa, \qquad a - a^2x \in \mathscr{A}^{\mathsf{qnil}}. \tag{1.2}$$

If there exists $x \in \mathscr{A}$ satisfying (1.2), the element a is said to be generalized Drazin invertible. Koliha proved in [12] that the set of generalized Drazin inverses of an element of a Banach algebra is or empty or a singleton. In case that a has a generalized Drazin inverse, we shall denote by a^{d} its unique generalized inverse. The subset of $\mathscr{A}$ consisting of elements that have a generalized Drazin inverse will be denoted by $\mathscr{A}^{\mathsf{d}}$. In fact, Koliha proved the following lemma which will be useful. (stated in a unital ring)

Lemma 1.1 ([12, Lemma 2.4])**.** *In a Banach algebra $\mathscr{A}$ with unit, an element $a \in \mathscr{A}$ is generalized Drazin invertible if and only if there is $p \in \mathscr{A}^{\bullet}$ such that*

$$pa = ap, \qquad ap \in \mathscr{A}^{\mathsf{qnil}}, \qquad a + p \in \mathscr{A}^{-1}.$$

In this case, the set consisting of generalized Drazin inverses of a is a singleton and its unique element a^{d} is given by

$$a^{\mathsf{d}} = (a+p)^{-1}(1-p).$$

Let $a \in \mathscr{A}$ be generalized Drazin invertible. If $a - a^2 a^{\mathsf{d}} \notin \mathscr{A}^{\mathsf{nil}}$, it is customary to say that $\mathrm{ind}(a) = \infty$. It is easily seen from the proof of [12, Lemma 2.4] that the idempotent p given in Lemma 1.1 is unique which explicit expression is $p = 1 - aa^{\mathsf{d}}$. We shall denote this idempotent by a^{π}.

Every $p \in \mathscr{A}^{\bullet}$ induces a matrix representation of any element $a \in \mathscr{A}$ given by (see [17, Chapter 5])

$$a = \begin{pmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{pmatrix}_p .$$

If $\mathscr{B}$ is a subalgebra of the unital algebra $\mathscr{A}$, for an element $b \in \mathscr{B}^{-1}$, we shall denote by $[b^{-1}]_{\mathscr{B}}$ the inverse of b in $\mathscr{B}$. Let us observe that in general $\mathscr{B}^{-1} \not\subset \mathscr{A}^{-1}$ (if we take $p \in \mathscr{A}^{\bullet}$, $p \neq 1$, and $\mathscr{B} = p\mathscr{A}p$, then $p \in \mathscr{B}^{-1}$ and $p \notin \mathscr{A}^{-1}$). But if the subalgebra $\mathscr{B}$ has unity, then $\mathscr{B}^{-1} \subset \mathscr{A}^{\mathsf{d}}$ and if $b \in \mathscr{B}^{-1}$, then $b^{\mathsf{d}} = [b^{-1}]_{\mathscr{B}}$: let e be the unity of $\mathscr{B}$, since $b[b^{-1}]_{\mathscr{B}} = [b^{-1}]_{\mathscr{B}} b = e$, it is easy to see $b[b^{-1}]_{\mathscr{B}} b = b$, $[b^{-1}]_{\mathscr{B}} b [b^{-1}]_{\mathscr{B}} = [b^{-1}]_{\mathscr{B}}$, and $[b^{-1}]_{\mathscr{B}} b = b[b^{-1}]_{\mathscr{B}}$.

It is known [3] that any $a \in \mathscr{A}^{\mathsf{d}}$ has the following matrix representation

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p,$$

$$p = 1 - a^{\pi} = aa^{\mathsf{d}}, \quad a_1 \in [p\mathscr{A}p]^{-1}, \quad a_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{qnil}}. \tag{1.3}$$

Then we have

$$a^{\mathsf{d}} = \begin{pmatrix} [{a_1}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p . \tag{1.4}$$

If a is Drazin invertible (instead of being generalized Drazin invertible), the only difference with the representation (1.3) is that a_2 is nilpotent instead of being quasinilpotent.

The following result [3] will be useful for our purposes.

Lemma 1.2 ([3, Theorem 2.3])**.** *Let $\mathscr{A}$ be a Banach algebra, $x, y \in \mathscr{A}$, and $p \in \mathscr{A}^{\bullet}$. Assume that*

$$x = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}_p , \qquad y = \begin{pmatrix} b & 0 \\ c & a \end{pmatrix}_{1-p} .$$

(i) *If $a \in (p\mathscr{A}p)^{\mathsf{d}}$ and $b \in ((1-p)\mathscr{A}(1-p))^{\mathsf{d}}$, then $x, y \in \mathscr{A}^{\mathsf{d}}$ and*

$$x^{\mathsf{d}} = \begin{pmatrix} a^{\mathsf{d}} & u \\ 0 & b^{\mathsf{d}} \end{pmatrix}_p , \qquad y^{\mathsf{d}} = \begin{pmatrix} b^{\mathsf{d}} & 0 \\ u & a^{\mathsf{d}} \end{pmatrix}_{1-p} , \tag{1.5}$$

where $u = \sum_{n=0}^{\infty} (a^{\mathsf{d}})^{n+2} c b^n b^{\pi} + \sum_{n=0}^{\infty} a^{\pi} a^n c (b^{\mathsf{d}})^{n+2} - a^{\mathsf{d}} c b^{\mathsf{d}}$.

(ii) *If $x \in \mathscr{A}^{\mathsf{d}}$ and $a \in (p\mathscr{A}p)^{\mathsf{d}}$, then $b \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$ and x^{d} is given by (1.5).*

Moreover, when an element $x \in \mathscr{A}^{\mathsf{d}}$ commutes with an idempotent $p \in \mathscr{A}$, the generalized Drazin inverse of x has a simple form in terms of the matrix representation relative to p as the following simple (but useful) result shows:

Lemma 1.3. *Let $\mathscr{A}$ be a unital Banach algebra and let $x \in \mathscr{A}, p \in \mathscr{A}^{\bullet}$. If $x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}_p$, then $x \in \mathscr{A}^{\mathsf{d}}$ if and only if $x_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and $x_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$. In this situation, one has $x^{\mathsf{d}} = \begin{pmatrix} x_1^{\mathsf{d}} & 0 \\ 0 & x_2^{\mathsf{d}} \end{pmatrix}_p$.*

The Drazin inverse is used in applications of many areas such that differential and difference equations, Markov chains and control theory [1, 2].

In recent years, the representations of the Drazin inverse of $a+b$ have been considered by many authors (see [4, 5, 6, 7, 8, 9, 11, 13, 14, 15]) (being a, b matrices, operators or elements in a infinite dimensional Banach algebra). In [10] the Drazin inverse in semigroups and associative rings was firstly introduced. The Drazin inverse in a Banach algebra was introduced in [9]. A formula is given for the Drazin inverse of a sum of two matrices in [11]. In [9], Djordjević and Wei considered additive results for the generalized Drazin inverse in a Banach space.

C.Y. Deng in [7] explored the Drazin inverse of bounded operators with commutativity up to a factor in a Banach space, being these extended by Cvetković-Ilić in [4]. In [18], the authors considered the Drazin inverse of a sum of two matrices and derived additive formulas under conditions weaker than those used in some recent papers on the subject. As an application they gave some new representations for the Drazin inverse of a block matrix.

In the rest of this section, we will give some key lemmas. In Section 2 we will discus the representations of the generalized Drazin inverse of $a+b$, $a+ab$ and $a+ba$ in terms of a, b a^d and b^d being elements of a unital Banach algebras under the conditions $a^3b = ba$ and $b^3a = ab$. We also will consider related results under the assumptions $a^3b = ba$, $\|a^{\mathsf{D}}b\| < 1$, and $\|(1 - aa^{\mathsf{D}})b^{\mathsf{D}}a\| < 1$.

We will give some lemmas in the following:

Lemma 1.4. *Let $\mathscr{A}$ be a ring and let $a, b \in \mathscr{A}$ satisfy $a^3b = ba$. If $n \in \mathbb{N}$, then*

(i) $a^{3n}b = ba^n$.

(ii) $(ab)^n = a^{(3^n-1)/2}b^n$.

(iii) $a^{3^n}b^n = b^n a$.

(iv) *If $b^3a = ab$, then $ab = a^{26n}(ab)b^{2n}$.*

Proof. The proofs of (i), (ii), and (iii) can be easily done by induction. To prove (iv), observe that by applying (iii) we have $ab = b^3a = a^{3^3}b^3 = a^{26}(ab)b^2$, and now, the equality $a^{26(n+1)}(ab)b^{2(n+1)} = a^{26}\left[a^{26n}(ab)b^{2n}\right]b^2$ permits finish the proof of (iv) by induction. □

Lemma 1.5. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}$ satisfy $a^3b = ba$. If $a \in \mathscr{A}^{\mathsf{qnil}}$ or $b \in \mathscr{A}^{\mathsf{qnil}}$, then $ab \in A^{\mathsf{qnil}}$ and $aba \in \mathscr{A}^{\mathsf{qnil}}$.*

Proof. Assume that a or b are quasinilpotent. By Lemma 1.4 we have

$$\|(ab)^n\|^{1/n} \leq \|a^{(3^n-1)/2}\|^{1/n}\|b^n\|^{1/n}.$$

Hence $ab \in \mathscr{A}^{\mathsf{qnil}}$. To prove $aba \in \mathscr{A}^{\mathsf{qnil}}$, observe that $aba = aa^3b$ and $(a^4)^3b = a^{3\cdot 4}b = ba^4$. By the first part of this lemma, (notice that $a \in \mathscr{A}^{\mathsf{qnil}} \Rightarrow a^4 \in \mathscr{A}^{\mathsf{qnil}}$) we get that aba is quasinilpotent. □

The proof of the following lemma is inspired by the proof of [4, Theorem 2.2].

Lemma 1.6. *Let $\mathscr{A}$ be a unital Banach algebra. Let $a, b \in \mathscr{A}$ be such that $a^3b = ba$. If $a \in \mathscr{A}^{\mathsf{d}}$, then $a^\pi b = ba^\pi$. If $b \in \mathscr{A}^{\mathsf{d}}$, then $b^\pi a = b^\pi ab^\pi$.*

Proof. Assume $a \in \mathscr{A}^{\mathsf{d}}$. Let $p = aa^{\mathsf{d}}$. To prove $bp = pb$ pick any $n \in \mathbb{N}$ and use Lemma 1.4 (iii). We have

$$\begin{aligned} pb - pbp &= pb(1-p) = p^{3n}b(1-p) = (a^{\mathsf{d}})^{3n}a^{3n}b(1-p) = (a^{\mathsf{d}})^{3n}ba^n(1-p)^n \\ &= (a^{\mathsf{d}})^{3n}ba^n(1-aa^{\mathsf{d}})^n = (a^{\mathsf{d}})^{3n}b[a(1-aa^{\mathsf{d}})]^n. \end{aligned}$$

Since $a(1-aa^{\mathsf{d}}) \in \mathscr{A}^{\mathsf{qnil}}$ we have, by making $n \to \infty$, that $pb = pbp$. Similarly we prove $bp = pbp$. Hence $bp = pbp$.

If $b \in \mathscr{A}^{\mathsf{d}}$, then by setting $q = bb^{\mathsf{d}}$ and by mimicking the above reasoning we have $qa = qaq$. □

2. Main results

Theorem 2.1. *Let $\mathscr{A}$ be a unital Banach algebra and let $a, b \in \mathscr{A}^{\mathsf{d}}$ such that $a^3b = ba$. Then*

(i) $ab \in \mathscr{A}^{\mathsf{d}}$ *and* $bb^{\mathsf{d}}(ab)^{\mathsf{d}} = b^{\mathsf{d}}a^{\mathsf{d}}$.

(ii) $aba \in \mathscr{A}^{\mathsf{d}}$ *and* $b^{\mathsf{d}}(aba)^{\mathsf{d}} = (b^{\mathsf{d}}a^{\mathsf{d}})^2$.

(iii) $bb^{\mathsf{d}}ab^{\mathsf{d}} = b^{\mathsf{d}}a^3$.

Proof. (i) and (ii): Since $a \in \mathscr{A}^{\mathsf{d}}$, the elements a and a^{d} can be represented as in (1.3) and (1.4), respectively. Let us represent $b = \begin{pmatrix} b_1 & b_4 \\ b_3 & b_2 \end{pmatrix}_p$. From Lemma 1.6, we have $pb = bp$, hence $b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p$. Therefore, $ab = \begin{pmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{pmatrix}_p$, $a^3b = \begin{pmatrix} a_1^3b_1 & 0 \\ 0 & a_2^3b_2 \end{pmatrix}_p$, and $ba = \begin{pmatrix} b_1a_1 & 0 \\ 0 & b_2a_2 \end{pmatrix}_p$. From $a^3b = ba$ we get $a_1^3b_1 = b_1a_1$ and $a_2^3b_2 = b_2a_2$. Since a_2 is quasinilpotent and by Lemma 1.5 a_2b_2 is quasinilpotent. i.e. $(a_2b_2)^{\mathsf{d}} = 0$. Let us observe that by Lemma 1.3, $b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$, $b_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$, and $b^{\mathsf{d}} = \begin{pmatrix} b_1^{\mathsf{d}} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p$.

Now we consider b_1:

(1) If b_1 is quasinilpotent, then $b_1^{\mathsf{d}} = 0$ and from Lemma 1.5 we get $(a_1b_1)^{\mathsf{d}} = 0$. By using $(a_2b_2)^{\mathsf{d}} = 0$ and Lemma 1.3 we get $ab \in \mathscr{A}^{\mathsf{d}}$ and $(ab)^{\mathsf{d}} = 0$. Since $b \in \mathscr{A}^{\mathsf{d}}$, from Lemma 1.3 we have $b_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$ and $b^{\mathsf{d}} = \begin{pmatrix} b_1^{\mathsf{d}} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p = \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p$. By that a^{d} is represented as in (1.4) we have $b^{\mathsf{d}}a^{\mathsf{d}} = 0$. Thus (i) holds.

By Lemma 1.5 we have $a_1b_1a_1$ and $a_2b_2a_2$ are quasinilpotent (in their respective subalgebras). By Lemma 1.3 we have $aba \in \mathscr{A}^{\mathsf{d}}$ and $(aba)^{\mathsf{d}} = 0$. Since

$$b^{\mathsf{d}}a^{\mathsf{d}} = \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p = 0,$$

(ii) holds.

(2) Assume now that b_1 is invertible (in the subalgebra $p\mathscr{A}p$). Since a_1 is also invertible (in the subalgebra $p\mathscr{A}p$), $a_1b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and $(a_1b_1)^{\mathsf{d}} = [a_1b_1{}^{-1}]_{p\mathscr{A}p} =$

$[b_1{}^{-1}]_{p\mathscr{A}p}[a_1{}^{-1}]_{p\mathscr{A}p}$. Recall that we have $(a_2b_2)^{\mathsf{d}} = 0$. By Lemma 1.3 we get

$$(ab)^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{p\mathscr{A}p}[a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p .$$

Since b_1 is invertible, $b^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p$. Thus, (i) holds.

Since $b_1 \in [p\mathscr{A}p]^{-1}$ (recall that a_1 always belongs to $[p\mathscr{A}p]^{-1}$), we have $a_1b_1a_1 \in [p\mathscr{A}p]^{-1}$. Since a_2 is quasinilpotent, by Lemma 1.5, the element $a_2b_2a_2$ is quasinilpotent. From Lemma 1.3 we get $aba \in \mathscr{A}^{\mathsf{d}}$ and $(aba)^{\mathsf{d}} = \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p}[b_1{}^{-1}]_{p\mathscr{A}p}[a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p$. Thus

$$b^{\mathsf{d}}(aba)^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p}[b_1{}^{-1}]_{p\mathscr{A}p}[a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p$$

and

$$b^{\mathsf{d}}a^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p .$$

Evidently, we have proved (ii).

(3) Assume that b_1 is neither invertible nor quasinilpotent. Setting $q = b_1b_1^{\mathsf{d}}$ we have the representation $b_1 = \begin{pmatrix} b_1' & 0 \\ 0 & b_1'' \end{pmatrix}_q$, where $b_1' \in [q\mathscr{A}q]^{-1}$ and $b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{qnil}}$ (recall that $q \in p\mathscr{A}p$ and p is the unity of the algebra $q \in p\mathscr{A}p$). Let us represent a_1 as follows:

$$a_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_q .$$

By Lemma 1.4 (iii)

$$\begin{aligned} qa_1 - qa_1q &= qa_1(p-q) = q^na_1(p-q)^n = (b_1^{\mathsf{d}})^n b_1^n a_1 (p-q)^n \\ &= (b_1^{\mathsf{d}})^n a_1^{3^n} b_1^n (p-q)^n \\ &= (b_1^{\mathsf{d}})^n a_1^{3^n} [b_1(p - b_1b_1^{\mathsf{d}})]^n. \end{aligned}$$

Hence $\|qa_1 - qa_1q\|^{1/n} \to 0$, i.e., $qa_1 = qa_1q$, thus

$$a_1 = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q . \tag{2.1}$$

Therefore,

$$a_1b_1 = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q \begin{pmatrix} b_1' & 0 \\ 0 & b_1'' \end{pmatrix}_q = \begin{pmatrix} a_{11}b_1' & 0 \\ a_{21}b_1' & a_{22}b_1'' \end{pmatrix}_q .$$

To prove $a_1b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ we shall apply Lemma 1.2 (i). To this end, we must prove that $a_{11}b_1' \in [q\mathscr{A}q]^{\mathsf{d}}$ and $a_{22}b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{d}}$. From $a_1 \in [p\mathscr{A}p]^{-1}$ and (2.1) we get $a_{11} \in [q\mathscr{A}q]^{-1}$ and recall that $b_1' \in [q\mathscr{A}q]^{-1}$ we have $a_{11}b_1' \in [q\mathscr{A}q]^{-1} \subset [q\mathscr{A}q]^{\mathsf{d}}$. Now, from $a_1^3 b_1 = b_1 a_1$ we get $a_{22}^3 b_1'' = b_1'' a_{22}$. By using that $b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{qnil}}$ and Lemma 1.5 we get $a_{22}b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{qnil}} \subset [(p-q)\mathscr{A}(p-q)]^{\mathsf{d}}$. Moreover, $(a_{22}b_1'')^{\mathsf{d}} = 0$. Therefore, from Lemma 1.2 we get $a_1b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and

$$(a_1b_1)^{\mathsf{d}} = \begin{pmatrix} [(a_{11}b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ x & 0 \end{pmatrix}_q,$$

where x is some element of $\mathscr{A}$ given by Lemma 1.2 (i). To prove $bb^{\mathsf{d}}(ab)^{\mathsf{d}} = b^{\mathsf{d}}a^{\mathsf{d}}$ it is enough to prove $b_1b_1^{\mathsf{d}}(a_1b_1)^{\mathsf{d}} = b_1^{\mathsf{d}}[a_1{}^{-1}]_{p\mathscr{A}p}$. But we have

$$\begin{aligned} b_1b_1^{\mathsf{d}}(a_1b_1)^{\mathsf{d}} &= \begin{pmatrix} b_1' & 0 \\ 0 & b_2 \end{pmatrix}_q \begin{pmatrix} [(b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \begin{pmatrix} [(a_{11}b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ x & 0 \end{pmatrix}_q \\ &= \begin{pmatrix} [(a_{11}b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \end{aligned}$$

and

$$\begin{aligned} b_1^{\mathsf{d}}[a_1{}^{-1}]_{p\mathscr{A}p} &= \begin{pmatrix} [(b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \\ &\cdot \begin{pmatrix} [a_{11}{}^{-1}]_{q\mathscr{A}q} & 0 \\ -[a_{22}{}^{-1}]_{(p-q)\mathscr{A}(p-q)}a_{21}[a_{11}{}^{-1}]_{q\mathscr{A}q} & [a_{22}{}^{-1}]_{q\mathscr{A}q} \end{pmatrix}_q \\ &= \begin{pmatrix} [(a_{11}b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q. \end{aligned} \tag{2.2}$$

We have

$$a_1b_1a_1 = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q \begin{pmatrix} b_1' & 0 \\ 0 & b_1'' \end{pmatrix}_q \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q = \begin{pmatrix} a_{11}b_1'a_{11} & 0 \\ y & a_{22}b_1''a_{22} \end{pmatrix}_q,$$

where y is some element in $\mathscr{A}$. The invertibility of $a_{11}b_1'a_{11}$ in $q\mathscr{A}q$ follows from the invertibility of a_{11} and b_1' $q\mathscr{A}q$. The quasinilpotency of $a_{11}b_1'a_{11}$ follows from $a_2^3b_1'' = b_1''a_2$, the quasinilpotency of b_1'', and by Lemma 1.5. By Lemma 1.2 (i) we have that $a_1b_1a_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and

$$(a_1b_1a_1)^{\mathsf{d}} = \begin{pmatrix} [a_{11}{}^{-1}]_{q\mathscr{A}q}[(b_1')^{-1}]_{q\mathscr{A}q}[a_{11}{}^{-1}]_{q\mathscr{A}q} & 0 \\ z & 0 \end{pmatrix}_q,$$

where z is an element of $\mathscr{A}$ given by Lemma 1.2. Having in mind the quasinilpotency of $a_2b_2a_2$ and that $aba = \begin{pmatrix} a_1b_1a_1 & 0 \\ 0 & a_2b_2a_2 \end{pmatrix}_p$ we get from Lemma 1.3 that $aba \in \mathscr{A}^{\mathsf{d}}$ and

$$(aba)^{\mathsf{d}} = \begin{pmatrix} (a_1b_1a_1)^{\mathsf{d}} & 0 \\ 0 & 0 \end{pmatrix}_p.$$

Thus, to prove (ii), it is enough to prove $b_1^{\mathsf{d}}(a_1b_1a_1)^{\mathsf{d}} = (b_1^{\mathsf{d}}[{a_1}^{-1}]_{p\mathscr{A}p})^2$. But, this expression follows easily from the above formula and the following computation:

$$\begin{aligned} b_1^{\mathsf{d}}(a_1b_1a_1)^{\mathsf{d}} &= \begin{pmatrix} [(b_1')^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \begin{pmatrix} [{a_{11}}^{-1}]_{q\mathscr{A}q}[(b_1')^{-1}]_{q\mathscr{A}q}[{a_{11}}^{-1}]_{q\mathscr{A}q} & 0 \\ z & 0 \end{pmatrix}_q \\ &= \begin{pmatrix} [(b_1')^{-1}]_{q\mathscr{A}q}[{a_{11}}^{-1}]_{q\mathscr{A}q}[(b_1')^{-1}]_{q\mathscr{A}q}[{a_{11}}^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q . \end{aligned}$$

(iii) As in the beginning of the proof but by changing the roles of a and b, we get that $a, b \in \mathscr{A}^{\mathsf{d}}$ can be represented by

$$a = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}_\pi, \qquad b = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}_\pi,$$

where $f_1, g_1 \in \pi\mathscr{A}\pi$, $f_2, g_2 \in (1-\pi)\mathscr{A}(1-\pi)$, g_1 is invertible and g_2 is quasinilpotent (in their respectives subalgebras) and π is the idempotent bb^{d}. From $a^3b = ba$ we get $f_i^3g_i = g_if_i$ for $i = 1, 2$. To prove (iii) we need to prove $g_ig_i^{\mathsf{d}}f_ig_i^{\mathsf{d}} = g_i^{\mathsf{d}}f_i^3$ for $i = 1, 2$. Since g_1 is invertible, then $g_1g_1^{\mathsf{d}}f_1g_1^{\mathsf{d}} = g_1^{\mathsf{d}}f_1^3$ follows from $f_1[{g_1}^{-1}]_{\pi\mathscr{A}\pi} = [{g_1}^{-1}]_{\pi\mathscr{A}\pi}f_1^3$. Since g_2 is quasinilpotent, then $g_2^{\mathsf{d}} = 0$, and thus, $g_2g_2^{\mathsf{d}}f_2g_2^{\mathsf{d}} = g_2^{\mathsf{d}}f_2^3$ trivially holds. □

Theorem 2.2. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}^{\mathsf{d}}$ such that $b^3a = ab$ and $a^3b = ba$. Then*

(i) $ab^{\mathsf{d}} = (b^3)^{\mathsf{d}}a = b^{\mathsf{d}}a^3$,

(ii) $b^{\mathsf{d}}a = b^2ab^{\mathsf{d}}$,

(iii) $ab \in \mathscr{A}^{\mathsf{d}}$ *and* $(ab)^{\mathsf{d}} = b^{\mathsf{d}}a^{\mathsf{d}}$.

Proof. We represent

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_q, \qquad b^{\mathsf{d}} = \begin{pmatrix} [{b_1}^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q,$$

where $q = bb^{\mathsf{d}}$, $b_1 \in [q\mathscr{A}q]^{-1}$ and $b_2 \in [(1-q)\mathscr{A}(1-q)]^{\mathsf{qnil}}$. From Lemma 1.6 we can write

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_q .$$

Moreover, from $a^3b = ba$ and $b^3a = ab$ we obtain $a_i^3b_i = b_ia_i$ and $b_i^3a_i = a_ib_i$ for $i = 1, 2$.

(i) and (ii): It is enough to prove: (a) $a_1[{b_1}^{-1}]_{q\mathscr{A}q} = ([{b_1}^{-1}]_{q\mathscr{A}q})^3a_1$; (b) $a_1[{b_1}^{-1}]_{q\mathscr{A}q} = [{b_1}^{-1}]_{q\mathscr{A}q}a_1^3$; and (c) $[{b_1}^{-1}]_{q\mathscr{A}q}a = b_1^2a_1[{b_1}^{-1}]_{q\mathscr{A}q}$. Expressions (a) and (c) follow from $b_1^3a_1 = a_1b_1$. Expression (b) follows from $b_1a_1 = b_1a_1^3$.

(iii): By Theorem 2.1, and the invertibility of b_1 (in $q\mathscr{A}q$) we get $a_1b_1 \in [q\mathscr{A}q]^{\mathsf{d}}$ and $(a_1b_1)^{\mathsf{d}} = [{b_1}^{-1}]_{q\mathscr{A}q}a_1^{\mathsf{d}}$. Since b_2 is quasinilpotent, from Lemma 1.5,

a_2b_2 is quasinilpotent, $a_2b_2 \in [(1-q)\mathscr{A}(1-q)]^{\mathsf{d}}$ and $(a_2b_2)^{\mathsf{d}} = 0$. From Lemma 1.3, we have $ab \in \mathscr{A}^{\mathsf{d}}$ and

$$(ab)^{\mathsf{d}} = \begin{pmatrix} [b_1{}^{-1}]_{q\mathscr{A}q} a_1^{\mathsf{d}} & 0 \\ 0 & 0 \end{pmatrix}_q = \begin{pmatrix} [b_1{}^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \begin{pmatrix} a_1^{\mathsf{d}} & 0 \\ 0 & a_2^{\mathsf{d}} \end{pmatrix}_q = b^{\mathsf{d}} a^{\mathsf{d}}.$$

□

Remark. Since the hypothesis of Theorem 2.2 are symmetric on the elements a and b, we have also valid formulas by changing the roles of a and b.

Lemma 2.1. *Let $\mathscr{A}$ be a Banach algebra and let $x, y \in \mathscr{A}$ be such that $x^3y = yx$ and $y^3x = xy$. If y is quasinilpotent, then $xy = yx = 0$.*

Proof. By Lemma 1.4 (iv), for an arbitrary positive integer n, we have

$$\|xy\|^{1/n} = \|x^{26n}(xy)(y)^{2n}\|^{1/n} \leq \|x\|^{26}\|xy\|^{1/n}\left[\|y^n\|^{1/n}\right]^2.$$

Making $n \to \infty$ and using that y is quasinilpotent we get $\lim_{n\to\infty} \|xy\|^{1/n} = 0$, i.e., $xy = 0$. Now, $yx = x^3y = x^2xy = 0$. □

Theorem 2.3. *Let $\mathscr{A}$ be a unital Banach algebra. Let $a, b \in \mathscr{A}^{\mathsf{d}}$ such that $a^3b = ba$ and $b^3a = ab$. Then*

(i) $(a+b)^{\mathsf{d}} = \frac{1}{8}bb^{\mathsf{d}}(3a^3 + 3b^3 - a - b)aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}$.

(ii) $(1 - aa^{\mathsf{d}}bb^{\mathsf{d}})(a+b)^{\mathsf{d}} = b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}$.

(iii) $(a+ab)^{\mathsf{d}} = \frac{3}{8}bb^{\mathsf{d}}\left[a^3 + a + ba + 3ab\right]aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}}$.

(iv) $(a+ba)^{\mathsf{d}} = \frac{3}{8}bb^{\mathsf{d}}\left[a^3 + a + ba + ab\right]aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}}$.

Proof. If $ab = 0$, then $ba = a^2ab = 0$. It is known [12, Theorem 5.7] that $ab = ba = 0$ and $a \in \mathscr{A}^{\mathsf{d}}$ and $b \in \mathscr{A}^{\mathsf{d}}$ imply $a+b \in \mathscr{A}^{\mathsf{d}}$ and $(a+b)^{\mathsf{d}} = a^{\mathsf{d}} + b^{\mathsf{d}}$. Similarly, $ba = 0$ leads to $a+b \in \mathscr{A}^{\mathsf{d}}$ and $(a+b)^{\mathsf{d}} = a^{\mathsf{d}} + b^{\mathsf{d}}$. Therefore, if $ab = 0$ or $ba = 0$, then the four formulas of this theorem hold.

(i) In the following we will assume $ab \neq 0$ and $ba \neq 0$. As in the proof of Theorem 2.1, we can represent a and b as follows:

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, \qquad b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p, \tag{2.3}$$

where $p = aa^{\mathsf{d}}$, $a_1 \in [p\mathscr{A}p]^{-1}$ and $a_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{qnil}}$. The hypotheses $a^3b = ba$ and $b^3a = ab$ imply that $a_i^3b_i = b_ia_i$ and $b_i^3a_i = a_ib_i$ for $i = 1, 2$. Moreover, a_2b_2 is quasinilpotent since a_2 is quasinilpotent. In the following we will prove the theorem depending on b_1 as in Theorem 2.1.

(1) If b_1 is neither invertible nor quasinilpotent, then we represent

$$b_1 = \begin{pmatrix} b_1' & 0 \\ 0 & b_1'' \end{pmatrix}_q, \tag{2.4}$$

where $q = b_1 b_1^{\mathsf{d}}$, $b_1' \in [q\mathscr{A}q]^{-1}$ and $b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{qnil}}$. If we set

$$a_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_q,$$

by Lemma 1.6 we get $a_{12} = 0$ and $a_{21} = 0$. Thus

$$a_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}_q. \tag{2.5}$$

From $a_1^3 b_1 = b_1 a_1$ and $b_1^3 a_1 = a_1 b_1$ we obtain $a_{11}^3 b_1' = b_1' a_{11}$, $(b_1')^3 a_{11} = a_{11} b_1'$, $a_{22}^3 b_1'' = b_1'' a_{22}$, and $(b_1'')^3 a_{22} = a_{22} b_1''$. Moreover, since $a_1 \in [p\mathscr{A}p]^{-1}$ we get $a_{11} \in [q\mathscr{A}q]^{-1}$ and $a_{22} \in [(p-q)\mathscr{A}(p-q)]^{-1}$.

Let us define

$$x = \frac{1}{8}\left(3a_{11}^3 + 3(b_1')^3 - a_{11} - b_1'\right). \tag{2.6}$$

By the definition of the generalized Drazin inverse, we shall prove that $a_{11} + b_1'$ is generalized Drazin invertible and $(a_{11} + b_1')^{\mathsf{d}} = x$. Before doing this, let us simplify some powers of $a_{11} + b_1'$, which will help us to prove $(a_{11} + b_1')^{\mathsf{d}} = x$. Evidently we have

$$(a_{11} + b_1')^2 = a_{11}^2 + a_{11} b_1' + b_1' a_{11} + (b_1')^2.$$

Observe that

$$a_{11} b_1' = (b_1')^3 a_{11} = (b_1')^2 b_1 a_{11} = (b_1')^2 a_{11}^3 b_1 = (b_1')^2 a_{11}^2 a_{11} b_1'.$$

The invertibility of $a_{11} b_1'$ yields

$$q = (b_1')^2 a_{11}^2 \tag{2.7}$$

(recall that the unity of the subalgebra $q\mathscr{A}q$ is q). Now we have

$$\begin{aligned} a_{11}^4 b_1' &= a_{11}(a_{11}^3 b_1') = a_{11} b_1' a_{11} = (a_{11} b_1') a_{11} = ((b_1')^3 a_{11}) a_{11} \\ &= b_1'((b_1')^2 a_{11}^2) = b_1'. \end{aligned} \tag{2.8}$$

The invertibility of b_1' leads to

$$a_{11}^4 = q. \tag{2.9}$$

Postmultplying (2.7) by a_{11}^2 and by using (2.9) we get

$$a_{11}^2 = (b_1')^2. \tag{2.10}$$

By inserting (2.10) into (2.7) we obtain

$$(b_1')^4 = q. \tag{2.11}$$

Moreover, from (2.10) we get

$$a_{11}^2 b_1' = (b_1')^3 = b_1' a_{11}^2, \qquad (b_1')^2 a_{11} = a_{11}^3 = a_{11}(b_1')^2. \tag{2.12}$$

Observe that the computations made (2.8) imply

$$a_{11} b_1' a_{11} = b_1'. \tag{2.13}$$

From (2.9) and (2.10) we have $b_1' a_{11} b_1' = (b_1' a_{11}) b_1' = (a_{11}^3 (b_1')) b_1' = a_{11}^3 (b_1')^2 = a_{11}^5 = a_{11}$. Thus,

$$b_1' a_{11} b_1' = a_{11}. \tag{2.14}$$

Expressions (2.10), (2.13), and (2.14) lead to

$$\begin{aligned}(a_{11} + b_1')^3 &= a_{11}^3 + a_{11} b_1' a_{11} + b_1' a_{11}^2 + (b_1')^2 a_{11} + a_{11}^2 b_1' + b_1' a_{11} b_1' \\ &\quad + a_{11}(b_1')^2 + (b_1')^3 = 3a_{11}^3 + 3(b_1')^3 + a_{11} + b_1'.\end{aligned} \tag{2.15}$$

Employing (2.9), (2.11), and recalling $a_{11}^3 b_1' = b_1' a_{11}$ and $(b_1')^3 a_{11} = a_{11} b_1'$ we have

$$\begin{aligned}(a_{11} + b_1')^4 &= (a_{11} + b_1')^3 (a_{11} + b_1') \\ &= (3a_{11}^3 + 3(b_1')^3 + (a_{11} + b_1'))(a_{11} + b_1') \\ &= 3a_{11}^4 + 3a_{11}^3 b_1' + 3(b_1')^3 a_{11} + 3(b_1')^4 + (a_{11} + b_1')^2 \\ &= 6q + 3a_{11} b_1' + 3b_1' a_{11} + (a_{11} + b_1')^2.\end{aligned}$$

Furthermore

$$\begin{aligned}(a_{11} + b_1')^5 &= (a_{11} + b_1')^4 (a_{11} + b_1') \\ &= \left[6q + 3a_{11} b_1' + 3b_1' a_{11} + (a_{11} + b_1')^2\right](a_{11} + b_1') \\ &= 6a_{11} + 6b_1' + 3a_{11} b_1' a_{11} + 3a_{11}(b_1')^2 + 3b_1' a_{11}^2 \\ &\quad + 3b_1' a_{11} b_1' + (a_{11} + b_1')^3,\end{aligned}$$

which by using (2.10), (2.13), (2.14), and (2.15) reduces to

$$\begin{aligned}(a_{11} + b_1')^5 &= 9a_{11} + 9b_1' + 3a_{11}^3 + 3b_1'^3 + (a_{11} + b_1')^3 \\ &= 2(a_{11} + b_1')^3 + 8(a_{11} + b_1').\end{aligned} \tag{2.16}$$

Now we have

$$\begin{aligned}(a_{11} + b_1')^7 &= [2(a_{11} + b_1')^3 + 8(a_{11} + b_1')](a_{11} + b_1')^2 \\ &= 2(a_{11} + b_1')^5 + 8(a_{11} + b_1')^3.\end{aligned} \tag{2.17}$$

In view of (2.6) and (2.15) we get

$$x = \frac{1}{8}(a_{11} + b_1')^3 - \frac{1}{4}(a_{11} + b_1'), \tag{2.18}$$

which trivially yields $x(a_{11} + b_1') = (a_{11} + b_1')x$. Furthermore, from (2.16) and (2.18)

$$\begin{aligned} x(a_{11}+b_1')x &= \left[\frac{1}{8}(a_{11}+b_1')^3 - \frac{1}{4}(a_{11}+b_1')\right](a_{11}+b_1')\left[\frac{1}{8}(a_{11}+b_1')^3 - \frac{1}{4}(a_{11}+b_1')\right] \\ &= \frac{1}{8}\left[(a_{11}+b_1')^5 - 2(a_{11}+b_1')^3\right]\left[\frac{1}{8}(a_{11}+b_1')^2 - \frac{1}{4}q\right] \\ &= (a_{11}+b_1')\left[\frac{1}{8}(a_{11}+b_1')^2 - \frac{1}{4}q\right] \\ &= \frac{1}{8}(a_{11}+b_1')^3 - \frac{1}{4}(a_{11}+b_1') = x. \end{aligned}$$

Now, by using (2.17) and (2.18), we have

$$\begin{aligned} (a_{11}+b_1')^6 x &= (a_{11}+b_1')^6\left[\frac{1}{8}(a_{11}+b_1')^3 - \frac{1}{4}(a_{11}+b_1')\right] \\ &= \frac{1}{8}(a_{11}+b_1')^2\left[(a_{11}+b_1')^7 - 2(a_{11}+b_1')^5\right] = (a_{11}+b_1')^5. \end{aligned}$$

Hence the expression $(a_{11} + b_1')^{5+1}x = (a_{11} + b_1')^5$ holds, and thus $a_{11} + b_1'$ is generalized Drazin invertible (in fact, is Drazin invertible) and $(a_{11} + b_1')^{\mathsf{d}} = x$.

Now, we shall study the generalized Drazin invertibility of $a_{22} + b_1''$. Recall that $a_{22}^3 b_1'' = b_1'' a_{22}$ and $(b_1'')^3 a_{22} = a_{22} b_1''$, $a_{22} \in [(p-q)\mathscr{A}(p-q)]^{-1}$, and $b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{qnil}}$. By Lemma 2.1 we get $a_{22}b_1'' = 0$. The invertibility of a_{22} leads to $b_1'' = 0$. Thus, $a_{22} + b_1'' = a_{22}$ and therefore, $(a_{22} + b_1'')^{\mathsf{d}} = a_{22}^{\mathsf{d}}$.

From (2.4), (2.5), $a_{12} = a_{21} = 0$, $a_{11} + b_1' \in [q\mathscr{A}q]^{\mathsf{d}}$, $a_{22} + b_1'' \in [(p-q)\mathscr{A}(p-q)]^{\mathsf{d}}$, and by Lemma 1.3, we have $a_1 + b_1 \in [p\mathscr{A}p]^{\mathsf{d}}$ and

$$\begin{aligned} (a_1+b_1)^{\mathsf{d}} * &= \begin{pmatrix} (a_{11}+b_1')^{\mathsf{d}} & 0 \\ 0 & (a_{22}+b_1'')^{\mathsf{d}} \end{pmatrix}_q \\ &= \begin{pmatrix} \frac{1}{8}\left(3a_{11}^3 + 3(b_1')^3 - a_{11} - b_1'\right) & 0 \\ 0 & a_{22}^{\mathsf{d}} \end{pmatrix}_q. \end{aligned}$$

By (2.5), (2.4), and by observing $q = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}_q$, we have

$$q(3a_1^3 + 3b_1^3 - a_1 - b_1) = \begin{pmatrix} 3a_{11}^3 + 3(b_1')^3 - a_{11} - b_1' & 0 \\ 0 & 0 \end{pmatrix}_q.$$

Recall that q is an idempotent in the subalgebra $p\mathscr{A}p$ (whose unity is p), and thus, $pq = qp = q$. So, $p - q = \begin{pmatrix} 0 & 0 \\ 0 & p-q \end{pmatrix}_q$, and therefore, from (2.5)

$$(p-q)[a_1{}^{-1}]_{p\mathscr{A}p} = \begin{pmatrix} 0 & 0 \\ 0 & a_{22}^{\mathsf{d}} \end{pmatrix}_q.$$

But $[a_1{}^{-1}]_{p\mathscr{A}p} \in p\mathscr{A}p$ and p is the unity of $p\mathscr{A}p$, thus $p[a_1{}^{-1}]_{p\mathscr{A}p} = [a_1{}^{-1}]_{p\mathscr{A}p}$ and $(p-q)[a_1{}^{-1}]_{p\mathscr{A}p} = (1-q)[a_1{}^{-1}]_{p\mathscr{A}p}$.

From the above computations we have

$$(a_1+b_1)^{\mathsf{d}} = \frac{1}{8}q(3a_1^3+3b_1^3-a_1-b_1)+(1-q)[a_1{}^{-1}]_{p\mathscr{A}p}.$$

From (2.3) we get $a_2^3b_2 = b_2a_2^3$ and $b_2^3a_2 = a_2b_2^3$. Recall that a_2 is quasinilpotent, hence by Lemma 2.1 we have that $a_2b_2 = b_2a_2 = 0$. Moreover, a_2 and b_2 are generalized Drazin invertible because a and b are generalized Drazin invertible (recall Lemma 1.3). By [12, Theorem 5.7] we get that a_2+b_2 is generalized Drazin invertible and $(a_2+b_2)^{\mathsf{d}} = a_2^{\mathsf{d}}+b_2^{\mathsf{d}}$, but $a^{\mathsf{d}} = 0$ since a_2 is quasinilpotent. Therefore, $(a_2+b_2)^{\mathsf{d}} = b_2^{\mathsf{d}}$. By Lemma 1.3 we get $a+b \in \mathscr{A}^{\mathsf{d}}$ and

$$\begin{aligned}(a+b)^{\mathsf{d}} &= \begin{pmatrix} (a_1+b_1)^{\mathsf{d}} & 0 \\ 0 & (a_2+b_2)^{\mathsf{d}} \end{pmatrix}_p \\ &= \begin{pmatrix} (a_1+b_1)^{\mathsf{d}} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p = \begin{pmatrix} (a_1+b_1)^{\mathsf{d}} & 0 \\ 0 & 0 \end{pmatrix}_p + \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \\ &= \begin{pmatrix} \frac{1}{8}q(3a_1^3+3b_1^3-a_1-b_1)+(1-q)[a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p + \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p. \qquad (2.19)\end{aligned}$$

Observe that q, a_1, b_1, and $[a_1{}^{-1}]_{p\mathscr{A}p}$ belong to the subalgebra $p\mathscr{A}p$ (whose unity is p), so $q(3a_1^3+3b_1^3-a_1-b_1)p = q(3a_1^3+3b_1^3-a_1-b_1)$ and $(1-q)[a_1{}^{-1}]_{p\mathscr{A}p}p = (1-q)[a_1{}^{-1}]_{p\mathscr{A}p}$. Hence by using (2.3) and recalling that $q = b_1b_1^{\mathsf{d}}$

$$\begin{aligned}&bb^{\mathsf{d}}(3a^3+3b^3-a-b)p \\ &\quad= \begin{pmatrix} b_1b_1^{\mathsf{d}} & 0 \\ 0 & b_2b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} 3a_1^3+3b_1^3-a_1-b_1 & 0 \\ 0 & 3a_1^3+3b_1^3-a_1-b_3 \end{pmatrix}_p \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}_p \\ &\quad= \begin{pmatrix} q(3a_1^3+3b_1^3-a_1-b_1) & 0 \\ 0 & 0 \end{pmatrix}_p \qquad (2.20)\end{aligned}$$

and

$$\begin{aligned} b^{\pi}a^{\mathsf{d}} &= a^{\mathsf{d}} - bb^{\mathsf{d}}a^{\mathsf{d}} \\ &= \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p - \begin{pmatrix} b_1b_1^{\mathsf{d}} & 0 \\ 0 & b_2b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p \\ &= \begin{pmatrix} (1-q)[a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p. \qquad (2.21)\end{aligned}$$

Since $b_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$ we get $b_2^{\mathsf{d}} \in (1-p)\mathscr{A}(1-p)$ and $b_2^{\mathsf{d}}(1-p) = b_2^{\mathsf{d}}$ because the unity of the subalgebra $(1-p)\mathscr{A}(1-p)$ is $1-p$. Thus

$$b^{\mathsf{d}}a^{\pi} = b^{\mathsf{d}}(1-aa^{\mathsf{d}}) = b^{\mathsf{d}}(1-p) = \begin{pmatrix} b_1^{\mathsf{d}} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p = \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p. \qquad (2.22)$$

From (2.19–2.22) we obtain

$$(a+b)^{\mathsf{d}} = \frac{1}{8} bb^{\mathsf{d}}(3a^3+3b^3-a-b)aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}.$$

(2) If b_1 is invertible, in the representation (2.4) we have $b_1 = b_1'$, $b_1'' = 0$, and $q = p$. The above computations work.

(3) If b_1 is quasinilpotent, in (2.4) we have $q = b_1 b_1^{\mathsf{d}} = 0$ (observe that in this case, the subalgebra $q\mathscr{A}q$ becomes the subalgebra $\{0\}$), $b_1 = b_1''$, and $b_1' = 0$. Also, the above computations are valid.

Therefore, (i) is proved.

(ii) Notice that $b^{\mathsf{d}}b^{\pi} = 0$. Furthermore,

$$aa^{\mathsf{d}}bb^{\mathsf{d}}b^{\mathsf{d}}a^{\pi} = pb^{\mathsf{d}}(1-p) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} b_1^{\mathsf{d}} & 0 \\ 0 & b_2^{\mathsf{d}} \end{pmatrix}_p \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p = 0.$$

By (2.3) and (2.20)

$$(aa^{\mathsf{d}}bb^{\mathsf{d}})\left[bb^{\mathsf{d}}(3a^3+3b^3-a-b)aa^{\mathsf{d}}\right] = bb^{\mathsf{d}}(3a^3+3b^3-a-b)aa^{\mathsf{d}}.$$

Thus, we obtain

$$\begin{aligned}(1-aa^{\mathsf{d}}bb^{\mathsf{d}})(a+b)^{\mathsf{d}} &= (1-aa^{\mathsf{d}}bb^{\mathsf{d}})\left[\frac{1}{8}bb^{\mathsf{d}}(3a^3+3b^3-a-b)aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}\right] \\ &= b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}.\end{aligned}$$

(iii) Let us decompose a and b as in (2.3), a_1 as in (2.5), and b_1 as in (2.4). We shall study the generalized Drazin invertibility of $a_{11} + a_{11}b_1'$, $a_{22} + a_{22}b_1''$, and $a_2 + a_2 b_2$.

We will apply properties (2.9), (2.10), (2.11), (2.13), (2.14) to simplify some powers of $a_{11} + a_{11}b_1'$. It is simple to see that

$$\begin{aligned}(a_{11}+a_{11}b_1')^2 &= 2a_{11}^2 + (b_1')^3 + b_1', \\ (a_{11}+a_{11}b_1')^3 &= 3a_{11}^3 + 3b_1'a_{11} + a_{11} + a_{11}b_1', \\ (a_{11}+a_{11}b_1')^4 &= 6q + 3b_1' + 3(b_1')^3 + (a_{11}+a_{11}b_1')^2, \\ (a_{11}+a_{11}b_1')^5 &= 8(a_{11}+a_{11}b_1') + 2(a_{11}+a_{11}b_1')^3, \\ (a_{11}+a_{11}b_1')^7 &= 8(a_{11}+a_{11}b_1')^3 + 2(a_{11}+a_{11}b_1')^5.\end{aligned}$$

As we did in the proof of (i), we have $(a_{11}+a_{11}b_1')^{\mathsf{d}} = \frac{1}{8}(a_{11}+a_{11}b_1')^3 + \frac{1}{4}(a_{11}+a_{11}b_1')$. Simplifying and using the above expression for the cube of $a_{11}+a_{11}b_1'$ we get

$$(a_{11}+a_{11}b_1')^{\mathsf{d}} = \frac{3}{8}\left[a_{11}^3 + a_{11} + b_1'a_{11} + 3a_{11}b_1'\right]. \tag{2.23}$$

Since b_1'' and a_2 are quasinilpotent and $a^3b = ba$, $b^3a = ab$ hold, by Lemma 2.1, we get $a_{22}b_1'' = 0$ and $a_2b_2 = 0$. Thus $a_{22}+a_{22}b_1''$ and $a_2+a_2b_2$ are generalized Drazin invertible and

$$(a_{22} + a_{22}b_1'')^{\mathsf{d}} = a_{22}^{\mathsf{d}} \qquad \text{and} \qquad (a_2 + a_2b_2)^{\mathsf{d}} = a_2^{\mathsf{d}} = 0, \tag{2.24}$$

the last equation be guaranteed by the quasinilpotency of a_2. By (2.23) and (2.24) and by the same argument as in (i), we get

$$(a + ab)^{\mathsf{d}} = \frac{3}{8}bb^{\mathsf{d}}\left[a^3 + a + ba + ab\right]aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}}$$

i.e., (iii) is proved. The proof of (iv) is similar to (iii). □

Theorem 2.4. *Let $\mathscr{A}$ be a unital Banach algebra and let a, b be Drazin invertible elements of $\mathscr{A}$. If $a^3b = ba$, $\|a^{\mathsf{D}}b\| < 1$, and $\|(1 - aa^{\mathsf{D}})b^{\mathsf{D}}a\| < 1$, then $a + b$ is generalized Drazin invertible and*

(i) $(a + b)^{\mathsf{d}} = \sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{D}}b)^na^{\mathsf{D}} + z + \sum_{n=0}^{\infty} a^{\pi}(a + b)^nb^{\pi}abb^{\mathsf{D}}z^{n+2}$.

(ii) $a^{\pi}bb^{\mathsf{D}}(a + b)^{\mathsf{d}} = z$,

where $z = a^{\pi}\sum_{n=0}^{\infty}(-1)^n(b^{\mathsf{D}}a)^nb^{\mathsf{D}}$.

Proof. If $ab = ba = 0$, the formulas of the Theorem hold by Theorem 5.7 of [12]. Thus, we can assume $ab \neq 0$ and $ba \neq 0$.

(i) Setting $p = aa^{\mathsf{D}}$, we represent a as in (1.3), where a_2 is nilpotent (since a is Drazin invertible instead of generalized Drazin invertible, we can get a stronger condition than the quasinilpotency). By Lemma 1.6 we have $pb = bp$, and thus we can represent b as follows

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p. \tag{2.25}$$

First we study the Drazin invertibility of $a_1 + b_1$. Since

$$a^{\mathsf{D}}b = \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p = \begin{pmatrix} [a_1{}^{-1}]_{p\mathscr{A}p}b_1 & 0 \\ 0 & 0 \end{pmatrix}_p,$$

we get $\|[a_1{}^{-1}]_{p\mathscr{A}p}b_1\| = \|a^{\mathsf{D}}b\| < 1$, and thus, $p + [a_1{}^{-1}]_{p\mathscr{A}p}b_1 \in [p\mathscr{A}p]^{-1}$. From $a_1 + b_1 = a_1(p + [a_1{}^{-1}]_{p\mathscr{A}p}b_1)$ and $a_1 \in [p\mathscr{A}p]^{-1}$ we obtain $a_1 + b_1 \in [p\mathscr{A}p]^{-1}$ and

$$\begin{aligned} [(a_1 + b_1)^{-1}]_{p\mathscr{A}p} &= [(p + [a_1{}^{-1}]_{p\mathscr{A}p}b_1)^{-1}]_{p\mathscr{A}p}[a_1{}^{-1}]_{p\mathscr{A}p} \\ &= \sum_{n=0}^{\infty}(-1)^n([a_1{}^{-1}]_{p\mathscr{A}p}b_1)^n[a_1{}^{-1}]_{p\mathscr{A}p}. \end{aligned}$$

From this, we have $a_1 + b_1 \in \mathscr{A}^{\mathsf{d}}$ and

$$(a_1 + b_1)^{\mathsf{d}} = \sum_{n=0}^{\infty} (-1)^n (a^{\mathsf{D}} b)^n a^{\mathsf{D}}. \tag{2.26}$$

Now, we study the Drazin invertibility of $a_2 + b_2$. To this end, let us define $q = b_2 b_2^{\mathsf{D}}$ and represent b_2 as follows:

$$b_2 = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}_q,$$

where $b_{11} \in [q\mathscr{A}q]^{-1}$ and b_{22} is nilpotent. Since $a^3 b = ba$ implies $a_2^3 b_2 = b_2 a_2$, by Lemma 1.4

$$qa_2 - qa_2 q = qa_2(1-q) = q^n a_2 (1-q) = (b_2^{\mathsf{D}})^n a_2^{3^n} b_2^n (1 - b_2 b_2^{\mathsf{D}})^n.$$

Since a_2 is nilpotent we get $qa_2 = qa_2 q$. Hence,

$$a_2 = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q, \qquad a_2 + b_2 = \begin{pmatrix} a_{11} + b_{11} & 0 \\ a_{21} & a_{22} + b_{22} \end{pmatrix}_q. \tag{2.27}$$

An evident induction argument shows that there exists $x_1, x_2, \ldots \in \mathscr{A}$ such that

$$a_2^n = \begin{pmatrix} a_{11}^n & 0 \\ x_n & a_{22}^n \end{pmatrix}_q, \qquad \forall\, n \in \mathbb{N}.$$

Since a_2 is nilpotent, we get that a_{11} and a_{22} are nilpotent.

In order to study the Drazin invertibility of $a_2 + b_2$, we use Lemma 1.2: We need prove that $a_{11} + b_{11}$ and $a_{22} + b_{22}$ are generalized Drazin invertible.

First we prove that $a_{11} + b_{11}$ is generalized Drazin invertible. Let us observe that

$$(1 - aa^{\mathsf{D}}) b^{\mathsf{D}} a = \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p \begin{pmatrix} b_1^{\mathsf{D}} & 0 \\ 0 & b_2^{\mathsf{D}} \end{pmatrix}_p \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p = \begin{pmatrix} 0 & 0 \\ 0 & b_2^{\mathsf{D}} a_2 \end{pmatrix}_p$$

and

$$b_2^{\mathsf{D}} a_2 = \begin{pmatrix} [{b_{11}}^{-1}]_{q\mathscr{A}q} & 0 \\ 0 & 0 \end{pmatrix}_q \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q = \begin{pmatrix} [{b_{11}}^{-1}]_{q\mathscr{A}q} a_{11} & 0 \\ 0 & 0 \end{pmatrix}_q,$$

which yields $\|[{b_{11}}^{-1}]_{q\mathscr{A}q} a_{11}\| = \|(1 - aa^{\mathsf{D}}) b^{\mathsf{D}} a\| < 1$. Now

$$a_{11} + b_{11} = b_{11} \left([{b_{11}}^{-1}]_{q\mathscr{A}q} a_{11} + q \right)$$

and $b_{11} \in [q\mathscr{A}q]^{-1}$ lead to $a_{11} + b_{11} \in [q\mathscr{A}q]^{-1}$ and

$$\begin{aligned} [(a_{11} + b_{11})^{-1}]_{q\mathscr{A}q} &= \sum_{n=0}^{\infty} (-1)^n ([{b_{11}}^{-1}]_{q\mathscr{A}q} a_{11})^n [{b_{11}}^{-1}]_{q\mathscr{A}q} = \sum_{n=0}^{\infty} (-1)^n (b_2^{\mathsf{D}} a_2)^n b_2^{\mathsf{D}} \\ &= (1 - aa^{\mathsf{D}}) \sum_{n=0}^{\infty} (-1)^n (b^{\mathsf{D}} a)^n b^{\mathsf{D}}. \end{aligned}$$

Since $q\mathscr{A}q$ is a subalgebra of $\mathscr{A}$ with unity, $a_{11}+b_{11}$ is generalized Drazin invertible and $(a_{11}+b_{11})^{\mathsf{d}} = [(a_{11}+b_{11})^{-1}]_{q\mathscr{A}q}$. Thus,

$$(a_{11}+b_{11})^{\mathsf{d}} = (1-aa^{\mathsf{D}})\sum_{n=0}^{\infty}(-1)^n(b^{\mathsf{D}}a)^n b^{\mathsf{D}}. \tag{2.28}$$

In the following, we study the Drazin invertibility of $a_{22}+b_{22}$. In fact, we are going to prove that $a_{22}+b_{22}$ is nilpotent. Since a_{22} and b_{22} are nilpotent, there exists $k\in\mathbb{N}$ such that $a_{22}^k = b_{22}^k = 0$. It is clear that if $a_{22}=b_{22}$, then $a_{22}+b_{22}$ is nilpotent; so we can assume $a_{22}\neq b_{22}$. Let us remark that for any $n\in\mathbb{N}$, we can write $(a_{22}+b_{22})^n = \sum_{j=1}^{2^n} c_{1j}\cdots c_{nj}$, where $c_{ij}\in\{a_{22},b_{22}\}$. Now, we are ready to prove $(a_{22}+b_{22})^{2k}=0$. To this end, we will prove that if $d_1,\ldots,d_{2k}\in\{a_{22},b_{22}\}$, then $d_1\cdots d_{2k}=0$. Let $A=\{r\in\{1,\ldots,2k\} : d_r = a_{22}\}$ and $B=\{r\in\{1,\ldots,2k\} : d_r=b_{22}\}$ (observe that $A\cup B=\{1,\ldots,2k\}$ and $A\cap B=\varnothing$). If $|\cdot|$ denotes the cardinal of a set, it is clear than $|A|\geq k$ or $|B|\geq k$. If $|A|\geq k$, by using $a_{22}^3b_{22}=b_{22}a_{22}$, then there exists $x\in\mathscr{A}$ such that $d_1\cdots d_{2k}=a_{22}^k x=0$. If $|B|\geq k$, by using again $a_{22}^3b_{22}=b_{22}a_{22}$, there exists $y\in\mathscr{A}$ such that $d_1\cdots d_{2k}=yb_{22}^k=0$. Thus, $a_{22}+b_{22}$ is nilpotent, and therefore, $a_{22}+b_{22}$ is Drazin invertible and $(a_{22}+b_{22})^{\mathsf{D}}=0$.

From the second expression of (2.27) and Lemma 1.2, we get a_2+b_2 is generalized Drazin invertible and

$$(a_2+b_2)^{\mathsf{d}} = \begin{pmatrix} (a_{11}+b_{11})^{\mathsf{d}} & 0 \\ u & (a_{22}+b_{22})^{\mathsf{d}} \end{pmatrix}_q,$$

where

$$\begin{aligned} u \;=\; & \sum_{n=0}^{\infty}\left[(a_{22}+b_{22})^{\mathsf{d}}\right]^{n+2} a_{21}(a_{11}+b_{11})^n(a_{11}+b_{11})^{\pi} + \\ & +\sum_{n=0}^{\infty}(a_{22}+b_{22})^{\pi}(a_{22}+b_{22})^n a_{21}\left[(a_{11}+b_{11})^{\mathsf{d}}\right]^{n+2} \\ & -(a_{22}+b_{22})^{\mathsf{d}}a_{21}(a_{11}+b_{11})^{\mathsf{d}}, \end{aligned}$$

which, having in mind that $(a_{22}+b_{22})^{\mathsf{d}}=0$, reduces to

$$(a_2+b_2)^{\mathsf{d}} = \begin{pmatrix} (a_{11}+b_{11})^{\mathsf{d}} & 0 \\ u & 0 \end{pmatrix}_q,$$

$$u = \sum_{n=0}^{\infty}(a_{22}+b_{22})^n a_{21}\left[(a_{11}+b_{11})^{\mathsf{d}}\right]^{n+2}. \tag{2.29}$$

From (2.29) we get

$$\begin{aligned} (a+b)^{\mathsf{d}} &= (a_1+b_1)^{\mathsf{d}} + (a_2+b_2)^{\mathsf{d}} \\ &= (a_1+b_1)^{\mathsf{d}} + (a_{11}+b_{11})^{\mathsf{d}} + \sum_{n=0}^{\infty}(a_{22}+b_{22})^n a_{21}\left[(a_{11}+b_{11})^{\mathsf{d}}\right]^{n+2}. \end{aligned} \tag{2.30}$$

In (2.26) and (2.28) we have expressed $(a_1+b_1)^{\mathsf{d}}$ and $(a_{11}+b_{11})^{\mathsf{d}}$, respectively, in terms of a, a^{D}, b, and b^{D}. We will express the remaining terms of (2.30) in terms of a, a^{D}, b, and b^{D}. To this end, let us remark that $q = b_2 b_2^{\mathsf{D}}$ is an element of the subalgebra $(1-p)\mathscr{A}(1-p)$ (whose unity is $1-p$) and thus, $(1-p)q = q(1-p) = q$. In other words, we have

$$pq = qp = 0.$$

Furthermore, recall that an element $x \in (1-p)\mathscr{A}(1-p)$ has the representation

$$x = \begin{pmatrix} qxq & qx(1-p-q) \\ (1-p-q)xq & (1-p-q)x(1-p-q) \end{pmatrix}_q .$$

Observe that from the expression of $a_2 + b_2$ in (2.27) we can write

$$(a_2+b_2)^n = \begin{pmatrix} (a_{11}+b_{11})^n & 0 \\ y_n & (a_{22}+b_{22})^n \end{pmatrix}_q ,$$

where $y_1, y_2, \ldots$ are elements in $\mathscr{A}$ (we are not interested in their explicit expressions). We have

$$\begin{aligned} &(1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}}) \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p \begin{pmatrix} (a_1+b_1)^n & 0 \\ 0 & (a_2+b_2)^n \end{pmatrix}_p \begin{pmatrix} p-b_1b_1^{\mathsf{D}} & 0 \\ 0 & 1-p-b_2b_2^{\mathsf{D}} \end{pmatrix}_p \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (a_2+b_2)^n(1-p-b_2b_2^{\mathsf{D}}) \end{pmatrix}_p . \end{aligned} \tag{2.31}$$

Having in mind that $(a_{22}+b_{22})^n \in (1-p-q)\mathscr{A}(1-p-q)$,

$$\begin{aligned} (a_2+b_2)^n(1-p-q) &= \begin{pmatrix} (a_{11}+b_{11})^n & 0 \\ y_n & (a_{22}+b_{22})^n \end{pmatrix}_q \begin{pmatrix} 0 & 0 \\ 0 & 1-p-q \end{pmatrix}_q \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (a_{22}+b_{22})^n \end{pmatrix}_q . \end{aligned}$$

Therefore,

$$(a_{22}+b_{22})^n = (1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}}). \tag{2.32}$$

Now, we are going to express a_{21} in terms of a, a^{D}, b and b^{D}. By $1-p$ is the unity of the subalgebra $(1-p)\mathscr{A}(1-p)$, one has

$$\begin{aligned} &(1-aa^{\mathsf{D}})(1-bb^{\mathsf{D}})abb^{\mathsf{D}} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p \begin{pmatrix} p-b_1b_1^{\mathsf{D}} & 0 \\ 0 & 1-p-b_2b_2^{\mathsf{D}} \end{pmatrix}_p \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p \begin{pmatrix} b_1b_1^{\mathsf{D}} & 0 \\ 0 & b_2b_2^{\mathsf{D}} \end{pmatrix}_p \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (1-p-q)a_2q \end{pmatrix}_p . \end{aligned}$$

Since

$$(1-p-q)a_2q = \begin{pmatrix} 0 & 0 \\ 0 & 1-p-q \end{pmatrix}_q \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}_q \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}_q = \begin{pmatrix} 0 & 0 \\ (1-p-q)a_{21}q & 0 \end{pmatrix}_q$$

and $a_{21} \in (1-p-q)\mathscr{A}q$ (this latter subset is not a subalgebra of $\mathscr{A}$) we have $(1-p-q)a_{21}q = a_{21}$. Thus, from the above computations, we have

$$a_{21} = (1-aa^{\mathsf{D}})(1-bb^{\mathsf{D}})abb^{\mathsf{D}}.$$

From (2.26), (2.28), (2.32), and (2.30) we get

$$(a+b)^{\mathsf{d}} = \sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{D}}b)^n a^{\mathsf{D}} + z + \sum_{n=0}^{\infty}(1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}})(1-aa^{\mathsf{D}})(1-bb^{\mathsf{D}})abb^{\mathsf{D}}z^{n+2}, \tag{2.33}$$

where $z = (1-aa^{\mathsf{D}})\sum_{n=0}^{\infty}(-1)^n(b^{\mathsf{D}}a)^n b^{\mathsf{D}}$. We can simplify expression of $(a+b)^{\mathsf{d}}$. By observing the expression of $(1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}})$ obtained in (2.31), easily we can see $(1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}})(1-aa^{\mathsf{D}}) = (1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}})$ and using that $1-bb^{\mathsf{D}}$ is an idempotent, (2.33) reduces to

$$(a+b)^{\mathsf{d}} = \sum_{n=0}^{\infty}(-1)^n(a^{\mathsf{D}}b)^n a^{\mathsf{D}} + z + \sum_{n=0}^{\infty}(1-aa^{\mathsf{D}})(a+b)^n(1-bb^{\mathsf{D}})abb^{\mathsf{D}}z^{n+2}.$$

(ii) By using (1.3) and (2.25) we have

$$(1-aa^{\mathsf{D}})bb^{\mathsf{D}}(a+b)^{\mathsf{d}} = \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}_p \begin{pmatrix} b_1b_1^{\mathsf{D}} & 0 \\ 0 & b_2b_2^{\mathsf{D}} \end{pmatrix}_p \begin{pmatrix} (a_1+b_1)^{\mathsf{d}} & 0 \\ 0 & (a_2+b_2)^{\mathsf{d}} \end{pmatrix}_p$$

and using that the unity of $(1-p)\mathscr{A}(1-p)$ is $1-p$ we get

$$(1-aa^{\mathsf{D}})bb^{\mathsf{D}}(a+b)^{\mathsf{d}} = b_2b_2^{\mathsf{D}}(a_2+b_2)^{\mathsf{d}} = q(a_2+b_2)^{\mathsf{d}}.$$

From (2.29) and using $q(a_{11}+b_{11})^d = (a_{11}+b_{11})^d$ (because $(a_{11}+b_{11})^d \in q\mathscr{A}q$ and q is the unity of the subalgebra $q\mathscr{A}q$),

$$q(a_2+b_2)^d = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}_q \begin{pmatrix} (a_{11}+b_{11})^{\mathsf{d}} & 0 \\ u & 0 \end{pmatrix}_q = \begin{pmatrix} (a_{11}+b_{11})^{\mathsf{d}} & 0 \\ 0 & 0 \end{pmatrix}_q.$$

Now, (2.28) leads to $(1-aa^{\mathsf{D}})bb^{\mathsf{D}}(a+b)^{\mathsf{d}} = (1-aa^{\mathsf{D}})\sum_{n=0}^{\infty}(-1)^n(b^{\mathsf{D}}a)^n b^{\mathsf{D}}$. □

If b is a special perturbation of a, then we have the following theorem.

Theorem 2.5. *Let $\mathscr{A}$ be a unital Banach algebra. If $a, b \in \mathscr{A}^{\mathsf{d}}$, $a^3b = ba$, and $b^3a = ab$, then*

(i) $\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\| \leq \frac{1}{8}\left(\|bb^{\mathsf{d}}\|\|aa^{\mathsf{d}}\|\|3a^3+3b^3-a-b\|\right) + \|bb^{\mathsf{d}}a^{\mathsf{d}}\| + \|b^{\mathsf{d}}a^{\pi}\|$.

(ii) $\|b^{\mathsf{d}}a^{\pi}\| \leq \|1 - aa^{\mathsf{d}}bb^{\mathsf{d}}\|\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\|$.

Proof. (i) From Theorem 2.3, we get

$$\begin{aligned}(a+b)^{\mathsf{d}} - a^{\mathsf{d}} &= \frac{1}{8}bb^{\mathsf{d}}(3a^3+3b^3-a-b)aa^{\mathsf{d}} + b^{\pi}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi} - a^{\mathsf{d}} \\ &= \frac{1}{8}bb^{\mathsf{d}}(3a^3+3b^3-a-b)aa^{\mathsf{d}} - bb^{\mathsf{d}}a^{\mathsf{d}} + b^{\mathsf{d}}a^{\pi}.\end{aligned}$$

Thus

$$\begin{aligned}\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\| &\leq \frac{1}{8}\|bb^{\mathsf{d}}(3a^3+3b^3-a-b)aa^{\mathsf{d}}\| + \|bb^{\mathsf{d}}a^{\mathsf{d}}\| + \|b^{\mathsf{d}}a^{\pi}\| \\ &\leq \frac{1}{8}\left(\|bb^{\mathsf{d}}\|\|aa^{\mathsf{d}}\|\|3a^3+3b^3-a-b\|\right) + \|bb^{\mathsf{d}}a^{\mathsf{d}}\| + \|b^{\mathsf{d}}a^{\pi}\|.\end{aligned}$$

(ii) By the proof of Theorem 2.3 we get $aa^{\mathsf{d}}bb^{\mathsf{d}} = bb^{\mathsf{d}}aa^{\mathsf{d}}$.

By Theorem 2.3, we have

$$\begin{aligned}b^{\mathsf{d}}a^{\pi} &= (1 - aa^{\mathsf{d}}bb^{\mathsf{d}})(a+b)^{\mathsf{d}} - b^{\pi}a^{\mathsf{d}} \\ &= (1 - bb^{\mathsf{d}}aa^{\mathsf{d}})(a+b)^{\mathsf{d}} - (1 - bb^{\mathsf{d}})a^{\mathsf{d}} \\ &= (1 - bb^{\mathsf{d}}aa^{\mathsf{d}})[(a+b)^{\mathsf{d}} - a^{\mathsf{d}}].\end{aligned}$$

Hence; $\|b^{\mathsf{d}}a^{\pi}\| \leq \|1 - bb^{\mathsf{d}}aa^{\mathsf{d}}\|\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\|$. □

In the following result we have a bound for another kind of perturbation.

Theorem 2.6. *Let $\mathscr{A}$ be a unital algebra and let $a, b \in \mathscr{A}$ be Drazin invertible satisfying $a^3b = ba$, $\|a^{\mathsf{D}}b\| < 1$, and $\|a^{\pi}b^{\mathsf{D}}a\| < 1$. Then*

(i) $\|(a+b)^{\mathsf{d}} - a^{\mathsf{D}}\| \leq \frac{\|a^{\mathsf{D}}\|\|a^{\mathsf{D}}b\|}{1-\|a^{\mathsf{D}}b\|} + \frac{\|b^{\mathsf{D}}\|}{1-\|a^{\pi}b^{\mathsf{D}}a\|} + \frac{\|abb^{\mathsf{D}}\|\|b^{\mathsf{D}}\|^2}{(1-\|a^{\pi}b^{\mathsf{D}}a\|)^2}\sum_{n=0}^{\infty}\|a^{\pi}(a+b)^n b^{\pi}\|$.

(ii) $\|z\| \leq \|a^{\pi}bb^{\mathsf{D}}\|\|(a+b)^{\mathsf{d}} - a^{\mathsf{D}}\|$, *where z is giving as in Theorem 2.4.*

Proof. (i) From Theorem 2.4, we have

$$(a+b)^{\mathsf{d}} - a^{\mathsf{D}} = \sum_{n=1}^{\infty}(-1)^n(a^{\mathsf{D}}b)^n a^{\mathsf{D}} + z + \sum_{n=0}^{\infty}a^{\pi}(a+b)^n b^{\pi}abb^{\mathsf{D}}z^{n+2},$$

where $z = a^{\pi}\sum_{n=0}^{\infty}(-1)^n(b^{\mathsf{D}}a)^n b^{\mathsf{D}}$. Hence, (in Theorem 2.4 we had proved that $\|a^{\mathsf{D}}b\| < 1$ and in (2.31) we have shown, by using the nilpotency of $a_{22} + b_{22}$, that there exists $k \in \mathbb{N}$ such that $a^{\pi}(a+b)^k b^{\pi} = 0$)

$$\begin{aligned}\|(a+b)^{\mathsf{d}} - a^{\mathsf{D}}\| &\leq \|a^{\mathsf{D}}\|\sum_{i=1}^{\infty}\|a^{\mathsf{D}}b\|^n + \|z\| + \|abb^{\mathsf{D}}\|\|z^2\|\sum_{n=0}^{\infty}\|a^{\pi}(a+b)^n b^{\pi}\| \\ &= \|a^{\mathsf{D}}\|\frac{\|a^{\mathsf{D}}b\|}{1-\|a^{\mathsf{D}}b\|} + \|z\| + \|abb^{\mathsf{D}}\|\|z\|^2\sum_{n=0}^{\infty}\|a^{\pi}(a+b)^n b^{\pi}\|.\end{aligned}$$

Now, we will find an upper bound for $\|z\|$. Observe that the proof of Theorem 2.4 distil $a^\pi(b^{\mathsf{D}}a)^n = (a^\pi b^{\mathsf{D}}a)^n$. Thus, $z = \sum_{n=0}^{\infty}(-1)^n(a^\pi b^{\mathsf{D}}a)^n b^{\mathsf{D}}$. By using that $\|a^\pi b^{\mathsf{D}}a\| < 1$, one has $\|z\| \le \|b^{\mathsf{D}}\| \sum_{n=0}^{\infty}\|a^\pi b^{\mathsf{D}}a\|^n = \|b^{\mathsf{D}}\|\frac{1}{1-\|a^\pi b^{\mathsf{D}}a\|}$. Hence, (i) is proved.

(ii) From Theorem 2.4 we have $a^\pi bb^{\mathsf{D}}(a+b)^{\mathsf{d}} = z$. Also, the from the proof of Theorem 2.4 we easily get $a^\pi bb^{\mathsf{D}}a^{\mathsf{D}} = 0$; hence $a^\pi bb^{\mathsf{D}}\left[(a+b)^{\mathsf{d}} - a^{\mathsf{D}}\right] = z$. The conclusion is obtained. □

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References

[1] A. Ben-Israel, T.N.E. Greville, *Generalized inverses: theory and applications*, 2nd ed., Springer-VerIag, New York, 2003.

[2] S.L. Campbell, C.D. Meyer, *Generalized inverses of linear transformations*, Dover, New York, 1991, Originally published: Pitman, London, 1979.

[3] N. Castro González, J.J. Koliha, *New additive results for the g-Drazin inverse*, Proc. Roy. Soc. Edinburgh Sect. A, 134 (2004), 1085-1097.

[4] D.S. Cvetković-Ilić, *The generalized Drazin inverse with commutativity up to a factor in Banach algebra*, Linear Algebra Appl., 431 (2009), 783-791.

[5] D.S. Cvetković-Ilić, D.S. Djordjević, Y. Wei, *Additive results for the generalzied Drazin inverse in a Banach algebra*, Linear Algebra Appl., 418 (2006), 53-61.

[6] A. Dajić, J.J. Koliha, *The weighted g-Drazin inverse for operators*, J. Austral. Math. Soc., 82 (2007), 163-181.

[7] C.Y. Deng, *The Drazin inverse of bounded operators with commutativity up to a factor*, Appl. Math. Comp., 206 (2008), 695-703.

[8] C. Deng, Y. Wei, *New additive results for the generalized Drazin inverse*, J. Math. Anal. Appl., 370 (2010), 313-321.

[9] D.S. Djordjević, Y. Wei, *Additive results for the generalized Drazin inverse*, J. Austral. Math., 73 (2002), 115-125.

[10] M.P. Drazin, *Pseudoinverse in associative rings and semigroups*, Amer. Math. Monthly, 65 (1958), 506-514.

[11] R.E. Hartwig, G. Wang, Y. Wei, *Some additive results on Drazin inverse*, Linear Algebra Appl., 322 (2001), 207-217.

[12] J.J. Koliha, *A generalized Drazin inverse*, Glasg. Math. J., 38 (1996), 367-381.

[13] X. Liu, L. Xu, Y. Yu, *The representations of the Drazin inverse of differences of two matrices*, Appl. Math. Comput., 216 (2010), 3652-3661.

[14] D. Mosić, D.S. Djordjević, *Additive results for the W g-Drazin inverse*, Linear Algebra Appl., 432 (2010), 2847-2860.

[15] P. Patrício, R.E. Hartwig, *Some additive results on Drazin Inverses,* Appl. Math. Comput., 215 (2009), 530-538.

[16] Y. Wei, G. Wang, *The perturbation theory for the Drazin inverse and its applications*, Linear Algebra Appl., 258 (1997), 179-186.

[17] D. S. Djordjević V. Rakočević, *Lectures on generalized inverse*, Faculty of Sciences and Mathematics, University of Niš, Niš, 2008.

[18] J. Ljubisavljević, D. S. Cvetković-Ilić, *Additive results for the Drazin inverse of block matrices and applications*, J. Comput. Appl. Math., 235 (2011), 3683-3690.

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