ON COMPUTING DIFFERENTIAL TRANSFORM OF NONLINEAR NON-AUTONOMOUS FUNCTIONS AND ITS APPLICATIONS

Essam R. El-Zahar*
Department of Mathematics
Faculty of Sciences and Humanities
Prince Sattam Bin Abdulaziz University
Alkharj, 11942
KSA
and
Department of Basic Engineering Science
Faculty of Engineering
Shebin El-Kom, 32511
Menofia University
Egypt
essam_zahar2006@yahoo.com

Abdelhalim Ebaid
Department of Mathematics
Faculty of Science
Tabuk University
P.O. Box 741, Tabuk 71491
KSA

Abstract. Although being powerful, the differential transform method yet suffers from a drawback which is how to compute the differential transform of nonlinear non-autonomous functions that can limit its applicability. In order to overcome this defect, we introduce in this paper, new general formulas and their related recurrence relations for computing the differential transform of any analytic nonlinear non-autonomous function with one or multi-variable. Several test examples for different types of nonlinear differential and integro-differential equations are solved to demonstrate the applicability and validity of the present method. The obtained results declare that the suggested approach not only effective but also a straightforward and powerful for solving differential and integro-differential equations with complex nonlinearities.

Keywords: differential transform method, nonlinear non-autonomous functions, nonlinear differential and integro-differential equations.

1. Introduction

The Differential Transform Method (DTM) which is based on Taylor series expansion was first introduced by Zhou [1] and has been successfully applied to a wide class of nonlinear problems arising in mathematical sciences and engineer-

*. Corresponding author

ing. The main advantage of DTM is that it can be applied directly to nonlinear differential equations with no need for linearization, discretization, or perturbation. Additionally, DTM does not generate secular terms (noise terms) and does not need to analytical integrations as other semi-analytical numerical methods such as HPM, HAM, ADM or VIM [2-7] and so DTM is an attractive tool for solving differential equations. Although this method has been proved to be an efficient tool for handling nonlinear differential and integro-differential equations, the nonlinear functions used in these equations are restricted to certain kinds of nonlinearities, e.g., polynomials and products with derivatives. For other types of nonlinear functions, Chang and Chang [8] construct a new algorithm based on obtaining a differential equation satisfied by this nonlinear function and then applying DTM to this obtained differential equation. Although their treatment was found effective for some forms of nonlinearity [9,10] it significantly increases the computational budget, especially if there are two or more nonlinear functions involved in the differential equation being investigated [11]. Moreover, in the case of complex nonlinearities, it may be quite difficult to obtain the differential equations satisfied by these nonlinear functions. To overcome this difficulty, a new formula has been derived [11-14] to calculate the differential transform of nonlinear autonomous one variable functions $f(y)$. Unfortunately, for nonlinear non-autonomous multi-variable functions $f\left(t, y_j(t)\right)$, $j = 1, 2, .., m$, no related formula has been given to calculate their transform functions. In order to overcome this defect, new general formulas and their related recurrence relations are deduced in this paper for computing the differential transform of any analytic nonlinear non-autonomous function with one or multi-variable. The proposed method deals directly with the nonlinear non-autonomous function in its form without any special kinds of transformations or algebraic manipulations. Also, there is no need to compute the differential transform of other functions to obtain the required one. For autonomous function, as a special case of the current study, these formulas and recurrence relations have the same mathematical structure as the Adomian polynomials but with constants instead of variable components. The applicability and validity of the present method are demonstrated through solving several test examples including nonlinear differential and integro-differential equations of different types.

2. New differential transform formulas

The basic definitions , fundamental theorems , convergence , error analysis of DTM and its applicability for various kinds of differential and integro-differential equations are given in [1, 15-31]. If a differential equation contains an analytic nonlinear non-autonomous function $f\left(t, y(t)\right)$ then the differential transform $F(n)$ of the function $f\left(t, y(t)\right)$ can be computed from the following theorems, where we assume that $f\left(t, y(t)\right) \neq f_0\left(t\right) f_1\left(y(t)\right)$.

Theorem 1. *The differential transform $F(n)$of any analytic nonlinear non-autonomous function $f(t, y(t))$ at a point t_0 can be computed from the formula*

$$F(n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(t_0 + \lambda, \sum_{i=0}^{n} Y(i)\lambda^i\right)\right]_{\lambda=0} \tag{1}$$

where $Y(i)$ is the differential transform of $y(t)$.

Proof. The differential transform$F(n)$ of $f(t, y(t))$ at a point t_0 is defined as

$$F(n) = \frac{1}{n!}\left[\frac{d^n}{dt^n} f(t, y(t))\right]_{t=t_0}. \tag{2}$$

And since $y(t)$ can be expressed as

$$y(t) = \sum_{i=0}^{\infty} Y(i)(t - t_0)^i, \tag{3}$$

where $Y(i)$ is the differential transform of $y(t)$ about t_0 , then we have

$$F(n) = \frac{1}{n!}\left[\frac{d^n}{dt^n} f\left(t, \sum_{i=0}^{\infty} Y(i)(t - t_0)^i\right)\right]_{t=t_0}. \tag{4}$$

Now, let $t - t_0 = \lambda$, then Eq. 4 becomes

$$F(n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(t_0 + \lambda, \sum_{i=0}^{\infty} Y(i)\lambda^i\right)\right]_{\lambda=0}.$$

And since $F(n)$ is a function of t_0 and $\{Y(i)\}_{i=0}^{n}$ only, then

$$F(n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(t_0 + \lambda, \sum_{i=0}^{n} Y(i)\lambda^i\right)\right]_{\lambda=0}.$$

By this way, the proof of Theorem 1 is completed.

Theorem 2. *The differential transform $F(n)$of any analytic nonlinear non-autonomous function $f(t, y(t))$ at a point t_0 , satisfies the analytic recurrence relation*

$$F(n) = \frac{1}{n}\left(\frac{\partial}{\partial t_0}F(n-1) + \sum_{i=0}^{n-1}(i+1)Y(i+1)\frac{\partial}{\partial Y(i)}F(n-1)\right), \ n \geq 1 \tag{5}$$

where $F(0) = f(t_0, \ Y(0))$.

Proof. Since we have

$$f^{(n)}(t, y(t)) = \frac{\partial}{\partial t} f^{(n-1)}(t, y(t)) + \frac{\partial}{\partial y(t)} f^{(n-1)}(t, y(t))\frac{dy(t)}{dt},$$

then

$$F(n) = \frac{1}{n!}\left[\frac{\partial}{\partial t} f^{(n-1)}\left(t, \sum_{i=0}^{\infty} Y(i)(t-t_0)^i\right)\right.$$
$$\left. + \sum_{i=0}^{\infty} \frac{\partial}{\partial Y(i)} f^{(n-1)}\left(t, \sum_{i=0}^{\infty} Y(i)(t-t_0)^i\right)(i+1)Y(i+1)\right]_{t=t_0}$$
$$= \frac{1}{n!}\left[(n-1)!\frac{\partial}{\partial t_0}F(n-1) + (n-1)!\sum_{i=0}^{\infty}(i+1)\,Y(i+1)\frac{\partial}{\partial Y(i)}F(n-1)\right]$$

and since $F(n-1)$is a function of $t_0$and$\{Y(i)\}_{i=0}^{n-1}$, then

$$F(n) = \frac{1}{n}\left(\frac{\partial}{\partial t_0}F(n-1) + \sum_{i=0}^{n-1}(i+1)Y(i+1)\frac{\partial}{\partial Y(i)}F(n-1)\right),\ n \geq 1.$$

By this way, the proof of Theorem 2 is completed.

Thus by Theorems 1 and 2 we have implemented a new algorithm for computing the one-dimensional differential transform of any analytic nonlinear non-autonomous function $f(t, y(t))$.

As a special case of the present study, i.e., for autonomous functions, the formula (1) and recurrence relation (5) are reduced to, respectively

$$F(n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{i=0}^{n} Y(i)\,\lambda^i\right)\right]_{\lambda=0}, \tag{6}$$

$$F(n) = \frac{1}{n}\left(\sum_{i=0}^{n-1}(i+1)\,Y(i+1)\frac{\partial}{\partial Y(i)}F(n-1)\right),\ n \geq 1, \tag{7}$$

where the special formula (6) is the present formula in [11-14]. In fact if a system of one-dimensional differential equations contains m coupled analytic nonlinearity function $f(t, y_j(t))$, $j = 1, 2, .., m$, where we assume that $f(t, y_j(t)) \neq f_0(t)\, f_1(y_1(t))\, f_2(y_2(t)) \ldots f_m(y_m(t))$, then Theorems 1 and 2 can be easily extended to multi-variable function and satisfy the following algorithms.

Corollary 1. *The differential transform $F(n)$of any analytic nonlinear non-autonomous function $f(t, y_j(t))$, $j = 1, 2, .., m$ at a point t_0 can be defined by*

$$F(n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(t_0 + \lambda, \sum_{i=0}^{n} Y_j(i)\,\lambda^i\right)\right]_{\lambda=0}, j = 1, 2, .., m, \tag{8}$$

where $Y_j(i)$ is the differential transform of $y_j(t)$.

Corollary 2. *The differential transform$F(n)$of any analytic nonlinear non-autonomous function $f\left(t, y_j(t)\right)$, $j = 1, 2, .., m$ at a point t_0 , satisfies the analytic recurrence relation*

$$F(n)=\frac{1}{n}\left(\frac{\partial}{\partial t_0}F(n-1)+\sum_{j=1}^{m}\sum_{i=0}^{n-1}(i+1)Y_j(i+1)\frac{\partial}{\partial Y_j(i)}F(n-1)\right), n\geq 1, \tag{9}$$

where $F(0) = f\left(t_0,\ Y_j(0)\right)$.

Moreover and as a special case of the present study, i.e., for autonomous function, the definition (8) and recurrence relation (9) are reduced, respectively, to

$$F(n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{i=0}^{n} Y_j(i)\,\lambda^i\right)\right]_{\lambda=0}, j = 1, 2, .., m, \tag{10}$$

$$F(n) = \frac{1}{n}\left(\sum_{j=1}^{m}\sum_{i=0}^{n-1}(i+1)Y_j(i+1)\frac{\partial}{\partial Y_j(i)}F(n-1)\right), \tag{11}$$

which have the same mathematical structure as the Adomian polynomials in [32] but with constants instead of variable components.

Table 1. Some fundamental operations of DTM.

Original function	Transformed function
$u(t) = \beta\,(\,v(t) \pm w(t)\,)$	$U(k) = \beta\,V(k) \pm \beta\,W(k)$
$u(t) = v(t)\ w(t)$	$U(k) = \sum_{\ell=0}^{k} V(\ell)\,W(k-\ell)$
$u(t) = \frac{d^m v(t)}{dt^m}$	$U(k) = \frac{(k+m)!}{k!}V(k+m)$
$u(t) = (\beta + t)^m$	$U(k) = H[m,k]\frac{m!}{k!\|(m-k)\|!}(\beta + t_0)^{m-k}$, $H[m,k] = \begin{cases} 1, & \text{if}\, m \geq k \\ 0, & \text{if}\, m < k \end{cases}$
$u(t) = e^{\lambda t}$	$U(k) = \frac{\lambda^k}{k!}e^{\lambda\, t_0}$
$u(t) = \sin(\omega t + \beta)$	$U(k) = \frac{\omega^k}{k!}\sin(\omega t_0 + \beta + \frac{k\pi}{2})$
$u(t) = \cos(\omega t + \beta)$	$U(k) = \frac{\omega^k}{k!}\cos(\omega t_0 + \beta + \frac{k\pi}{2})$
$u(t) = \int_{t_0}^{t} v(t)dt$	$U(k) = \frac{V(k-1)}{k}, k \geq 1, U(0) = 0$
$u(t) = \int_{t_0}^{t} v(t)w(t)$	$U(k) = \frac{1}{k}\sum_{l}^{k-1} V(l)W(k-l-1), k \geq 1, k \geq 1, U(0) = 0.$

3. Applications

In this section, we have solved different types of differential and integro-differential problems with different forms of nonlinear non-autonomous functions that the formula in the literature [11-14] was found not applicable.

Example 1. Consider the nonlinear initial-value problem

$$y'(t) - y(t) = \ln\left(t + y(t)\right), t \in [1, 2], y(t) \geq 0, y(1) = 0. \tag{12}$$

Using the basic properties of DTM, Table.1, and taking the transform of equations in (12) result in

$$(k + 1)Y(k + 1) - Y(k) = F(k), Y(0) = 0, k = 0, 1, 2, ..., \tag{13}$$

where$F(k)$is the differential transform of the nonlinear term $\ln(t+y(t))$. $F(k)$is computed using the present method and given by

$$\begin{aligned}
&F(0) = \ln(1 + Y(0)),\ \ F(1) = \frac{1 + Y(1)}{1 + Y(0)},\\
&F(2) = \frac{Y(2)}{1 + Y(0)} - \frac{1}{2}\frac{(1 + Y(1))^2}{(1 + Y(0))^2},\\
&F(3) = \frac{Y(3)}{1 + Y(0)} - \frac{Y(2)\,(1 + Y(1))}{(1 + Y(0))^2} + \frac{(1 + Y(1))^3}{3\,(1 + Y(0))^3}\\
&F(4) = \frac{Y(4)}{1 + Y(0)} - \frac{Y(3)\,(1 + Y(1))}{(1 + Y(0))^2} + \frac{Y(2)\,(1 + Y(1))^2}{(1 + Y(0))^3}\\
&\qquad - \frac{Y(2)^2}{2\,(1 + Y(0))^2} - \frac{(1 + Y(1))^4}{4\,(1 + Y(0))^4}.
\end{aligned} \tag{14}$$

Therefore, a combination of the recurrence relation(13) and the computed $F(k)$ in (14) with the solution formula (13) results in the series solution $\tilde{y}(t) = \frac{1}{2}(t-1)^2 + \frac{1}{6}(t-1)^3 + \frac{1}{24}(t-1)^4 + \frac{1}{120}(t-1)^5 + ...$ For sufficiently large number of terms, the closed form of the obtained series solution is $\tilde{y}(t) = \mathrm{e}^{t-1} - t$, which is the exact solution.

Table 2: Numerical results for Example 2.

	$\mid y(t_i) - \tilde{y}(t_i) \mid$		
t_i	$N = 5$	$N = 10$	$N = 15$
0.0	0.0000+e00	0.0000+e00	0.0000+e00
0.2	7.3689e-006	8.2100e-010	9.6316e-011
0.4	4.5510e-004	1.3793e-006	2.3291e-008
0.6	4.8955e-003	1.1150e-004	3.0230e-006
0.8	2.5675e-002	2.4110e-003	2.6524e-004
1.0	9.0957e-002	2.5516e-002	8.3823e-003

Example 2. Consider the nonlinear initial-value problem

$$y'(t) + \varepsilon y(t)^2 = \varepsilon \sin(\,ty(t)\,), t \in [0, 1]\, y(0) = 1. \tag{15}$$

Taking the differential transform of equations in (15) results in

$$(k+1)Y(k+1)+\varepsilon\sum_{\ell=0}^{k}Y(\ell)\,Y(k-\ell)=\varepsilon F(k),Y(0)=1,k=0,1,2,..., \tag{16}$$

where $F(k)$ is the differential transform of the nonlinear term $\sin(ty(t))$.$F(k)$ is computed using the present method and given by

$$\begin{aligned} &F(0)=0,\;\; F(1)=Y(0),\;\; F(2)=Y(1),\;\; F(3)=\frac{-Y(0)^3}{6}+Y(2),\\ &F(4)=y(3)-\frac{1}{2}Y(0)^2y(1),\\ &F(5)=\frac{1}{120}Y(0)^5-\frac{1}{2}Y(0)Y(1)^2-\frac{1}{2}Y(0)^2Y(2)+Y(4). \end{aligned} \tag{17}$$

Using (16) and (17), the series solution is obtained and given at $\varepsilon=0.1$ by

$$\tilde{y}(t)=1-\frac{1}{10}t+\frac{3}{50}t^2-\frac{23}{3000}t^3-\frac{119}{60000}t^4+\frac{247}{300000}t^5-\frac{2233}{4500000}t^6+....$$

The presented results are compared with those obtained using MATLAB built-in solver ode45 in Table 2. The ode45 solver integrates ODEs using explicit 4th & 5th Runge-Kutta (4, 5) formula [33]. In order to guarantee a good numerical reference, ode45 is configured using an absolute error of $10^{-12}$and relative error of 10^{-8}.

Example 3. Consider the nonlinear first order Volterra integro-differential equation

$$\begin{aligned} &y'(t)=\cos t-\frac{t^2}{2}+1+\int_0^t \sin^{-1}(1-\tau+y(\tau))\,d\tau,\\ &t\in[0,1]\,,y\,(0)=-1,y'\,(0)=2. \end{aligned} \tag{18}$$

The differential transform of equations in (18) are

$$\begin{aligned} &(k+1)Y(k+1)=\frac{1}{k!}\cos\left(\frac{k\pi}{2}\right)-\frac{\delta(k-2)}{2}+\delta(k)+\frac{F(k-1)}{k},\\ &Y(0)=-1\,,Y(1)=2\,,\,k\geq 1, \end{aligned}$$

where $F(k)$ is the differential transform of the nonlinear term $\sin^{-1}(1-\tau+y(\tau))$. By applying the present method to the nonlinear function $f(\tau,y(\tau))=\sin^{-1}(1-\tau+y(\tau))$ at $\tau_0=0$, $Y(0)=-1$ and $Y(1)=2$, and by using the principal values of the square root and inverse trigonometric functions, we get

$$
\begin{aligned}
&F(0)=0,\ F(1)=1,\ F(2)=Y(2),\ F(3)=Y(3)+\frac{1}{6},\\
&F(4)=Y(4)+\frac{1}{2}Y(2),\\
&F(5)=Y(5)+\frac{3}{40}+\frac{1}{2}Y(3)+\frac{1}{2}Y(2)^2,\\
&F(6)=\frac{3}{8}Y(2)+\frac{1}{2}Y(4)+Y(2)Y(3)+Y(6)+\frac{1}{6}Y(2)^3\\
&F(7)=\frac{1}{112}\left(56Y(3)+84\right)Y(2)^2+Y(2)Y(4)+\frac{3}{8}Y(3)\\
&\quad+\frac{1}{2}Y(5)+Y(7)+\frac{1}{2}Y(3)^2+\frac{5}{112}
\end{aligned}
\tag{19}
$$

Hence, the series solution is obtained and given by

$$\tilde{y}(t)=-1+2t-\frac{1}{6}t^3+\frac{1}{120}t^5-\frac{1}{5040}t^7+\frac{1}{362880}t^9+...$$

For sufficiently large number of terms, the closed form of the obtained series solution is $\tilde{y}(t)=\sin(t)+t-1$, which is the exact solution.

Example 4. Consider the nonlinear second order Volterra integro-differential equation

$$y''(t)-2y(t)y'(t)=-t+\int_0^t\frac{\sec^2\tau}{1+y(\tau)^2}\,d\tau, t\in[0,1]\,, y(0)=0, y'(0)=1. \tag{20}$$

The differential transform of equations in (20) are $(k+2)(k+1)Y(k+2)=2\sum_{\ell=0}^{k}(\ell+1)\,Y(\ell+1)\,Y(k-\ell)-\delta(k-1)+\frac{F(k-1)}{k}$, $Y(0)=0$, $Y(1)=1$, $Y(2)=0$, $k\geq 1$,where $F(k)$ is the differential transform of the nonlinear term $\frac{\sec^2\tau}{1+y(\tau)^2}$ obtained using the present method at $\tau_0=0$ and given by

$$
\begin{aligned}
&F(0)=1,\ F(1)=0,\ F(2)=0,\ F(3)=0,\\
&F(4)=\frac{2}{3}-2Y(3),\ Y(5)=-2Y(4),\\
&F(6)=\frac{-13}{45}-2Y(5)+2Y(3)-Y(3)^2,\\
&F(6)=(2-2Y(3))Y(4)-2Y(6),\\
&F(7)=\frac{17}{35}+5Y(3)^2-\frac{10}{3}Y(3)-2Y(3)Y(5)+2Y(5)-2Y(7)-Y(4)^2
\end{aligned}
\tag{21}
$$

Hence, the approximate series solution is obtained and given by

$$\tilde{y}(t)=t+\frac{1}{3}t^3+\frac{2}{15}t^5+\frac{17}{315}t^7+\frac{62}{2835}t^9+...$$

For sufficiently large number of terms, the closed form of the obtained series solution is $\tilde{y}(t) = \tan t$, which is the exact solution.

Example 5. Consider the nonlinear Volterra integro-differential equation with proportional delay

$$y'\left(\frac{t}{2}\right) = \frac{1}{2} - t\sin t + \int_0^t \frac{y\,(3\tau)^2 \sin t}{(3\tau+1)^2}\,d\tau, y(0) = 1, y'(0) = 1. \tag{22}$$

The differential transform of equations in (22) are

$$(k+1)Y(k+1)\left(\frac{1}{2}\right)^{k+1} = \frac{1}{2}\delta(k) - \sum_{\ell=0}^{k}\frac{1}{\ell!}\sin\left(\frac{\ell\pi}{2}\right)\delta(k-\ell-1)$$
$$+\sum_{\ell=0}^{k-1}\frac{1}{\ell!}\sin\left(\frac{\ell\pi}{2}\right)\frac{F(k-\ell-1)}{k-\ell}\ ,\ Y(0) = 1\ ,\ Y(1) = 1\ ,\ k \geq 1,$$

where $F(k)$ is the differential transform of the nonlinear term $\frac{w(\tau)^2}{(3\tau+1)^2}$, $w\,(\tau) = \sum_{k=0}^{\infty} 3^k Y(k)\tau^k$, obtained using the present method and given by

$$\begin{aligned} &F(0) = 1,\ F(1) = 0,\ F(2) = 18Y(2),\ F(3) = -54Y(2) + 54Y(3), \\ &F(4) = 81Y(2)^2 + 162Y(2) - 162Y(3) + 162Y(4), \\ &F(5) = -486Y(2)^2 + (486Y(3) - 486)Y(2) - 486Y(4) + 486Y(5) + 486Y(3). \end{aligned} \tag{23}$$

Hence, the series solution is obtained and given by

$$\tilde{y}(t) = t + 1$$

which is the exact solution.

Example 6. Consider the following nonlinear non-autonomous initial-value ODE system

$$\begin{aligned} &y_1'(t) = -y_1(t) + t + \ln\left(y_1(t) - \frac{1}{t+y_2(t)}\right) \\ &y_2'(t) = -y_2(t) - 1 + \frac{4}{y_1(t)} - \ln\left(t + y_2(t)\right) \\ &y_1(0) = 2\ ,\quad y_2(0) = 1. \end{aligned} \tag{24}$$

Applying the differential transform to (24), results in

$$\begin{aligned} &(k+1)\,Y_1(k+1) = \delta(K-1) - Y_1(k) + F_1(k) \\ &(k+1)\,Y_2(k+1) = -Y_2(k) - \delta(K) + F_2(k) \\ &Y_1(0) = 2\ ,\quad Y_2(0) = 1. \end{aligned} \tag{25}$$

where $F_1(k)$ and $F_2(k)$ are the differential transform of the nonlinear functions $f_1 = \ln\left(y_1(t) - \frac{1}{t+y_2(t)}\right)$ and $f_2 = \frac{4}{y_1(t)} - \ln(t + y_2(t))$, respectively. $F_1(k)$ and $F_2(k)$ are computed using the present method and given by

$$
\begin{aligned}
&F_1(0) = 0,\ \ F_1(1) = Y_1(1) + 1 + Y_2(1),\\
&F_1(2) = Y_1(2) - (1 + Y_2(1))^2 + Y_2(2) - \frac{1}{2}(Y_1(1) + 1 + Y_2(1))^2\\
&F_1(3) = Y_1(3) + (1 + Y_2(1))^3 - 2(1 + Y_2(1))Y_2(2) + Y_2(3)\\
&\quad - (Y_1(2) - 1 - 2Y_2(1) - Y_2(1)^2 + Y_2(2))(Y_1(1)\\
&\quad + 1 + Y_2(1)) + \frac{1}{3}(Y_1(1) + 1 + Y_2(1))^3,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&F_2(0) = 2,\ \ F_2(1) = -Y_1(1) - Y_2(1) - 1\\
&F_2(2) = \frac{1}{2}Y_1(1)^2 - Y_1(2) - Y_2(2) + \frac{1}{2}(1 + Y_2(1))^2\\
&F_2(3) = -\frac{1}{4}Y_1(1)^3 + Y_1(2)Y_1(1) - Y_1(3) - Y_2(3)\\
&\quad + (1 + Y_2(1))\,Y_2(2) - \frac{1}{3}(1 + Y_2(1))^3.
\end{aligned} \tag{27}
$$

Hence, the series solutions are obtained and given as

$$\tilde{y}_1(t) = 2 - 2t + t^2 - \frac{1}{3}t^3 + \frac{1}{12}t^4 - \frac{1}{60}t^5 + ...$$

$$\tilde{y}_2(t) = 1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + ...$$

For sufficiently large number of terms, the closed form of the obtained series solutions are $\tilde{y}_1(t) = 2e^{-t}$, $\tilde{y}_2(t) = e^t - t$, which are the exact solutions.

4. Conclusions

In this paper, new general formulas and recurrence relations have been derived for computing the differential transform of any analytic nonlinear non-autonomous functions with one or multi-variable. As a special case of the present study, i.e., for autonomous function, the current formulas and recurrence relations have the same mathematical structure as the Adomian polynomials but with constants instead of variable components. The main advantage of the present method is that it can deal directly with nonlinear non-autonomous functions in their forms without any special transformation or algebraic manipulations and can be directly implemented in any symbolic language such as MATHEMATICA or MAPLE. The suggested modified DTM has been successfully applied on different types of differential and integro-differential equations with nonlinear non-autonomous functions that no related formula in the literature has been given to calculate their transform functions. The method is applied directly to the nonlinear equations with no need to analytical integration which is essential for other semi-analytical numerical methods such as HPM, HAM, ADM or VIM. Moreover, the obtained series solutions declare that the suggested method is a straight forward even in solving different types of differential and integro-differential equations with complex nonlinearities. Finally, the authors believe that the present study should be extended to include similar differential and intego-differential equations in the applied sciences, which increases its applicability.

References

[1] J.K. Zhou, *Differential transformation and its applications for electrical circuits*, Huazhong University Press, Wuhan–China 1986.

[2] H. Hosseinzadeh, H. Jafari, M.R. Gholami, D.D. Ganji, *Solving a class of boundary value problems in structural engineering and fluid mechanics using homotopy perturbation and adomian decomposition methods*, Italian Journal of Pure and Applied Mathematics, 37 (2017), 687-698.

[3] A. Ebaid, R. Rach, E. El-Zahar, *A new analytical solution of the hyperbolic Kepler equation using the Adomian decomposition method,* Acta Astronautica, 138 (2017), 1-9.

[4] H. Ghaneai, M.M. Hosseini, *Solving differential-algebraic equations through variational iteration method with an auxiliary parameter*, Applied Mathematical Modelling, 40 (2016), 3991-4001.

[5] Y. Zhao, Z. Lin, Z. Liu, S. Liao, *The improved homotopy analysis method for the Thomas–Fermi equation*, Applied Mathematics and Computation, 218(2012), 8363-8369.

[6] S. Al-Shara, F. Awawdeh, S. Abbasbandy, *An automatic scheme on the homotopy analysis method for solving nonlinear algebraic equations*, Italian journal of pure and applied mathematics, 37 (2017), 5-14.

[7] J. Biazar, M.A. Asadi, F. Salehi, *Rational Homotopy Perturbation Method for solving stiff systems of ordinary differential equations*, Applied Mathematical Modelling, 39(2015), 1291-1299.

[8] S.H.Chang, I.L. Chang, *A new algorithm for calculating one-dimensional differential transform of nonlinear functions*, Applied Mathematics and Computation, 195 (2008), 799-808.

[9] A.E.H. Ebaid, *Approximate periodic solutions for the non-linear relativistic harmonic oscillator via differential transformation method*, Communications in Nonlinear Science and Numerical Simulation, 15(2010), 1921-1927.

[10] A. E. Ebaid, *A reliable aftertreatment for improving the differential transformation method and its application to nonlinear oscillators with fractional nonlinearities*, Communications in Nonlinear Science and Numerical Simulation, 16 (2011), 528-536.

[11] A. E. Ebaid, *On a new differential transformation method for solving nonlinear differential equations*, Asian-European Journal of Mathematics, 6 (2013), 1350057.

[12] S.H. Behiry, *Differential transform method for nonlinear initial-value problems by adomian polynomials*, Journal of Applied & Computational Mathematics, 2012.

[13] H.S. Nik, F. Soleymani, *A Taylor-type numerical method for solving nonlinear ordinary differential equations*, Alexandria Engineering Journal, 52 (2013), 543-550.

[14] H. Fatoorehchi, H. Abolghasemi, *Improving the differential transform method: a novel technique to obtain the differential transforms of nonlinearities by the Adomian polynomials*, Applied Mathematical Modelling, 37 (2013), 6008-6017.

[15] V.S. Erturk, Z.M. Odibat, S. Momani, *Application of multi-step differential transform method for the analytical and numerical solutions of the density dependent Nagumo telegraph equation*, Romanian Journal of Physics, 57 (2012), 1065-1078.

[16] M. Abdulkawi, *Solution of Cauchy type singular integral equations of the first kind by using differential transform method*, Applied Mathematical Modelling, 39 (2015), 2107-2118.

[17] E.R. EL-Zahar, *Approximate analytical solutions of singularly perturbed fourth order boundary value problems using differential transform method*, Journal of King Saud University (Science), 25 (2013), 257–265.

[18] E.R. El-Zahar, *Applications of adaptive multi step differential transform method to singular perturbation problems arising in science and engineering*, Applied Mathematics and Information Sciences, 9 (2015), 223-232.

[19] E.R. El-Zahar, *Piecewise approximate analytical solutions of high order singular perturbation problems with a discontinuous source term*, International Journal of Differential Equations, 2016, Article ID 1015634, 12 pages, 2016. doi:10.1155/2016/1015634.

[20] A. Gökdoğan, M. Merdan, A. Yildirim, *Adaptive multi-step differential transformation method to solving nonlinear differential equations*, Mathematical and Computer Modelling, 55 (2012), 761-769.

[21] A. Arikoglu, I. Ozkol, *Solutions of integral and integro-differential equation systems by using differential transform method*, Computers & Mathematics with Applications, 56 (2008), 2411-2417.

[22] M. Al-Smadi, A. Freihat, O.A. Arqub, N. Shawagfeh, *A novel multistep generalized differential transform method for solving fractional-order Lü chaotic and hyperchaotic systems*, Journal of Computational Analysis & Applications, 19 (2015), 713-724.

[23] K. Moaddy, A. Freihat, M. Al-Smadi, E. Abuteen, I. Hashim, *Numerical investigation for handling fractional-order Rabinovich–Fabrikant model using the multistep approach*, Soft Computing, 1-10, 2016.

[24] M. Al-Smadi, A. Freihat, H. Khalil, S. Momani, R. Ali Khan, *Numerical multistep approach for solving fractional partial differential equations*, International Journal of Computational Methods, 14 (2017), 1750029.

[25] A. El-Ajou, O.A. Arqub, M. Al-Smadi, *A general form of the generalized Taylor's formula with some applications*, Applied Mathematics and Computation, 256 (2015), 851-859.

[26] E.R. El-Zahar, H.M. Habib, M.M. Rashidi, I.M. El-Desoky, *A Comparison of Explicit Semi-Analytical Numerical Integration Methods for Solving Stiff ODE Systems*, American Journal of Applied Sciences, 12 (2015), 304.

[27] Z.M. Odibat, S. Kumar, N. Shawagfeh, A. Alsaedi, T. Hayat, *A study on the convergence conditions of generalized differential transform method*, Mathematical Methods in the Applied Sciences, 40 (2017), 40-48.

[28] C. Bervillier, *Status of the differential transformation method*, Applied Mathematics and Computation, 218 (2012), 10158-10170.

[29] R. Barrio, *Performance of the Taylor series method for ODEs/DAEs*, Applied Mathematics and Computation, 163 (2005), 525-545.

[30] E. R. EL-Zahar, *An Adaptive Step-Size Taylor Series Based Method and Application to Nonlinear Biochemical Reaction Model*, Trends in Applied Sciences Research, 7 (2012), 901-912.

[31] M. Thongmoon, S. Pusjuso, *The numerical solutions of differential transform method and the Laplace transform method for a system of differential equations*, Nonlinear Analysis: Hybrid Systems, 4 (2010), 425-431.

[32] J.S. Duan, *Convenient analytic recurrence algorithms for the Adomian polynomials*, Applied Mathematics and Computation, 217 (2011), 6337-6348.

[33] J.R. Dormand, P.J. Prince, *A family of embedded Runge-Kutta formulae*, Journal of computational and applied mathematics, 6 (1980), 19-26.

Accepted: 17.05.2018