ROUGH APPROXIMATIONS IN KU-ALGEBRAS

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Abstract. In this paper, the concept of roughness in KU-algebras is introduced. We study the lower and upper approximations of KU-subalgebras and KU-ideals and proved that the lower/upper approximation of KU-subalgebra (resp., KU-ideals) is a KU-subalgebra (resp., KU-ideals). A connection between rough sets and KU-Algebras with their weak and strong ideals have also been taken under consideration and some related results have been shown.

Keywords: KU-subalgebras, KU-ideals, lower approximations, upper approximations, definable.

1. Introduction

The notion of rough sets was introduced by Pawlak in his paper [21]. The theory of rough sets has emerged as another major powerfull mathematical approach for managing and handling different types of uncertainty in information systems that arises from inexact, noisy, or incomplete information. It is turning out to be methodologically significant to the domains of artificial intelligence and cognitive sciences, especially in the representation of and reasoning with vague and/or imprecise knowledge, data classification, data analysis, machine learning, pattern recognition and knowledge discovery. In connection with algebraic structures, Biswas and Nanda [4] gave the notion of rough subgroups, and Kuroki [13] introduced rough ideals in semigroups. Xiao and Zhang [27] introduced rough prime ideals and rough fuzzy prime ideals in semigroups. Ameri et al. [3] applied rough set theory to hyper BCK-algebra. Dudek et al. [7] and Ma [15] applied rough set theory to BCI-algebras. Jun et al. considered roughness in BCK-algebra [10], lattice implication algebras [11] and BCC-algebra [12]. Mao and Zhou [16] studied the rough set theory in Pseudo-BCK-algebra. Torkzadeh and Ghorbani [26] studied rough filters in B-Algebras.

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Prabpayak and Leerawat [22] introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some related properties in [22, 23].

The concept of fuzzy sets was introduced by Zadeh [29]. There are several authors who considered KU-algebras in terms of different types of fuzzy sets, for instance, Mostafa et al. [17] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals, also see [18, 19]. Akram et al. [1] studied interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideals of KU-algebras. In [2, 28], the author applied the concept of cubic sets to KU-algebras. Davvaz et al. [6] introduced neutrosophic ideals of neutrosophic KU-algebras. Gulistan et al. [8, 9] studied the generalized version fuzzy KU-ideals of KU-algebras. Muhiuddin [20] studied bipolar fuzzy KU-subalgebras/ideals of KU-algebras. Senapati et al. [24, 25] introduced T-fuzzy KU-subalgebras and intuitionistic fuzzy bi-normed KU-ideals of a KU-algebra. As a work in computer science Chen et al.[5] worked on data mining framework based on rough set theory to improve location selection decisions as a case study of a restaurant chain whereas Karimi [14] studied rough sets and Gray sets.

2. Preliminaries

In this section we shall define some basic concepts including KU-algebras, KU-subalgebras, KU-ideals and shall provide examples based on them.

Definition 1 ([23]). *By a KU-algebra we mean an algebra* $(X, *, 0)$ *of type* $(2, 0)$ *with a single binary operation* $*$ *that satisfies the following identities: for any* $x, y, z \in X$,

$(ku1) : (x * y) * [(y * z) * (x * z)] = 0,$
$(ku2) : x * 0 = 0,$
$(ku3) : 0 * x = x,$
$(ku4) : x * y = 0 = y * x$ *implies* $x = y$.

In what follows, let $(X, *, 0)$ denote a KU-algebra unless otherwise specified. For brevity we also call X a KU-algebra. In X we can define a binary relation $\leq$ by : $x \leq y$ if and only if $y * x = 0$.

Definition 2 ([23]). $(X, *, 0)$ *is a KU-algebra if and only if it satisfies:*

$(ku5) : (y * z) * (x * z) \leq (x * y),$
$(ku6) : 0 \leq x,$
$(ku7) : x \leq y,\ y \leq x$ *implies* $x = y$,
$(ku8) : x \leq y$ *if and only if* $y * x = 0$.

Definition 3. *In a KU-algebra, the following identities are true [17]:*

(1) $z * z = 0,$
(2) $z * (x * z) = 0,$
(3) $x \leq y$ *imply* $y * z \leq x * z$,
(4) $z * (y * x) = y * (z * x)$, *for all* $x, y, z \in X$,

(5) $y * [(y * x) * x] = 0$.

Example 1 ([17])**.** Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by the following table

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

It is easy to see that X is KU-algebra.

Definition 4 ([23])**.** *A subset S of KU-algebra X is called a KU-subalgebra of X if $x * y \in S$, whenever $x, y \in S$.*

Definition 5 ([23])**.** *A non-empty subset A of a KU-algebra X is called a KU-ideal of X if it satisfies the following conditions:*
(1) $0 \in A$,
(2) $x * (y * z) \in A$, $y \in A$ *implies* $x * z \in A$, *for all* $x, y, z \in X$.

Example 2. Let $X = \{0, 1, 2, 3, 4, 5\}$ in which $*$ is defined by the following table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	2	4	5
2	0	0	0	1	4	5
3	0	0	0	0	4	5
4	0	0	0	1	0	5
5	0	0	0	0	0	0

Clearly $(X, *, 0)$ is a KU-algebra. It is easy to show that $A = \{0, 1\}$ and $B = \{0, 1, 2, 3, 4\}$ are KU-ideals of X.

Definition 6. *Let A be a nonempty subset of a KU-algebra X and $0 \in A$. Then,*

(1) *A is called a weak KU-ideal of X if $y * x \in A$ and $y \in A$ imply that $x \in A$, for all $x, y \in X$;*

(2) *A is called a strong KU-ideal of X if $(y * x) \cap A \neq \emptyset$ and $y \in A$ imply that $x \in A$, for all $x, y \in X$.*

3. Roughness in KU-algebras

Let V be a set and E an equivalence relation on V and let $P(V)$ denote the power set of V. For all $x \in V$, let $[x]_E$ denote the equivalence class of x with respect to E. Define the functions $\underline{E}, \overline{E} : P(V) \to P(V)$ as follows: $\forall\, S \in P(V)$,

$$\underline{E}(S) = \{x \in V : [x]_E \subseteq S\}$$

and

$$\overline{E}(S) = \{x \in V : [x]_E \cap S \neq \emptyset\}.$$

The pair (V, E) is called an approximation space. Let S be a subset of V. Then S is said to be definable if $\underline{E}(S) = \overline{E}(S)$ and rough otherwise. $\underline{E}(S)$ is called the lower approximation of S while $\overline{E}(S)$ is called the upper approximation.

Throughout this section X is a KU-algebra. Let A be a KU-ideal of X. Define a relation Θ on X by $(x, y) \in \Theta$ if and only if $x * y \in A$ and $y * x \in A$. Then Θ is an equivalence relation on X related to a KU-ideal A of X. Moreover satisfies $(x, y) \in \Theta$ and $(u, v) \in \Theta$ imply $(x * u, y * v) \in \Theta$.

Hence Θ is a congruence relation on X. Let A_x denote the equivalence class of x with respect to the equivalence relation Θ related to a KU-ideal A of X, and X/A denote the collection of all equivalence classes, that is, $X/A = \{A_x : x \in X\}$. Then $A_0 = A$. If $A_x * A_y$ is defined as the class containing $x * y$, that is, $A_x * A_y = A_{x*y}$, then $(X/A, *, A_0)$ is a KU-algebra. Let Θ be an equivalence relation on X related to a KU-ideal A of X. For any nonempty subset S of X, the lower and upper approximation of S are denoted by $\underline{\Theta}(A, S)$ and $\overline{\Theta}(A, S)$ respectively, that is,

$$\underline{\Theta}(A, S) = \{x \in X : A_x \subseteq S\}$$

and

$$\overline{\Theta}(A, S) = \{x \in x : A_x \cap S \neq \emptyset\}.$$

If $A = S$, then $\underline{\Theta}(A, S)$ and $\overline{\Theta}(A, S)$ are denoted by $\underline{\Theta}(A)$ and $\overline{\Theta}(A)$, respectively.

Definition 7 ([21])**.** *Given an approximation space (U, Θ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough set in (U, Θ) if and only if $(A, B) = Apr(X)$ for some $X \in P(U)$.*

Definition 8 ([21])**.** *Let (U, Θ) be an approximation space and X be a non-empty subset of U.*

(*i*) *If $\underline{Apr}(X) = \overline{Apr}(X)$, then X is called definable.*

(*ii*) *If $\underline{Apr}(X) = \emptyset$, then X is called empty interior.*

(*iii*) *If $Apr(X) = U$, then X is called empty exterior.*

Example 3. Let $X = \{0, 1, 2, 3, 4\}$ be a KU-algebra with the Cayley's table as follows (see [28]).

·	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	1
2	0	3	0	3	4
3	0	1	2	0	1
4	0	0	0	0	0

Let $A = \{0, 1\}$ be a KU-ideal of X $(A \lhd X)$ and let Θ be an equivalence relation on X related to A. Then $A_0 = A_1 = A$, $A_2 = \{2\}$, $A_3 = \{3\}$, and $A_4 = \{4\}$.

Hence

$$\begin{aligned}\underline{\Theta}(A,\{0,1\}) &= \{0,1\} \lhd X\\ \underline{\Theta}(A,\{0,2\}) &= \{2\}\\ \underline{\Theta}(A,\{0,3\}) &= \{3\}\\ \underline{\Theta}(A,\{0,1,2,3\}) &= \{0,1,2,3\} \lhd X\end{aligned}$$

and

$$\begin{aligned}\overline{\Theta}(A,\{0,1\}) &= \{0,1\} \lhd X\\ \overline{\Theta}(A,\{0\}) &= \{0,1\}\\ \overline{\Theta}(A,\{2\}) &= \{0,2\}\\ \overline{\Theta}(A,\{1,2,3\}) &= \{0,1,2,3\} \lhd X\\ \overline{\Theta}(A,\{0,2,3\}) &= \{0,1,2,3\} \lhd X\\ \overline{\Theta}(A,\{1,2,3,4\}) &= \{0,1,2,3,4\} \lhd X.\end{aligned}$$

In Example 3, we know that there exists a non-KU-ideal S of X such that their lower and upper approximation are KU-ideals of X. Also we choose some non-KU-ideals S of X such that their lower and upper approximation are KU-ideals of X.

Proposition 1. *Let Θ and Ξ be equivalence relations on X related to KU-ideals A and B of X, respectively. If S and T are nonempty subsets of X. Then*

(1) $\underline{\Theta}(A,S) \subseteq S \subseteq \overline{\Theta}(A,S)$;
(2) $\underline{\Theta}(A,\emptyset) = \emptyset = \overline{\Theta}(A,\emptyset)$
(3) $\overline{\Theta}(A,S\cup T) = \overline{\Theta}(A,S) \cup \overline{\Theta}(A,T)$;
(4) $\underline{\Theta}(A,S\cap T) = \underline{\Theta}(A,S) \cap \underline{\Theta}(A,T)$;
(5) $S \subseteq T$ *implies* $\underline{\Theta}(A,S) \subseteq \underline{\Theta}(A,T)$ *and* $\overline{\Theta}(A,S) \subseteq \overline{\Theta}(A,T)$;
(6) $\underline{\Theta}(A,S) \cup \underline{\Theta}(A,T) \subseteq \underline{\Theta}(A,S\cup T)$;
(7) $\overline{\Theta}(A,S\cap T) \subseteq \overline{\Theta}(A,S) \cap \overline{\Theta}(A,T)$;
(8) $\Theta \subseteq \Xi$ *and* $A \subseteq B$ *implies* $\underline{\Xi}(B,S) \subseteq \overline{\Theta}(A,S)$ *and* $\overline{\Theta}(A,S) \subseteq \overline{\Xi}(B,S)$.

Proof. (1) If $x \in \underline{\Theta}(A,S)$, then $x \in A_x \subseteq S$. Hence $\underline{\Theta}(A,S) \subseteq S$. Next, if $x \in S$, then, since $x \in A_x$, we have $A_x \cap S \neq \phi$, and so $x \in \overline{\Theta}(A,S)$. Thus $S \subseteq \overline{\Theta}(A,S)$.

(2) is straightforward.

(3) Note that

$$\begin{aligned}x \in \overline{\Theta}(A,S\cup T) &\iff A_x \cap (S\cup T) \neq \phi\\ &\iff (A_x\cap S)\cup(A_x\cap T) \neq \phi\\ &\iff A_x\cap S \neq \phi \;\; or \;\; A_x \cap T \neq \phi\\ &\iff x \in \overline{\Theta}(A,S) \; or \; a \in \overline{\Theta}(A,T)\\ &\iff x \in \overline{\Theta}(A,S)\cup\overline{\Theta}(A,T).\end{aligned}$$

Thus

$$\overline{\Theta}(A, S \cup T) = \overline{\Theta}(A, S) \cup \overline{\Theta}(A, T).$$

(4) Note that

$$\begin{aligned} x \in \underline{\Theta}(A, S \cap T) &\iff A_x \subseteq S \cap T \\ &\iff A_x \subseteq S \ \ and \ \ A_x \subseteq T \\ &\iff x \in \underline{\Theta}(A, S) \ and \ x \in \underline{\Theta}(A, T) \\ &\iff x \in \underline{\Theta}(A, S) \cap \underline{\Theta}(A, T). \end{aligned}$$

Thus

$$\underline{\Theta}(A, S \cap T) = \underline{\Theta}(A, S) \cap \underline{\Theta}(A, T).$$

(5) Since $S \subseteq T$ if and only if $S \cap T = S$, by (3) we have

$$\underline{\Theta}(A, S) = \underline{\Theta}(A, S \cap T) = \underline{\Theta}(A, S) \cap \underline{\Theta}(A, T).$$

This implies that $\underline{\Theta}(A, S) \subseteq \underline{\Theta}(A, T)$. Note also that $S \subseteq T$ if and only if $S \cup T = T$, by (2) we have

$$\overline{\Theta}(A, T) = \overline{\Theta}(A, S \cup T) = \overline{\Theta}(A, S) \cup \overline{\Theta}(A, T).$$

This implies that $\overline{\Theta}(A, S) \subseteq \overline{\Theta}(A, T)$.

(6) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, by (4) we have

$$\underline{\Theta}(A, S) \subseteq \underline{\Theta}(A, S \cup T) \quad \text{and} \quad \underline{\Theta}(A, T) \subseteq \underline{\Theta}(A, S \cup T).$$

This implies $\underline{\Theta}(A, S) \cup \underline{\Theta}(A, T) \subseteq \underline{\Theta}(A, S \cup T)$.

(7) Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, by (4) we have

$$\overline{\Theta}(A, S \cap T) \subseteq \overline{\Theta}(A, S) \quad \text{and} \quad \overline{\Theta}(A, S \cap T) \subseteq \overline{\Theta}(A, T).$$

This implies $\overline{\Theta}(A, S \cap T) \subseteq \overline{\Theta}(A, S) \cap \overline{\Theta}(A, T)$.

(8) Since $\Theta \subseteq \Xi$. If $x \in \underline{\Xi}(B, S)$, then $B_x \subseteq S$. But $\Theta \subseteq \Xi$, then $A_x \subseteq B_x \subseteq S$, that is, $A_x \subseteq S$. Thus $x \in \underline{\Theta}(A, S)$. Hence

$$\underline{\Xi}(B, S) \subseteq \underline{\Theta}(A, S).$$

Now let x be any element of $\overline{\Theta}(S)$. So $A_x \cap S \neq \phi$, then there exists $y \in A_y \cap S$ such that $y \in A_y$ and $y \in S$. Hence $(y, x) \in \Theta$, that is $y * x \in A$. Since $A \subseteq B$, it follows that $y * x \in B$ and $x * y \in B$ so that $(y, x) \in \Xi$, that is, $y \in B_x$. Therefore $y \in B_x \cap S$, which means that $x \in \Xi(B, S)$. This completes the proof. □

Proposition 2. *Let Θ be an equivalence relation on X related to a KU-ideal A of X. If S is a nonempty subset of X. Then*

(1) $\underline{\Theta}(A, \underline{\Theta}(A, S)) = \underline{\Theta}(A, S)$;
(2) $\overline{\Theta}(A, \overline{\Theta}(A, S)) = \overline{\Theta}(A, S)$;
(3) $\overline{\Theta}(A, \underline{\Theta}(A, S)) = \underline{\Theta}(A, S)$;
(4) $\underline{\Theta}(A, \overline{\Theta}(A, S)) = \overline{\Theta}(A, S)$;
(5) $\underline{\Theta}(A, S) = (\overline{\Theta}(A, S^c))^c$;
(6) $\overline{\Theta}(A, S) = (\underline{\Theta}(A, S^c))^c$;
(7) $\underline{\Theta}(A, A_x) = X = \overline{\Theta}(A, A_x)$, *for all $x \in X$.*

Proof. The proof is straightforward. □

Proposition 3. *Let Θ be an equivalence relation on X related to a KU-ideal A of X. If S is a nonempty subset of X. Then*
(1) $\overline{\Theta}(A,S) * \overline{\Theta}(A,T) \subseteq \overline{\Theta}(A,S*T)$;
(2) If Θ is congruence relation, then $\underline{\Theta}(A,S) * \underline{\Theta}(A,T) \subseteq \underline{\Theta}(A,S*T)$.

Proof. (1) Let c be any element of $\overline{\Theta}(A,S) * \overline{\Theta}(A,T)$. Then $c = p*q$ with $p \in \overline{\Theta}(A,S)$ and $q \in \overline{\Theta}(A,T)$. Thus there exist elements $x, y \in S$ such that

$$x \in A_p \cap S \quad \text{and} \quad y \in A_q \cap T.$$

Thus $x \in A_p$, $y \in A_q$, $x \in S$, and $y \in T$. Since Θ is a congruence on S, it follows that

$$x * y \in A_p * A_q \in A_{p*q}.$$

On the other hand, since $x*y \in S*T$. We have $x*y \in A_{p*q} \cap S*T$, and so $c = p*q \in \overline{\Theta}(A,S*T)$. Thus we have

$$\overline{\Theta}(A,S) * \overline{\Theta}(A,T) \subseteq \overline{\Theta}(A,S*T).$$

(2) Assume that Θ is complete, let c be any element of $\underline{\Theta}(A,S) * \underline{\Theta}(A,T)$. Then $c = p*q$ with $p \in \underline{\Theta}(A,S)$ and $q \in \underline{\Theta}(A,T)$. It follows that $A_p \subseteq S$ and $A_q \subseteq T$. Since Θ is a congruence relation on S, we have

$$A_{p*q} = A_p * A_q \subseteq S*T.$$

So $c = p*q \in \underline{\Theta}(A,S*T)$. Thus

$$\underline{\Theta}(A,S) * \underline{\Theta}(A,T) \subseteq \underline{\Theta}(A,S*T).$$

This completes the proof. □

Proposition 4. *Let Θ and Ξ be equivalence relations on X related to KU-ideals A and B of X, respectively. If S and T are nonempty subsets of X. Then*
(1) $\overline{\Theta \cap \Xi}(A \cap B, S) \subseteq \overline{\Theta}(A,S) \cap \overline{\Xi}(B,S)$;
(2) $\underline{\Theta \cap \Xi}(A \cap B, S) \supseteq \underline{\Theta}(A,S) \cap \underline{\Xi}(B,S)$.

Proof. (1) Note that $\Theta \cap \Xi$ is also a congruence relation on S. Let $c \in \overline{\Theta \cap \Xi}(A \cap B, S)$, then $[A \cap B]_c \cap S \neq \phi$. Then there exists an element $x \in [A \cap B]_c \cap S$. Since $(x,c) \in \Theta \cap \Xi$, we have

$$(x,c) \in \Theta \text{ and } (x,c) \in \Xi.$$

Thus we have $x \in A_c$ and $x \in B_c$. Since $x \in S$, we have $x \in A_c$, $x \in S$ and $x \in B_c$, $x \in S$. This implies that

$$x \in A_c \cap S \quad \text{and} \quad x \in B_c \cap S$$

$$A_c \cap S \neq \phi \quad \text{and} \quad B_c \cap S \neq \phi.$$

So $c \in \overline{\Theta}(A, S)$ and $c \in \overline{\Xi}(B, S)$, hence $c \in \overline{\Theta}(A, S) \cap \overline{\Xi}(B, S)$. Thus we obtain

$$\overline{\Theta \cap \Xi}(A \cap B, S) \subseteq \overline{\Theta}(A, S) \cap \overline{\Xi}(B, S).$$

(2) Since $\Theta \cap \Xi \subseteq \Theta$ and $\Theta \cap \Xi \subseteq \Xi$, which implies that

$$\underline{\Theta}(A, S) \subseteq \underline{\Theta \cap \Xi}(A \cap B, S) \quad \text{and} \quad \underline{\Xi}(B, S) \subseteq \underline{\Theta \cap \Xi}(A \cap B, S)$$

$$\Longrightarrow \underline{\Theta}(A, S) \cap \underline{\Xi}(B, S) \subseteq \underline{\Theta \cap \Xi}(A \cap B, S).$$

This completes the proof. □

Theorem 1. *Let (X, Θ) be an approximation space. Then*
(1) *for every $S \subseteq X$, $\underline{\Theta}(A, S)$ and $\overline{\Theta}(A, S)$ are definable sets,*
(2) *for every $x \in X$, A_x is definable set.*

Proof. (1) By Proposition 2 (1) and (3), we have

$$\underline{\Theta}(A, \underline{\Theta}(A, S)) = \underline{\Theta}(A, S) = \overline{\Theta}(A, \underline{\Theta}(A, S)).$$

Hence $\underline{\Theta}(A, S)$ is definable. On the other hand by Proposition 2 (2) and (4), we have

$$\overline{\Theta}(A, \overline{\Theta}(A, S)) = \overline{\Theta}(A, S) = \underline{\Theta}(A, \overline{\Theta}(A, S)).$$

Therefore $\overline{\Theta}(A, S)$ is a definable set.
(2) By Proposition 2 (7) the proof is clear. □

Definition 9. *A nonempty subset S of X is called an upper (resp. a lower) rough KU-subalgebra of X if the upper (resp. nonempty lower) approximation of S is a KU-subalgebra of X. If S is both an upper and a lower rough KU-subalgebra of X, we say that S is a rough KU-subalgebra of X.*

Theorem 2. *Let Θ be an congruence relation on X related to a KU-ideal A of X. If S is a KU-subalgebra of X, then*
(1) *$\overline{\Theta}(A, S)$ is a KU-subalgebra of X.*
(2) *$\underline{\Theta}(A, S)$ is a KU-subalgebra of X.*

Proof. (1) Let $x, y \in \overline{\Theta}(A, S)$. Then

$$A_x \cap S \neq \emptyset \text{ and } A_y \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in A_x$ and $b \in A_y$. It follows that $(a, x) \in \Theta$ and $(b, y) \in \Theta$. Since Θ is a congruence relation on X, we have $(a * b, x * y) \in \Theta$. Hence $a * b \in A_{x*y}$. Since S is a KU-subalgebra of X, we get $a * b \in S$, and therefore $a * b \in A_{x*y} \cap S$, that is, $A_{x*y} \cap S \neq \emptyset$. This shows that $x * y \in \overline{\Theta}(A, S)$, and consequently $\overline{\Theta}(A, S)$ is a KU-subalgebra of X.

(2) Let $x, y \in \underline{\Theta}(A, S)$. Then $A_x \subseteq S$ and $A_y \subseteq S$. Since S is a KU-subalgebra of X, it follows that

$$A_{x*y} = A_x * A_y \subseteq S$$

so that $x * y \in \underline{\Theta}(A, S)$. Hence $\underline{\Theta}(A, S)$ is a KU-subalgebra of X. □

The following example shows that the converse of Theorem 2(1) may not be true.

Example 4. Let $X = \{0, 1, 2, 3, 4\}$ be a KU-algebra with the Cayley's table as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	2	4
2	0	0	0	1	4
3	0	0	0	0	4
4	0	1	1	1	0

Let $A = \{0, 1, 2\}$ be a KU-ideal of X $(A \lhd X)$ and let Θ be an equivalence relation on X related to A. Then $A_0 = A_1 = A_2 = A$, $A_3 = \{3\}$, and $A_4 = \{4\}$. Note that $S = \{1, 3\}$ is not a KU-subalgebra of X, but $\overline{\Theta}(A, S) = \{0, 1, 2, 3\}$ is KU-subalgebra of X.

Definition 10. *A nonempty subset S of X is called an upper (resp. a lower) rough KU-ideal of X if the upper (resp. nonempty lower) approximation of S is a KU-ideal of X. If S is both an upper and a lower rough KU-ideal of X, we say that S is a rough KU-ideal of X.*

Theorem 3. *Let Θ be an congruence relation on X related to a KU-ideal A of X. If S is a KU-ideal of X containing A, then*
(1) $\overline{\Theta}(A, S)$ *is a KU-ideal of X.*
(2) $\underline{\Theta}(A, S)$ *is a KU-ideal of X.*

Proof. (1) Let S be a KU-ideal of X containing A. Obviously $0 \in \overline{\Theta}(A, S)$. Let $x, y, z \in X$ be such that $y \in \overline{\Theta}(A, S)$ and $x * (y * z) \in \overline{\Theta}(A, S)$. Then

$$A_y \cap S \neq \emptyset \text{ and } A_{x*(y*z)} \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in A_y$ and $b \in A_{x*(y*z)}$. Hence $(a, y) \in \Theta$ and $(b, (x*(y*z))) \in \Theta$, which implies $y*a \in A \subseteq S$ and $(x*(y*z))*b \in A \subseteq S$. Since $a, b \in S$ and S is a KU-ideal, we get

$$y \in S \text{ and } x * (y * z) \in S,$$

it follows from Definition 5(2) that $x * z \in S$. Note that $x * z \in A_{x*z}$, thus $x * z \in A_{x*z} \cap S$, that is, $A_{x*z} \cap S \neq \emptyset$. Hence $x * z \in \overline{\Theta}(A, S)$ and therefore $\overline{\Theta}(A, S)$ is a KU-ideal of X.

(2) Let S be a KU-ideal of X containing A. Let $x \in A_0$. Then $x \in A \subseteq S$, and so $A_0 \subseteq S$. Hence $0 \in \underline{\Theta}(A, S)$. Let $x, y, z \in X$ be such that $y \in \underline{\Theta}(A, S)$ and $x * (y * z) \in \underline{\Theta}(A, S)$. Then

$$A_y \in S \text{ and } A_x * (A_y * A_z) = A_{x*(y*z)} \subseteq S.$$

Let $w \in A_{x*z} = A_x * A_z$. Then $w = A_x * A_z$ for some $a \in A_x$ and $c \in A_z$. From $a \in A_x$ and $c \in A_z$, we have $(a, x) \in \Theta$ and $(c, z) \in \Theta$. Taking $b \in A_y$ then we get $(b, y) \in \Theta$. Since Θ is a congruence relation, we get

$$(a * (b * c), x * (y * z)) \in \Theta \text{ and so } a * (b * c) \in A_{x*(y*z)} \subseteq S.$$

Since S is a KU-ideal of X, it follows from Definition 5(2) that $w \in a * c \in S$, so that $A_{x*z} \subseteq S$. Hence $x * z \in \underline{\Theta}(A, S)$ and therefore $\underline{\Theta}(A, S)$ is a KU-ideal of X. □

Theorem 4. *Let Θ be an congruence relation on X related to a KU-ideal A of X. If S is a weak KU-ideal of X containing A, then*

(1) *$\overline{\Theta}(A, S)$ is a weak KU-ideal of X.*

(2) *$\underline{\Theta}(A, S)$ is a weak KU-ideal of X.*

Proof. (1) Let S be a weak KU-ideal of X containing A. Obviously $0 \in \overline{\Theta}(A, S)$. Let $x, y \in X$ be such that $y \in \overline{\Theta}(A, S)$ and $y * x \in \overline{\Theta}(A, S)$. Then

$$A_y \cap S \neq \emptyset \text{ and } A_{y*x} \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in A_y$ and $b \in A_{y*x}$. Hence $(a, y) \in \Theta$ and $(b, (y * x)) \in \Theta$, which implies

$$y * a \in A \subseteq S \text{ and } (y * x) * b \in A \subseteq S.$$

Since $a, b \in S$ and S is a weak KU-ideal, we get $y \in S$ and $y * x \in S$, it follows from Definition 6(1) that $x \in S$. Note that $x \in A_x$, thus $x \in A_x \cap S$, that is, $A_x \cap S \neq \emptyset$. Hence $x \in \overline{\Theta}(A, S)$ and therefore $\overline{\Theta}(A, S)$ is a weak KU-ideal of X.

(2) Let S be a weak KU-ideal of X containing A. Let $x \in A_0$. Then $x \in A \subseteq S$, and so $A_0 \subseteq S$. Hence $0 \in \underline{\Theta}(A, S)$. Let $x, y \in X$ be such that $y \in \underline{\Theta}(A, S)$ and $y * x \in \underline{\Theta}(A, S)$. Then

$$A_y \in S \text{ and } A_y * A_x = A_{y*x} \subseteq S.$$

Let $w \in A_x$. Then $w = A_x$ for some $a \in A_x$. From $a \in A_x$, we have $(a, x) \in \Theta$. Taking $b \in A_y$ then we get $(b, y) \in \Theta$. Since Θ is a congruence relation, we get

$$(b * a, y * x) \in \Theta \text{ and } b * a \in A_{y*x} \subseteq S.$$

Since S is a weak KU-ideal of X, it follows from Definition 6(1) that $w = a \in S$, so that $A_x \subseteq S$. Hence $x \in \underline{\Theta}(A, S)$ and therefore $\underline{\Theta}(A, S)$ is a weak KU-ideal of X. □

Theorem 5. *Let Θ be an congruence relation on X related to a KU-ideal A of X. If S is a strong KU-ideal of X containing A, then*

(1) *$\overline{\Theta}(A, S)$ is a strong KU-ideal of X.*

(2) *$\underline{\Theta}(A, S)$ is a strong KU-ideal of X.*

Proof. (1) Let $x, y \in X$ be such that

$$(y * x) \cap \overline{\Theta}(A, S) \neq \emptyset \text{ and } y \in \overline{\Theta}(A, S).$$

Then $A_y \cap S \neq \emptyset$ and so there exist $z \in X$ such that $z = y * x$ and $z \in \overline{\Theta}(A, S)$. Hence $A_z \cap S \neq \emptyset$ and so there exist $c, d \in X$ such that

$$c \in A_z \cap S \text{ and } d \in A_y \cap S.$$

Hence $c\Theta z$ and $d\Theta y$ where $c, d \in S$. Thus we $z * c \in A \subseteq S$ and $y * d \in A \subseteq S$. Since S is a strong KU-ideal and $c, d \in S$, we have $z \in S$ and $y \in S$. Thus we have proved that $(y * x) \cap A \neq \emptyset$ and $y \in A$. Since S is a strong KU-ideal, we have $x \in S$ and so $A_x \cap S \neq \emptyset$ which means that $\overline{\Theta}(A, S)$ is a strong KU-ideal of S.

(2) Let $x, y \in X$ be such that

$$(y * x) \cap \underline{\Theta}(A, S) \neq \emptyset \text{ and } y \in \underline{\Theta}(A, S).$$

Let $a \in A_x$ and $b \in A_y$. Then $a\Theta x$ and $b\Theta y$. Since Θ is a congruence relation on X, $b * a\Theta y * x$. Since $(y * x) \cap \underline{\Theta}(A, S) \neq \emptyset$, then there exist $t \in X$ such that $t \in y * x$ and $t \in \underline{\Theta}(A, S)$. Now, $t \in b * a\Theta y * x$ implies that there exist $z \in b * a$ such that $z\Theta t$ and so $A_t = A_z \subseteq S$. Hence $z \in S$ and so $(b * a) \cap S \neq \emptyset$. On the other hand, we have $b \in A_y \subseteq S$. Since S is a strong KU-ideal of X, then we have $a \in S$ which implies $A_x \subseteq S$ that means $x \in \underline{\Theta}(A, S)$. Therefore, $\underline{\Theta}(A, S)$ is a strong KU-ideal of S. □

4. Conclusion

Roughness is one of the important method to tackle the uncertainty and vagueness in information system. It is widely used in database management system more basically in data mining and big data. In recent years roughness has been applied and used successfully in a number of challenging fields such as pure and applied algebras, computer science, engineering, medical science and soft computing method in biology and and computer science.

This paper connects KU-algebras (KU-ideals) with roughness through definitions, examples and results based on lower and upper approximations of this logical algebras. We expect that this connection may invoke some new directions with other related and similar concepts in logical algebras and different types of KU-algebras including soft KU-algebras and hyper soft KU-algebras togather with some reasoning and logic. The next step in the direction of this work could be the results based on rough set approach to handle medical related data.

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