BAYESIAN ESTIMATION AND PREDICTION BASED ON EXPONENTIAL RESIDUAL TYPE II CENSORED LIFE DATA

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Abstract. In this paper, we consider statistical inference problems for the residual life data come from exponential model based on type II censored data. Maximum likelihood and Bayesian approaches are used to estimate the scale parameter for exponential model also we construct symmetric credible intervals. Further, we propose to estimate the posterior predictive density of the future ordered observations and then obtain the corresponding predictors and we obtain the predictive survival function to compute the predictive interval for the missing order statistics. Numerical comparisons are conducted to assess the performance of the estimators of the parameter as well as the predictors of future ordered data.
Keywords: residual life data, exponential distribution, type II censored data, Maximum likelihood estimation, Bayes estimation, Bayes prediction.

1. Introduction

Exponential distribution plays an important role in lifetime data analysis and is a commonly used distribution in reliability engineering. In the last few years, many researchers have developed inference procedures for exponential model. Sarhan (2003) obtained the empirical Bayes estimators of exponential model. Janeen (2004) discussed the empirical Bayes estimators of the exponential distribution parameter based on record values. Yimin and Weian (2010) estimated the scale parameter for two parameters exponential distribution using empirical Bayes procedure under the type I censoring life test. Chen and Lio (2010) considered the parameter estimation for exponential distribution under progressive type I interval censoring. Also, several authers studied Bayesian inference and prediction. Kundu (2008) studied the Bayesian inference of the Weibull

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parameters when the data are progressively censored under the squared error loss function. Kundu and Howlader (2010) described the Bayesian inference and prediction of the inverse Weibull distribution for type II censored data, they obtained the Bayes estimatiors based on the square error loss function. Pradham and Kundu (2011) studied the Bayesian estimation and prediction of the two parameter Gamma distribution, they considered the Bayes estimation under square error loss function with the assumption that the scale parameter has a Gamma prior and the shape parameter has any log-cocave prior. Al-Hussaini (1999) studied the Bayesian prediction problem for a large class of lifetime distribution. For more details, one can refer to Balakrishnan et al. (2005).

The main aim of this paper is to compute the MLE and Bayes estimate of the unknown parameter based on residual exponential life data under square error loss function and predicte the residual life time for the missing items which is important especially in actuarial, medical and engineering sciences.

Suppose that n items are kept under observation until failure. These items could be some systems, components, or computer chips in reliability study experiments, or they could be patients put under certain drug or clinical conditions and their lifetimes $\underset{\sim}{X} = (X_1, X_2, ..., X_n)$ follow the exponential distribution with the probability density function (pdf)

$$f(x;\theta) = \begin{cases} \theta.e^{-\theta x}, & \text{if } x > 0, \theta > 0 \\ 0, & \text{if } x \leq 0, \end{cases} \tag{1}$$

and cumulative distribution function (cdf)

$$F(x;\theta) = 1 - e^{-\theta x}, x > 0, \theta > 0. \tag{2}$$

Here $\theta > 0$ is the scale parameter.

The residual life random variable at age t is defined as $Y = X - t \mid X > t,\ t > 0$. These residual life data and its ordering can be effectively applied in reliability theory and play an important role, especially if one observes only the residual lifetime. This type of data do arise naturally in survival actuarial studies.

The cdf of the residual life variable can be obtained as

$$G(y) = \Pr(X - t < y \mid X > t) = \frac{F(t+y) - F(t)}{1 - F(t)}\ , \quad y > 0, \tag{3}$$

and the corresponding pdf is

$$g(y) = \frac{f(t+y)}{1 - F(t)}, \quad y > 0. \tag{4}$$

Now, for the exponential distribution we get

$$G(y) = 1 - e^{-\theta y},\ y > 0,\ \theta > 0, \tag{5}$$

and

$$g(y) = \theta e^{-\theta y},\ y > 0, \theta > 0. \tag{6}$$

Now, for some reason or other, one may terminate the experiment at the $r-th$ failure, that is, at time $X_{r:n}$, we obtain type II censored sample. Here r is fixed, while $X_{r:n}$, the duration of the experiment is random. The likelihood function in this case is

$$L(\theta \mid \underset{\sim}{X}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} g\left(x_i \mid \theta\right)\left[1 - G(x_r;\theta)\right]^{n-r}\ ,\ x_1 < x_2 < ... < x_r. \tag{7}$$

Consider we have complete data for the residual life times $Y_1, Y_2, ..., Y_n$, where $Y_i = X_i - t \mid X_i > t$, $i = 1, 2, ..., n$ are the residual lifetime random variables.

2. Maximum likelihood estimation

In this section, we derive the maximum likelihood estimator (MLE) of the parameter θ of the exponential model based on type II censored data using the residual life observations. Let $Y_{1:n} < Y_{2:n} < ... < Y_{r:n}$ be a type II censored sample of size r $(1 < r < n)$. On using Eq.(5), Eq.(6) and Eq.(7), the likelihood function is given by

$$L\left(\theta \mid data\right) = \frac{n!}{(n-r)!}\theta^r e^{-\theta\left[y_r(n-r)+\sum_{i=1}^{r} y_i\right]}.$$

By differentiating the natural logarithm of the likelihood function with respect to θ and equating the resulting term to zero, we get

$$\theta = \frac{r}{y_r\left(n-r\right) + \sum_{i=1}^{r} y_i}. \tag{8}$$

The MLE of θ, say $\overset{\wedge}{\theta}$, is obtained as a solution of Eq.(8). For more details about the existence of these MLEs and uniqueness, see Balakrishnan and Kateri (2008).

3. Bayes estimation

For the Bayesian inference and life testing plan, we need to assume some prior distributions to obtain the Bayes estimates (BEs) of θ and the corresponding credible intervals based on type II censored data, the natural choice of the scale parameter θ has a conjugate gamma prior $Gamma\left(a, b\right)$, with pdf

$$\pi_1\left(\theta \mid a, b\right) = \begin{cases} \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}, & \text{if } \theta > 0 \\ 0, & \text{if } \theta \leq 0, \end{cases}$$

with the hyper-parameters $a > 0$, $b > 0$ and $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$.

For more details, see Kundu (2008), and Kundu and Raqab (2012). For computing the Bayes estimates, we assume mainly a squared error loss (SEL) function only.

Based on a Type II censored data $Y_{1:n} < Y_{2:n} < ... < Y_{r:n}, (1 \le r \le n)$ and the prior of θ, we obtain the conditional posterior of θ using Eq.(5) and Eq.(6) as

$$
\begin{aligned}
(9) \qquad \pi(\theta \mid data) &\propto \prod_{i=1}^{r} g(y_i \mid \theta) \left[1 - G(y_r; \theta)\right]^{n-r} \pi_1(\theta) \\
&\propto \theta^{r+a-1} e^{-\theta\left(b + y_r(n-r) + \sum_{i=1}^{r} y_i\right)}.
\end{aligned}
$$

The conditional posterior distribution of θ given data, $\pi(\theta \mid data)$ is

$$
(10) \qquad Gamma\left(r + a, b + y_r(n-r) + \sum_{i=1}^{r} y_i\right).
$$

Therefore, the BE of θ, say $\hat{\theta}_{Bayes}$, is

$$
(11) \qquad \hat{\theta}_{Bayes} = \frac{r+a}{b + y_r(n-r) + \sum_{i=1}^{r} y_i}.
$$

Also, the $(1-\beta)\,100\%$ Bayesian credible interval for θ is given by (θ_L, θ_U) such that

$$
\int_{\theta_L}^{\theta_U} \pi(\theta \mid data)\, d\theta = 1 - \beta.
$$

So,

$$
(12) \qquad \Pr(\lambda_L < \theta < \infty) = 1 - \frac{\beta}{2} \text{ and } \Pr(\lambda_U < \theta < \infty) = \frac{\beta}{2}.
$$

Using Eq.(10) and the incomplete gamma function which is defined as $\Gamma(a, c) = \int_c^\infty x^{a-1} e^{-x} dx$, $a > 0, c > 0$, we obtain

$$
(13) \qquad \Gamma(r + a, u\theta_L) = \left(1 - \frac{\beta}{2}\right) \Gamma(r + a) \text{ and } \Gamma(r + a, u\theta_U) = \frac{\beta}{2} \Gamma(r + a),
$$

where $u = b + y_r(n-r) + \sum_{i=1}^{r} y_i$.

Now, by using a suitable numerical method to solve equations (13), we obtain (θ_L, θ_U).

4. Bayes prediction

The Bayes prediction of the unknown observation from the future sample based on current a vailable sample, known as informative sample, is an important feature in Bayes analysis. We mainly consider the estimation of posterior predictive density of a future observation based on the current data. The objective is to provide an estimate of the future observation of an residual experiment based on the results obtained from an informative experiment.

Let $y_{1:n} < y_{2:n} < ... < y_{r:n}$ be the observed sample known as informative sample and $y_{r+1:n} < y_{r+2:n} < ... < y_{n:n}$ be the unobserved future sample. Our goal is to predict $Y_{s:n}$, $r < s \le n$. The posterior predictive density of $Y_{s:n}$ given the observed data $\underset{\sim}{Y} = (y_{1:n}, y_{2:n}, ..., y_{r:n})$ is defined as

$$\pi_{Y_s}(y \mid data) = \int_0^\infty g_{Y_s|data}(y \mid \theta)\, \pi(\theta \mid data)\, d\theta \ ,\ y_s > y_r,$$

where $g_{Y_s|data}(y \mid \theta)$ is the conditional density of Y_s given θ and data $\underset{\sim}{Y}$, see for example Chen, Shao and Ibrahim (2000).

Using the Markovian property of the conditional order statistics, see David and Nagaraja (2003), the conditional pdf of $Y_{s:n}$ given $\underset{\sim}{Y}$ is just the conditional pdf of $Y_{s:n}$ given $Y_{r:n}$, $(r+1 \le s \le n)$ that is,

$$\begin{aligned} (14) \qquad g_{Y_s|data}(y_s \mid \alpha,\lambda) &= g_{Y_s|Y_r}(y_s \mid \alpha,\lambda) = \frac{g_{r;s:n}(y_r, y_s)}{g_{r:n}(y_r)} \\ &= c\,\theta \left[1 - e^{-\theta(y-y_r)}\right]^{s-r-1} e^{-\theta(y-y_r)(n-s+1)}, \end{aligned}$$

where $g_{r;s:n}(y_r, y_s)$ is the joint pdf of the $r-th$ and $s-th$ order statistics from a sample of size n from the parent distribution $G(.)$. One can observe that the conditional density of $Y_{s:n}$ given $Y_{r:n}$ is just the marginal density of $(s-r)-th$ order statistics from a sample of size $(n-r)$ from the left truncated distribution of $G(.)$ at y_r. By using the binomial expansian, we have

$$(15) \qquad g_{Y_s|data}(y \mid \alpha,\lambda) = c\,\theta \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i\, e^{-\theta(y-y_r)[n+i-s+1]},$$

where $c = \frac{(n-r)!}{(s-r-1)!(n-s)!}$. So, the posterior predictive density of $Y_{s:n}$ at any point $y > y_r$ is

$$\begin{aligned} \pi_{Y_s}(y \mid data) &= c \int_0^\infty \theta \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i e^{-\theta(y-y_r)[n+i-s+1]}\ \pi(\theta \mid data) d\theta \\ &= c \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \int_0^\infty \theta^{r+a} e^{-\theta[(y-y_r)(n+i-s+1)+y_r(n-r)+\sum_{i=1}^r y_i+b]} d\theta \\ &= c \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \frac{\Gamma(r+a+1)}{[(y-y_r)(n+i-s+1)+y_r(n-r)+\sum_{i=1}^r y_i+b]^{r+a+1}}. \end{aligned}$$

The Bayes predictor (BP) of $Y_{s:n}$ under SEL function is

$$\begin{aligned} Y_{s:n}^{BP} &= \int_{y_r}^{\infty} y\ \pi_{Y_s}(y \mid data)\, dy \\ &= c\int_{y_r}^{\infty} y \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \\ &\quad \frac{\Gamma(r+a+1)}{[(y-y_r)(n+i-s+1)+y_r(n-r)+\sum_{i=1}^{r} y_i + b]^{r+a+1}} dy. \end{aligned}$$

By using the transformation $v=(y-y_r)(n+i-s+1)+y_r(n-r)+\sum_{i=1}^{r} y_i+b$, we get

$$\begin{aligned} (16)\quad Y_{s:n}^{BP} &= c\sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \frac{1}{(n-s+i+1)^2} \\ &\quad [\frac{b+\sum_{i=1}^{r} y_i + y_r(n-r)+y_r(n+i-s+1)(r+a-1)}{(r+a-1)}]. \end{aligned}$$

Another important problem is to construct a two sided predictive interval of the order statistics Y_s. For this, we need to obtain the predictive survival function of Y_s which is defined as

$$\begin{aligned} (17)\quad S_{Y_s|data}(y \mid \theta) &= \Pr(Y>y) = \int_{y}^{\infty} g_{Y_s|data}(z \mid \theta)\, dz \\ &= c\sum_{i=0}^{s-r-1} \binom{s-r-1}{i} \frac{(-1)^i}{(n-s+i+1)} e^{-\theta(y-y_r)(n-s+i+1)}. \end{aligned}$$

Under the SEL function, the predictive survival function of Y_s is

$$\begin{aligned} & S_{Y_s|data}^{P}(y \mid \alpha, \lambda) \\ (18)\quad & = \int_{0}^{\infty} c\sum_{i=0}^{s-r-1} \binom{s-r-1}{i} \frac{(-1)^i}{(n-s+i+1)} e^{-\theta(y-y_r)(n-s+i+1)} \\ & \pi(\theta \mid data)\, d\theta. \\ & = c\sum_{i=0}^{s-r-1} \binom{s-r-1}{i} \frac{(-1)^i}{(n-s+i+1)} \\ & \frac{[y_r(n-r)+\sum_{i=1}^{r} y_i + b]^{r+a}}{[(y-y_r)(n-s+i+1)+y_r(n-r)+\sum_{i=1}^{r} y_i+b]^{r+a}}. \end{aligned}$$

Now, the $(1-\beta)\,100\%$ predictive interval of Y_s can be found by solving the following non -linear equations (19) for the lower bound (L) and upper bound (U) using a suitable numerical technique

$$(19)\qquad \hat{S}^{P}_{Y_s|data}(L) = 1-\frac{\beta}{2} \text{ and } \hat{S}^{P}_{Y_s|data}(U) = \frac{\beta}{2}.$$

5. Simulations and data analysis

5.1 Simulations

We report some numerical experiments performed to evaluate the behavior of the proposed methods for different sampling schemes and different priors based on type II censored data. We have assumed $\theta = 2$ to generate exponential residual life data at $t = 0.2$, $t = 0.5$, and $t = 0.9$. To compute the BEs under SEL function, we have assumed $\pi_1(\theta)$, the prior of θ, has gamma density function with shape and scale parameters a and b respectively. For the computations of BEs, we consider types of prior for θ: first prior is the non-informative prior, *i.e* $a = b = 0$, we call this prior as $Prior$ 0, second prior is the informative prior, her we use three cases namely $(Prior\ 1:\ a = 1, b = 2)$, $(Prior\ 2:\ a = 1, b = 3)$, and $(Prior\ 3:\ a = 1, b = 4)$.From Eq.8 and Eq.11, we notice that the MLE and BE are equals under $Prior$ 0. In each sampling schemes, we compute the MLE and the BE of θ under SEL function and 95% credible intervals of θ based on 10,000 samples. We report the average Bayes estimates, mean squared errors (MSEs), coverage percentages lengths and average credible intervals lenghts for θ based on 10,000 replications.

Table 1: MLEs and Bayes estimates with respect to SEL function based on residual type II censored data, when $t = 0.2$.

Scheme	MLE	$Bayes(Prior1)$	$Bayes(Prior2)$	$Bayes(Prior3)$
	θ	θ	θ	θ
Scheme 1: $n = 25, r = 10$	2.1846 (0.5848)	1.6744 (0.2436)	1.4269 (0.3878)	1.2260 (0.5426)
Scheme 2: $n = 25, r = 15$	2.0850 (0.3933)	1.7407 (0.2101)	1.6311 (0.2472)	1.4420 (0.3695)
Scheme 3: $n = 25, r = 20$	2.1844 (0.2926)	1.8432 (0.1352)	1.6445 (0.1905)	1.5803 (0.2320)
Scheme 4: $n = 40, r = 10$	2.3275 (0.9030)	1.6942 (0.2552)	1.3856 (0.4404)	1.2412 (0.6294)
Scheme 5: $n = 40, r = 20$	2.1413 (0.2658)	1.7811 (0.1757)	1.6714 (0.1827)	1.5249 (0.2585)
Scheme 6: $n = 40, r = 30$	1.9813 (0.1200)	1.8752 (0.0865)	1.7477 (0.1297)	1.6773 (0.1109)

Note: The first entry represents the point estimate, while the corresponding MSE is given between the parentheses.

From tables 1,2 and 3, it is evident that the MLEs and Bayes estimates of θ improve when more information is available (r gets large), this observations is valid for priors $(Prior\ 1, Prior\ 2\ and\ Prior\ 3)$ when $t = 0.2$, $t = 0.5$, and $t = 0.9$ and the Bayes estimates of θ under informative priors get better than the MLEs of θ. The Bayes estimates of θ under informative priors perform less better when the variance gets small. From table 4, we observe that the average credible intervals lenghts for θ become smaller when r increases and n is fixed for all priors when $t = 0.2$, $t = 0.5$, and $t = 0.9$.

For computing the predictors, based on exponential residual data we have obtained the point predictors and 95% predictive intervals (PIs) at $t = 0.2$, $t = 0.5$, and $t = 0.5$ for the missing order statistics Y_s, $r < s \leq n$ using $Prior$ 1. From table 5 and 6, we observe that the predicted values for the missing order statistics $Y_{s:n}$ are quite close to each other and fall in their PIs for all schemes.

Table 2: MLEs and Bayes estimates with respect to SEL function based on residual type II censored data, when $t = 0.5$.

Scheme	MLE	$Bayes(Prior1)$	$Bayes(Prior2)$	$Bayes(Prior3)$
	θ	θ	θ	θ
Scheme 1: $n = 25, r = 10$	2.1529 (0.5083)	1.6004 (0.3032)	1.4559 (0.3646)	1.2141 (0.4532)
Scheme 2: $n = 25, r = 15$	2.1741 (0.3508)	1.8137 (0.1919)	1.5602 (0.2752)	1.4170 (0.3230)
Scheme 3: $n = 25, r = 20$	2.1056 (0.2423)	1.7677 (0.1604)	1.6984 (0.1718)	1.5445 (0.2066)
Scheme 4: $n = 40, r = 10$	2.1689 (0.5192)	1.6156 (0.2661)	1.4082 (0.4266)	1.2768 (0.4884)
Scheme 5: $n = 40, r = 20$	2.1633 (0.2758)	1.7927 (0.1742)	1.6738 (0.1789)	1.5455 (0.2161)
Scheme 6: $n = 40, r = 30$	2.0712 (0.1819)	1.9185 (0.1016)	1.7328 (0.1340)	1.6936 (0.1353)

Note: The first entry represents the point estimate, while the corresponding MSE is given between the parentheses.

Table 3: MLEs and Bayes estimates with respect to SEL function based on residual type II censored data, when $t = 0.9$.

Scheme	MLE	$Bayes(Prior1)$	$Bayes(Prior2)$	$Bayes(Prior3)$
	θ	θ	θ	θ
Scheme 1: $n = 25, r = 10$	2.1661 (0.4285)	1.6435 (0.2821)	1.4305 (0.3883)	1.2974 (0.4123)
Scheme 2: $n = 25, r = 15$	2.2071 (0.3329)	1.8032 (0.1496)	1.5639 (0.2441)	1.4454 (0.3094)
Scheme 3: $n = 25, r = 20$	2.0603 (0.1848)	1.8125 (0.1243)	1.7065 (0.1397)	1.5159 (0.1809)
Scheme 4: $n = 40, r = 10$	2.1492 (0.5191)	1.6915 (0.2422)	1.4051 (0.4252)	1.2488 (0.5043)
Scheme 5: $n = 40, r = 20$	2.1784 (0.2374)	1.8448 (0.1534)	1.6361 (0.1853)	1.5587 (0.2117)
Scheme 6: $n = 40, r = 30$	2.0036 (0.1430)	1.8822 (0.1093)	1.7688 (0.1250)	1.6280 (0.1316)

Note: The first entry represents the point estimate, while the corresponding MSE is given between the parentheses.

Table 4: Average credible intervals lengths of θ based on residual type II censored data.

$Schemes$		$Prior1$	$Prior2$	$Prior3$
$n = 25, r = 10$	$t = 0.2$	2.7801 (0.94)	2.8156 (0.95)	2.8116 (0.94)
	$t = 0.5$	2.8632 (0.95)	2.8016 (0.91)	2.8068 (0.93)
	$t = 0.9$	2.8630 (0.92)	2.8713 (0.95)	2.7282 (0.95)
$n = 25, r = 15$	$t = 0.2$	2.1333 (0.95)	2.1384 (0.95)	2.1870 (0.97)
	$t = 0.5$	2.1600 (0.92)	2.2269 (0.93)	2.1665 (0.96)
	$t = 0.9$	2.1862 (0.95)	2.1852 (0.94)	2.1661 (0.93)
$n = 25, r = 20$	$t = 0.2$	1.8204 (0.96)	1.8177 (0.96)	1.8049 (0.95)
	$t = 0.5$	1.8020 (0.95)	1.8675 (0.92)	1.8371 (0.93)
	$t = 0.9$	1.8312 (0.94)	1.8265 (0.93)	1.8231 (0.92)
$n = 40, r = 10$	$t = 0.2$	2.7981 (0.95)	1.8483 (0.94)	2.8470 (0.93)
	$t = 0.5$	2.8390 (0.94)	2.7808 (0.96)	2.7822 (0.94)
	$t = 0.9$	2.7631 (0.92)	2.8666 (0.96)	2.8301 (0.95)
$n = 40, r = 20$	$t = 0.2$	1.8381 (0.94)	1.8337 (0.94)	1.8256 (0.94)
	$t = 0.5$	1.8494 (0.96)	1.8459 (0.94)	1.8910 (0.95)
	$t = 0.9$	1.8298 (0.94)	1.9649 (0.95)	1.8303 (0.93)
$n = 40, r = 30$	$t = 0.2$	1.4673 (0.95)	1.4863 (0.94)	1.4298 (0.96)
	$t = 0.5$	1.4764 (0.94)	1.4638 (0.96)	1.4657 (0.97)
	$t = 0.9$	1.4839 (0.95)	1.4300 (0.95)	1.4722 (0.94)

Note: The first entry represents the average lenght of the credible intervals, while the corresponding coverage percentages is given between the parentheses.

Table 5: Point predictors and PIs for the missing order statistics $\mathrm{Y}_{s:n}$, $r+1 \leq s \leq n$.

$Schemes$	$Predicted\ Values$	$Predicted\ Values$	$Predicted\ Values$
	(95% PI)	(95% PI)	(95% PI)
$n = 25, r = 10$	$t = 0.2$	$t = 0.5$	$t = 0.9$
$Y_{11:n}$	0.2370 (0.1982, 0.3556)	0.2927 (0.2731, 0.6551)	0.2504 (0.2109, 0.3713)
$Y_{14:n}$	0.3751 (0.2379, 0.6571)	0.4508 (0.2937, 0.7737)	0.3912 (0.2513, 0.6787)
$Y_{17:n}$	0.5551 (0.3119, 1.0245)	0.6569 (0.3784, 1.1944)	0.5748 (0.3268, 1.0533)
$Y_{19:n}$	0.7147 (0.3804, 1.3505)	0.8397 (0.4569, 1.5677)	0.7375 (0.3967, 1.3857)
$Y_{21:n}$	0.9333 (0.4731, 1.8044)	1.0899 (0.5630, 2.0875)	0.9603 (0.4911, 1.8485)
$Y_{23:n}$	1.2809 (0.6124, 2.5571)	1.4880 (0.7225, 2.9494)	1.3148 (0.6332, 2.6159)
$Y_{25:n}$	2.1749 (0.9016, 4.7886)	2.5117 (1.0537, 5.5046)	2.2262 (0.9280, 4.8910)
$n = 25, r = 15$			
$Y_{16:n}$	0.4719 (0.4175, 0.6329)	0.5385 (0.4729, 0.7324)	0.5369 (0.4703, 0.7341)
$Y_{18:n}$	0.6035 (0.4508, 0.9153)	0.6970 (0.5131, 1.0727)	0.6980 (0.5111, 1.0801)
$Y_{20:n}$	0.7759 (0.5173, 1.2597)	0.9047 (0.5932, 1.4878)	0.9093 (0.5925, 1.5021)
$Y_{21:n}$	0.8873 (0.5632, 1.4822)	1.0390 (0.6485, 1.7559)	1.0458 (0.6488, 1.7748)
$Y_{23:n}$	1.2122 (0.6954, 2.1528)	1.4306 (0.8077, 2.5640)	1.4439 (0.8106, 2.5964)
$Y_{24:n}$	1.4908 (0.7997, 2.7692)	1.7662 (0.9334, 3.3068)	1.7852 (0.9385, 3.3517)
$Y_{25:n}$	2.0478 (0.9738, 4.1686)	2.4374 (1.1433, 4.9930)	2.4677 (1.1518, 5.0663)
$n = 25, r = 20$			
$Y_{21:n}$	1.0750 (0.9405, 1.4664)	0.7690 (0.6696, 1.0582)	1.1499 (0.9983, 1.5910)
$Y_{22:n}$	1.2472 (0.9721, 1.8594)	0.8963 (0.6930, 1.3486)	1.3440 (1.0339, 2.0339)
$Y_{23:n}$	1.4769 (1.0380, 2.3601)	1.0660 (0.7417, 1.7186)	1.6028 (1.1082, 2.5981)
$Y_{24:n}$	1.8214 (1.1470, 3.1327)	1.3205 (0.8222, 2.2894)	1.9910 (1.2310, 3.4687)
$Y_{25:n}$	2.5104 (1.3436, 4.8671)	1.8296 (0.9675, 3.5710)	2.7675 (1.4526, 5.4233)

Note: The first entry represents the point predictor, while the corresponding PIs is given between the parentheses.

Table 6: Point predictors and PIs for the missing order statistics $Y_{s:n}$, $r+1 \leq s \leq n$.

Schemes	*Predicted Values*	*Predicted Values*	*Predicted Values*
	(95% *PI*)	(95% *PI*)	(95% *PI*)
$n = 40, r = 10$	$t = 0.2$	$t = 0.5$	$t = 0.9$
$Y_{11:n}$	0.1817 (0.1573, 0.2561)	0.1434 (0.1228, 0.2065)	0.1836 (0.1591, 0.2581)
$Y_{14:n}$	0.2620 (0.1808, 0.4285)	0.2114 (0.1427, 0.3523)	0.2640 (0.1827, 0.4307)
$Y_{17:n}$	0.3519 (0.2197, 0.6059)	0.2875 (0.1756, 0.5023)	0.3544 (0.2216, 0.6083)
$Y_{20:n}$	0.4542 (0.2675, 0.8040)	0.3438 (0.2018, 0.6117)	0.4564 (0.2695, 0.8067)
$Y_{23:n}$	0.5726 (0.3243, 1.0327)	0.4056 (0.2312, 0.7311)	0.5750 (0.3264, 1.0355)
$Y_{26:n}$	0.7134 (0.3923, 1.3044)	0.4742 (0.2641, 0.8633)	0.7160 (0.3944, 1.3077)
$Y_{29:n}$	0.8868 (0.4757, 1.6414)	0.5511 (0.3012, 1.0116)	0.8895 (0.4780, 1.6451)
$Y_{32:n}$	1.1130 (0.5829, 2.0854)	0.6386 (0.3434, 1.1810)	1.1162 (0.5853, 2.0897)
$Y_{35:n}$	1.4384 (0.7323, 2.7372)	0.7400 (0.3922, 1.3784)	1.4418 (0.7349, 2.7423)
$Y_{38:n}$	2.0249 (0.9818, 3.9715)	0.8611 (0.4497, 1.6152)	2.0291 (0.9842, 3.9781)
$Y_{40:n}$	3.1481 (1.3623, 6.7199)	1.0105 (0.5198, 1.9111)	3.1537 (1.3657, 6.7298)
$n = 40, r = 20$			
$Y_{21:n}$	0.3643 (0.3366, 0.4450)	0.4397 (0.4066, 0.5359)	0.4822 (0.4444, 0.5924)
$Y_{23:n}$	0.4257 (0.3530, 0.5695)	0.5129 (0.4262, 0.6843)	0.5661 (0.4668, 0.7622)
$Y_{25:n}$	0.4947 (0.3823, 0.6951)	0.5951 (0.4612, 0.8341)	0.6602 (0.5069, 0.9337)
$Y_{27:n}$	0.5731 (0.4203, 0.8337)	0.6887 (0.5064, 0.9994)	0.76730 (0.5586, 1.1228)
$Y_{29:n}$	0.6642 (0.4665, 0.9928)	0.7973 (0.5616, 1.1891)	0.8915 (0.6218, 1.3401)
$Y_{31:n}$	0.7726 (0.5227, 1.1825)	0.9266 (0.6285, 1.4153)	1.0395 (0.6984, 1.5989)
$Y_{33:n}$	0.9068 (0.5921, 1.4192)	1.0865 (0.7113, 1.6975)	1.2226 (0.7931, 1.9219)
$Y_{35:n}$	1.0826 (0.6813, 1.7355)	1.2962 (0.8177, 2.0747)	1.4626 (0.9149, 2.3536)
$Y_{37:n}$	1.3383 (0.8054, 2.2130)	1.6011 (0.9657, 2.6441)	1.8115 (1.0841, 3.0053)
$Y_{40:n}$	2.3799 (1.2058, 4.5515)	2.8430 (1.4431, 5.4325)	3.2331 (1.6307, 6.1969)
$n = 40, r = 30$			
$Y_{31:n}$	0.6022 (0.5554, 0.7362)	0.9975 (0.9352, 1.1758)	0.8688 (0.8042, 1.0535)
$Y_{33:n}$	0.7155 (0.5856, 0.9644)	1.1483 (0.9754, 1.4796)	1.0251 (0.8459, 1.3684)
$Y_{35:n}$	0.8641 (0.6470, 1.2395)	1.3461 (1.0571, 1.8458)	1.2305 (0.9305, 1.7479)
$Y_{37:n}$	1.0800 (0.7434, 1.6425)	1.6335 (1.1855, 2.3823)	1.5279 (1.0636, 2.3039)
$Y_{40:n}$	1.9598 (1.0722, 3.6069)	2.8046 (1.6231, 4.9973)	2.7417 (1.5171, 5.0142)

Note: The first entry represents the point predictor, while the corresponding PIs is given between the parentheses.

5.2 Data analysis

We analyze the residual real lifetimes data (in minutes) to breakdown of an insulating fluid at voltage 40 kv is usually assumed to be exponential in engineering theory. The real lifetimes data has been recently considered by Nelson (1982). The sample consists of the smallest nine observations of tweleve times to breakdown as follows: $1, 1, 2, 3, 12, 25, 46, 56, 68$. Before we analyze the data, we divide all the points by 100.

The MLE of θ based on the above data is determined to be 1.4887. The Bayes estimate of θ using $Prior1$ is 1.1273. Also, the 95% cerdible interval of θ is $(0.9737, 2.0889)$. Now, we consider the prediction of the $10-th, 11-th$ and $12-th$ order statistics which are missing. The predicted values and the 95% PI of the $10-th, 11-th$ and $12-th$ order statistics are presented in Table 7. It is observed that all predicted values with respect to SEL function are all ordered and fall in their corresponding PI.

Table 7: Point predictors and PIs for the real data.

$n = 12, r = 9$	$Y_{10:n}$	$Y_{11:n}$	$Y_{12:n}$
$Predicted\ Values$	103.40	319.23	411.35
(95% PI)	$(79.63, 118.81)$	$(304.42, 335.28)$	$(394.90, 431.83)$

References

[1] E.K. Al-Hussaini, *Predicting observable from a general class of distributions*, Journal of Statistical Planning and Inference, 79 (1999), 79-81.

[2] N. Balakrishnan, C.T. Lin, P.S. Chan, *A comparison of two simple prediction intervals for exponential distribution*, IEEE T. Reliab., 54 (2005), 27-33.

[3] N. Balakrishnan, M. Kateri, *On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data*, Statistics & Probability Letters, 78 (2008), 2971-2975.

[4] D.G. Chen, Y.L. Lio, *Parameter estimation for generalized exponential distribution under progressive Type I interval censoring*, Computational Statistics and Data Analysis, 54 (2010), 1581-1591.

[5] M.H. Chen, Q.M. Shao, J.G. Ibrahim, *Monte Carlo Methods in Bayesian Computation*, Springer-Verlag, New York, 2000.

[6] H.A. David, H.N. Nagaraja, *Order Statistics*, 3^{rd} Edition, Wiley, New York, 2003.

[7] Z.F. Janeen, *Empirical Bayes analysis of record statistics based on LINEX and quadratic loss functions*, Comput. Math. Appl., 47 (2004), 947-954.

[8] M.P. Kaminskiy, V.V. Krivtsov, *A Simple Proccedure for Bayesian Estimation of the Weibull Distribution*, IEEE Transactions on Reliability Analysis, 54 (2005), 612-616.

[9] D. Kundu, *Bayesian Inference and Life Testing Plan for the Weibull Distribution in Presence of Progressive Censoring*, Technometrics, 50 (2008), 144-154.

[10] D. Kundu, H. Howlader, *Bayesian Inference and Prediction of the Inverse Weibull Distribution for Type-II Censored Data*, Computational Statistics and Data Analysis, 54 (2010),1547-1558.

[11] D. Kundu, M.Z. Raqab, *Bayesian Inference and Prediction for a Type-II censored Weibull Distribution*, Journal of Statistical Planning and Inference, 142 (2012), 41-47.

[12] W. Nelson, *Applied Life Data Analysis*, John Wiley and Sons, New York, 1982.

[13] B. Pradhan, D. Kundu, *Bayes Estimation and Prediction of the Two-Parameter Gamma Distribution*, Journal of Statistical Computation and Simulation, 81 (2011), 1187-1198.

[14] A.A. Sarhan, *Empirical Bayes estimetes in exponential reliability model*, Appl. Math. Comput., 135 (2003), 319-332.

[15] S. Yimin, Y. Weian, *The EB Estimation of Scale-parameter for Two parameter Exponential Distribution Under the Type-I Censoring Life Test*, Journal of Physical Sciences, 14 (2010), 25-30.

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