NONEXISTENCE OF GLOBAL SOLUTIONS TO A FRACTIONAL NONLINEAR ULTRA-PARABOLIC SYSTEM

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Abstract. In this work, we study the sufficient conditions for that ensure the nonexistence of global solutions to a Cauchy problem for a fractional nonlinear ultra-parabolic system. The Blowing-up solutions is also presented. Our method of proof relies on a suitable choice of a test function and the weak formulation approach of the sought for solutions.

Keywords: fractional derivatives, nonlinear ultra-parabolic equations, nonexistence, test-function, blowing-up solutions.

1. Introduction

The main objective of this work is to improve the results of Kerbal and Kirane [6] by considering fractional in time and space for the nonlinear ultra-parabolic

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system

(1.1) $D_{0|t_1}^{\alpha_1}(u-u_2)+D_{0|t_2}^{\alpha_2}(u-u_1)+(-\Delta)^{\frac{\alpha}{2}}(|u|)=k_1|u|^{p_1}|v|^{q_1},\ k_1=\text{const.}$

(1.2) $D_{0|t_1}^{\beta_1}(v-v_2)+D_{0|t_2}^{\beta_2}(v-v_1)+(-\Delta)^{\frac{\beta}{2}}(|v|)=k_2|u|^{p_2}|v|^{q_2},\ k_2=\text{const.}$

posed for $(t_1,t_2,x)\in Q=\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^N$, and supplemented with the initial conditions

(1.3) $$u(t_1,0,x)=u_1(t_1,x),\,u(0,t_2,x)=u_2(t_2,x),$$

(1.4) $$v(t_1,0,x)=v_1(t_1,x),\,v(0,t_2,x)=v_2(t_2,x).$$

Here $p_1\geq 0, p_2>1, q_1>1, q_2\geq 0,\ 0<\alpha_1,\alpha_2<1,\ 0<\beta_1,\beta_2<1,\ 1\leq\alpha,\beta\leq 2$ are constants and $D_{0|t_j}^{\alpha_j},\ D_{0|t_j}^{\beta_j},\ j=1,2$ are the fractional derivatives in the sense of Riemann-Liouville. The operator $D_{0|t}^{\alpha}$ is defined, for an absolutely continuous function $g:\mathbb{R}_+\longrightarrow\mathbb{R}$, by $\left(D_{0|t}^{\alpha}\right)g(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{g(\tau)}{(t-\tau)^{\alpha}}d\tau$, and $\Gamma(\alpha)=\int_0^{\infty}r^{\alpha-1}e^{-r}dr$ is the Euler gamma function. The fractional power of the Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ $(1\leq\alpha\leq 2)$ stands for diffusion in media with impurities and is defined as $(-\Delta)^{\frac{\alpha}{2}}v(x)=\mathcal{F}^{-1}\left(|\zeta|^{\alpha}\mathcal{F}(v)(\zeta)\right)(x)$, where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse. In addition, it satisfies the following condition $\forall v;\mathbb{R}^N\longrightarrow\mathbb{R}$ we have $(-\Delta)^{\frac{\alpha}{2}}v\in L^{\frac{p}{p-m}}\left(\mathbb{R}^N\right)$ and the operator $D_{0|t}^{\alpha}$ counts for the abnormal diffusion, a recently very much studied topic in probability, physics, chemistry, biology, image processing, etc, see for instance [7,8] and their references. Classical multi-time or ultraparabolic problems have received a special interest and attention by authors due to their application in real life problems, see for example [2,5,9,13], while the fractional analogs are in their preliminary steps.

2. Preliminaries

The right-sided Riemann-Liouville derivatives of order $0<\alpha<1$ for an absolutely continuous function $g:\mathbb{R}_+\longrightarrow\mathbb{R}$, is defined by:

$$\left(D_{t|T}^{\alpha}\right)g(t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T\frac{g(\tau)}{(\tau-t)^{\alpha}}d\tau.$$

Note that for a differentiable function g, the left-sided Caputo derivatives of order α is defined as:

$$D_{0|t}^{\alpha}(g-g(0))(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{g'(\tau)}{(\tau-t)^{\alpha}}d\tau.$$

Finally, taking into account the following integration by parts formula:

$$\int_0^T f(t)\,D_{0|t}^{\alpha}g(t)\,dt=\int_0^T D_{t|T}^{\alpha}f(t)\,g(t)\,dt,$$

where $f,\ g \in C^{\infty}(J,\mathbb{R}),\ J \subset \mathbb{R}$.

We also need some preparatory lemmas based on the function

$$\phi(t) = \begin{cases} \left(1 - \dfrac{t}{T}\right)^{\lambda}, & 0 \le t \le T \\ 0, & t > T \end{cases} \tag{2.1}$$

where $\lambda \ge 2$.

We define the regular function $0 \le \psi \le 1$:

$$\psi(\xi) = \begin{cases} 1, & \text{if } 0 \le \xi \le 1, \\ \text{decreasing}, & \text{if } 1 \le \xi \le 2, \\ 0, & \text{if } \quad \xi \ge 2, \end{cases} \tag{2.2}$$

which will be used hereafter.

3. Results

We consider the system with a two-dimensional fractional time (1.1)-(1.2) and let us set

$$I_0 = \int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi dP + \int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi dP,$$

$$J_0 = \int_Q v_2 D^{\beta_1}_{t_1|T}\varphi dP + \int_Q v_1 D^{\beta_2}_{t_2|T}\varphi dP.$$

Definition 1. *Let $Q_T = (0,T) \times (0,T) \times \mathbb{R}^N,\ 0 < T < +\infty$, and we say that $(u,v) \in \left(L^1_{loc}(Q_T)\right)^2$ is a local weak solution to problem* (1) *on Q_T, if $u^{p_i}v^{q_i} \in L^1_{loc}(Q_T),\ i = 1,2$, and it is such that*

$$\begin{aligned} &\int_Q k_1 |u|^{p_1}|v|^{q_1}\varphi dP + I_0 \\ &= \int_Q u D^{\alpha_1}_{t_1|T}\varphi dP + \int_Q u D^{\alpha_2}_{t_2|T}\varphi dP + \int_Q |u| \left(-\Delta\right)^{\frac{\alpha}{2}} \varphi dP, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\int_Q k_2 |u|^{p_2}|v|^{q_2}\varphi dP + J_0 \\ &\quad = \int_Q v D^{\beta_1}_{t_1|T}\varphi dP + \int_Q v D^{\beta_2}_{t_2|T}\varphi dP + \int_Q |v| \left(-\Delta\right)^{\frac{\beta}{2}} \varphi dP, \end{aligned} \tag{3.2}$$

for any test function $\varphi \in C^{\infty}$, such that $\varphi(T,t_2,x) = \varphi(t_1,T,x) = 0$, and $P = (t_1,t_2,x)$.

Now, set

$$\sigma_1 = -\frac{q_1\beta_1 - (N+2)(p_2q_1 - 1) + p_2q_1\alpha_1}{p_2q_1 - 1},$$

$$\sigma_2 = -\frac{q_1\beta_2 - (N+2)(p_2q_1-1) + p_2q_1\alpha_1}{p_2q_1-1},$$

$$\sigma_3 = -\frac{q_1\beta - (N+2)(p_2q_1-1) + p_2q_1\alpha_1}{p_2q_1-1},$$

$$\sigma_4 = -\frac{q_1\beta_1 - (N+2)(p_2q_1-1) + p_2q_1\alpha_2}{p_2q_1-1},$$

$$\sigma_5 = -\frac{q_1\beta_2 - (N+2)(p_2q_1-1) + p_2q_1\alpha_2}{p_2q_1-1},$$

$$\sigma_6 = -\frac{q_1\beta - (N+2)(p_2q_1-1) + p_2q_1\alpha_2}{p_2q_1-1},$$

$$\sigma_7 = -\frac{q_1\beta_1 - (N+2)(p_2q_1-1) + p_2q_1\alpha}{p_2q_1-1},$$

$$\sigma_8 = -\frac{q_1\beta_2 - (N+2)(p_2q_1-1) + p_2q_1\alpha}{p_2q_1-1},$$

$$\sigma_9 = -\frac{q_1\beta - (N+2)(p_2q_1-1) + p_2q_1\alpha}{p_2q_1-1}.$$

Theorem 1. *Let* $p_1 \geq 0$, $q_2 \geq 0$, $p_2 > 1$, $q_1 > 1$. *Let* $u_0, v_0 \in L^\infty(\mathbb{R}^N)$, *suth that* $u_0 \geq 0$, $v_0 \geq 0$, *and assume that,*

$$\int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi^\mu dP > 0, \quad \int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi^\mu dP > 0,$$
$$\int_Q v_2 D^{\beta_1}_{t_1|T}\varphi^\mu dP > 0, \quad \int_Q v_1 D^{\beta_2}_{t_2|T}\varphi^\mu dP > 0,$$

then solutions to system (1.1)-(1.2) *blow-up whenever*

$$\max\{\sigma_1, \cdots, \sigma_9, \delta_1, \cdots, \delta_9\} \leq 0.$$

Proof. Assume that the solution is nontrivial and global. Next, replacing φ by φ^μ in (3.1) and then using Holder's inequality to estimate the I_u and I_v (As we shall see later), we obtain the following estimates:

$$\int_Q u|D^{\alpha_1}_{t_1|T}\varphi^\mu| \leq \left(\int_Q k_2|u|^{p_2}|v|^{q_2}\varphi^\mu\right)^{\frac{1}{p_2}}$$
$$\cdot\left(\int_Q k_2^{-\frac{1}{p_2-1}}|D^{\alpha_1}_{t_1|T}\varphi^\mu|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}}, \tag{3.3}$$

$$\int_Q u|D^{\alpha_2}_{t_2|T}\varphi^\mu| \leq \left(\int_Q k_2|u|^{p_2}|v|^{q_2}\varphi^\mu\right)^{\frac{1}{p_2}}$$
$$\cdot\left(\int_Q k_2^{-\frac{1}{p_2-1}}|D^{\alpha_2}_{t_2|T}\varphi^\mu|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}}, \tag{3.4}$$

$$\int_Q u|(-\Delta)^{\frac{\alpha}{2}}\varphi^\mu| \le \left(\int_Q k_2|u|^{p_2}|v|^{q_2}\varphi^\mu\right)^{\frac{1}{p_2}}$$

$$\cdot\left(\int_Q k_2^{-\frac{1}{p_2-1}}|(-\Delta)^{\frac{\alpha}{2}}\varphi^\mu|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}}. \tag{3.5}$$

Similarly, we have

$$\int_Q v|D^{\beta_1}_{t_1|T}\varphi^\mu| \le \left(\int_Q k_1|u|^{p_1}|v|^{q_1}\varphi^\mu\right)^{\frac{1}{q_1}}$$

$$\cdot\left(\int_Q k_1^{-\frac{1}{q_1-1}}|D^{\beta_1}_{t_1|T}\varphi^\mu|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}} \tag{3.6}$$

$$\int_Q v|D^{\beta_2}_{t_2|T}\varphi^\mu| \le \left(\int_Q k_1|u|^{p_1}|v|^{q_1}\varphi^\mu\right)^{\frac{1}{q_1}}$$

$$\cdot\left(\int_Q k_1^{-\frac{1}{q_1-1}}|D^{\beta_2}_{t_2|T}\varphi^\mu|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}}, \tag{3.7}$$

$$\int_Q v|(-\Delta)^{\frac{\beta}{2}}\varphi^\mu| \le \left(\int_Q k_1|u|^{p_1}|v|^{q_1}\varphi^\mu\right)^{\frac{1}{q_1}}$$

$$\cdot\left(\int_Q k_1^{-\frac{1}{q_1-1}}|(-\Delta)^{\frac{\beta}{2}}\varphi^\mu|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}} \tag{3.8}$$

If we set

$$I_u = \left(\int_Q k_2|u|^{p_2}|v|^{q_2}\varphi^\mu\right)^{\frac{1}{p_2}},\ I_v = \left(\int_Q k_1|u|^{p_1}|v|^{q_1}\varphi^\mu\right)^{\frac{1}{q_1}},$$

$$A(p_2) = \left(\int_Q k_2^{-\frac{1}{p_2-1}}|D^{\alpha_1}_{t_1|T}\varphi^\mu|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}},$$

$$A(q_1) = \left(\int_Q k_1^{-\frac{1}{q_1-1}}|D^{\beta_1}_{t_1|T}\varphi^\mu|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}},$$

$$B(p_2) = \left(\int_Q k_2^{-\frac{1}{p_2-1}}|D^{\alpha_2}_{t_2|T}\varphi^\mu|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}},$$

$$B(q_1) = \left(\int_Q k_1^{-\frac{1}{q_1-1}}|D^{\beta_2}_{t_2|T}\varphi^\mu|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}},$$

$$C(p_2) = \left(\mu\int_Q k_2^{-\frac{1}{p_2-1}}|(-\Delta)^{\frac{\alpha}{2}}\varphi|^{\frac{p_2}{p_2-1}}|v|^{-\frac{q_2}{p_2-1}}\varphi^{-\frac{\mu}{p_2-1}}\right)^{\frac{p_2-1}{p_2}},$$

$$C(q_1) = \left(\mu\int_Q k_1^{-\frac{1}{q_1-1}}|(-\Delta)^{\frac{\beta}{2}}\varphi|^{\frac{q_1}{q_1-1}}|u|^{-\frac{p_1}{q_1-1}}\varphi^{-\frac{\mu}{q_1-1}}\right)^{\frac{q_1-1}{q_1}},$$

$$I_0^\mu = \int_Q u_2 D_{t_1|T}^{\alpha_1}\varphi^\mu + \int_Q u_1 D_{t_2|T}^{\alpha_2}\varphi^\mu,$$
$$J_0^\mu = \int_Q v_2 D_{t_1|T}^{\beta_1}\varphi^\mu + \int_Q v_1 D_{t_2|T}^{\beta_2}\varphi^\mu,$$

then, using estimates (3.2), (3.7), we can write (3.1) as

$$I_v + I_0^\mu \leq I_u^{\frac{1}{p_2}}\left(A(p_2)+B(p_2)+C(p_2)\right),$$
$$I_u + J_0^\mu \leq I_v^{\frac{1}{q_1}}\left(A(q_1)+B(q_1)+C(q_1)\right).$$

Since I_0^μ , $J_0^\mu > 0$ we have

$$I_v \leq I_u^{\frac{1}{p_2}}\left(A(p_2)+B(p_2)+C(p_2)\right), \tag{3.9}$$
$$I_u \leq I_v^{\frac{1}{q_1}}\left(A(q_1)+B(q_1)+C(q_1)\right). \tag{3.10}$$

Now, from (3.8) and (3.9), we have

$$I_v + I_0^\mu \leq I_v^{\frac{1}{p_2 q_1}}\left(A^{\frac{1}{p_2}}(q_1)+B^{\frac{1}{p_2}}(q_1)+C^{\frac{1}{p_2}}(q_1)\right)\left(A(p_2)+B(p_2)+C(p_2)\right).$$

Then Young's inequality implies

$$\begin{aligned} I_v + I_0^\mu \leq K \Bigg[&\left(A^{\frac{1}{p_2}}(q_1)A(p_2)\right)^{\frac{p_2q_1}{p_2q_1-1}} + \left(B^{\frac{1}{p_2}}(q_1)A(p_2)\right)^{\frac{p_2q_1}{p_2q_1-1}} + \left(C^{\frac{1}{p_2}}(q_1)A(p_2)\right)^{\frac{p_2q_1}{p_2q_1-1}} \\ &+ \left(A^{\frac{1}{p_2}}(q_1)B(p_2)\right)^{\frac{p_2q_1}{p_2q_1-1}} + \left(B^{\frac{1}{p_2}}(q_1)B(p_2)\right)^{\frac{p_2q_1}{p_2q_1-1}} + \left(C^{\frac{1}{p_2}}(q_1)B(p_2)\right)^{\frac{p_2q_1}{p_2q_1-1}} \\ &+ \left(A^{\frac{1}{p_2}}(q_1)C(p_2)\right)^{\frac{p_2q_1}{p_2q_1-1}} + \left(B^{\frac{1}{p_2}}(q_1)C(p_2)\right)^{\frac{p_2q_1}{p_2q_1-1}} + \left(C^{\frac{1}{p_2}}(q_1)C(p_2)\right)^{\frac{p_2q_1}{p_2q_1-1}} \Bigg], \end{aligned}$$

for some positive constant K. We choose the test function $\varphi(t_1,t_2,x)$ in the form

$$\varphi(t_1,t_2,x) = \varphi_1(t_1)\,\varphi_2(t_2)\,\varphi_3(x),$$

where $\varphi_1(t_1) = (1-t_1/T)_+^\lambda$, [1], $\varphi_2(t_2) = (1-t_2/T)_+^\lambda$ and $\varphi_3(x) = \psi\left(|x|^2/T^2\right)$, and let us now pass to the new variables

$$\tau_1 = T^{-1}t_1,\ \tau_2 = T^{-1}t_2,\ \ y = T^{-1}x;$$

we have,

$$A(p_2) = CT^{-\alpha_1+(N+2)\left(1-\frac{1}{p_2}\right)},$$
$$A(q_1) = CT^{-\beta_1+(N+2)\left(1-\frac{1}{q_1}\right)},$$

1. $(1-t_1/T)_+^\lambda = \sup\left\{0,(1-t_1/T)^\lambda\right\}$

$$B(p_2) = CT^{-\alpha_2+(N+2)\left(1-\frac{1}{p_2}\right)},$$

$$B(q_1) = CT^{-\beta_2+(N+2)\left(1-\frac{1}{q_1}\right)},$$

$$C(p_2) = CT^{-\alpha+(N+2)\left(1-\frac{1}{p_2}\right)},$$

$$C(q_1) = CT^{-\beta+(N+2)\left(1-\frac{1}{q_1}\right)}.$$

For some positive constant C. Hence, we obtain

$$I_v + I_0^\mu \leq K\left[T^{\sigma_1} + T^{\sigma_2} + \cdots + T^{\sigma_9}\right]. \tag{3.11}$$

Similarly, we obtain for I_u the estimate

$$I_u + J_0^\mu \leq K\left[T^{\delta_1} + T^{\delta_2} + \cdots + T^{\delta_9}\right]. \tag{3.12}$$

Finally, passing to the limit as $T \to +\infty$, we observe that:

Either $\max\{\sigma_1, \cdots, \sigma_9, \delta_1, \cdots, \delta_9\} < 0$ and in this case, the right hand side tends to zero while the left hand side is strictly positive. Hence, we obtain a contradiction. Or $\max\{\sigma_1, \cdots, \sigma_9, \delta_1, \cdots, \delta_9\} = 0$ and in this case, following the similar analysis used in [6], we prove a contradiction. □

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