EXPONENTIAL STABILITY OF NONLINEAR SYSTEMS VIA ALTERNATE CONTROL

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Abstract. In this paper, the exponential stability of nonlinear systems via alternate control is considered. Our result avoids solving linear matrix inequalities. A numerical example is given to show effectiveness of the result.

Keywords: exponential stability, nonlinear systems, alternate control.

1. Introduction

Throughout this paper, let $\|x\|$ be the Euclidean norm of the vector x. $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ and A^T are the largest, the smallest eigenvalue and the transpose of a real symmetric matrix A, respectively. The symmetrical positive definite matrix A is represented by $A > 0$. I represents the proper dimension identity matrix. $f\left(x\left(t_0^-\right)\right)$ is defined by $f\left(x\left(t_0^-\right)\right) = \lim_{t\to t_0^-} f\left(x\left(t\right)\right)$.

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Recently, Feng et al.[2] considered nonlinear systems via alternate control as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + B_1 x(t), & mT \le t < mT + \tau, \\ \dot{x}(t) = Ax(t) + f(x(t)) + B_2 x(t), & mT + \tau \le t < (m+1)T, \end{cases} \tag{1.1}$$

where $x(t) \in R^n$ denotes the state vector, $f : R^n \to R^n$ is said to be a continuous nonlinear function if $f(0) = 0$, there exists a constant $l \ge 0$ such that $\|f(x)\| \le l\|x\|$, $A, B_1, B_2 \in R^{n\times n}$ are constant matrices, $T > 0$ denotes the control period, $\tau \in (0, T)$ is a constant.

Meanwhile, Feng et al.[2] presented three conditions to guarantee system (1.1) to be exponentially stable. Two conditions are to find $g_1 > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, $g_2 \in R$ and $0 < P \in R^{n\times n}$ satisfying

$$PA + A^T P + PB_1 + B_1^T P + \epsilon_1 P^2 + \epsilon_1^{-1} L + g_1 P \le 0 \tag{1.2}$$

and

$$PA + A^T P + PB_2 + B_2^T P + \epsilon_2 P^2 + \epsilon_2^{-1} L - g_2 P \le 0. \tag{1.3}$$

Inequalities (1.2) and (1.3) are equivalent to the linear matrix inequalities as follows:

$$\begin{bmatrix} PA + A^T P + PB_1 + B_1^T P + \epsilon_1^{-1} L + g_1 P & -P \\ -P & -\epsilon_1^{-1} I \end{bmatrix} \le 0$$

and

$$\begin{bmatrix} PA + A^T P + PB_2 + B_2^T P + \epsilon_2^{-1} L - g_2 P & -P \\ -P & -\epsilon_2^{-1} I \end{bmatrix} \le 0.$$

Although linear matrix inequalities can be solved in polynomial-time, the computation amount of solving linear matrix inequalities is not very small [1]. Therefore, our purpose is to find $h_1 < 0$, $h_2 \in R$ and $0 < P \in R^{n\times n}$ avoiding solving linear matrix inequalities such that the system (1.1) is exponentially stable. For more information on this topic and its applications have been presented in the literature, for instance, see [5-7, 9, 10].

2. Main result

We need two lemmas which play a major role in the proof of theorem.

Lemma 2.1 ([4]). *Let $x, y \in R^n$. Then*

$$\left|x^T y\right| \le \|x\| \|y\|.$$

Lemma 2.2 ([4]). *Let $A \in R^{n\times n}$ be a symmetric matrix. Then for all $x \in R^n$,*

$$\lambda_{\min}(A)\, x^T x \le x^T A x \le \lambda_{\max}(A)\, x^T x.$$

Theorem 2.1. *Let $0 < P \in R^{n\times n}$ such that the following two conditions are satisfied:*

(1) $h_1 < 0$,

(2) $h_1\tau + h_2(T-\tau) < 0$,

where $\beta_1 = \lambda_{\max}(P^{-1}(PA + A^TP + PB_1 + B_1^TP))$, $\beta_2 = \lambda_{\max}(P)$, $\beta_3 = \lambda_{\min}(P)$, $\beta_4 = \lambda_{\max}(P^{-1}(PA + A^TP + PB_2 + B_2^TP))$, $h_1 = \beta_1 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$, $h_2 = \beta_4 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$. Then, system (1.1) is exponentially stable at origin.

Proof. Let us construct the following Lyapunov function:

$$V(x(t)) = x^T(t)Px(t).$$

Let $t \in [mT, mT+\tau)$, by Lemma 2.1 and 2.2, we have

$$\begin{aligned} D^+(V(x(t))) &= 2x^T(t)P(Ax(t) + f(x(t)) + B_1x(t)) \\ &= x^T(t)\left(PA + A^TP + PB_1 + B_1^TP\right)x(t) + 2x^T(t)Pf(x(t)) \\ &\le \beta_1x^T(t)Px(t) + 2\sqrt{x^T(t)Px(t)f^T(x(t))Pf(x(t))} \\ &\le \beta_1x^T(t)Px(t) + 2\sqrt{x^T(t)Px(t)\beta_2f^T(x(t))f(x(t))} \\ &\le \beta_1x^T(t)Px(t) + 2\sqrt{x^T(t)Px(t)\beta_2l^2x^T(t)x(t)} \\ &\le \beta_1x^T(t)Px(t) + 2l\sqrt{x^T(t)Px(t)\frac{\beta_2}{\beta_3}x^T(t)Px(t)} \\ &= h_1V(x(t)), \end{aligned}$$

which means

$$V(x(t)) \le V\left(x(mT)^-\right)e^{h_1(t-mT)}. \tag{2.1}$$

Similarly, let $t \in [mT+\tau, (m+1)T)$, we have

$$\begin{aligned} D^+(V(x(t))) &= 2x^T(t)P(Ax(t) + f(x(t)) + B_2x(t)) \\ &= x^T(t)\left(PA + A^TP + PB_2 + B_2^TP\right)x(t) + 2x^T(t)Pf(x(t)) \\ &\le \beta_4x^T(t)Px(t) + 2\sqrt{x^T(t)Px(t)f^T(x(t))Pf(x(t))} \\ &\le h_2V(x(t)), \end{aligned}$$

which infers

$$V(x(t)) \le V\left(x(mT+\tau)^-\right)e^{h_2(t-mT-\tau)}. \tag{2.2}$$

When $m = 0$, let $t \in [0, \tau)$, from (2.1), we have

$$V(x(t)) \le V(x(0))e^{h_1t},$$

so

$$V\left(x\left(\tau^-\right)\right) \le V(x(0))e^{h_1\tau}. \tag{2.3}$$

Let $t \in [\tau, T)$, by (2.2) and (2.3), we have

$$\begin{aligned} V\left(x\left(t\right)\right) &\leq V\left(x\left(\tau^{-}\right)\right) e^{h_2(t-\tau)} \\ &\leq V\left(x\left(0\right)\right) e^{h_1\tau+h_2(t-\tau)}, \end{aligned}$$

so

$$V\left(x\left(T^{-}\right)\right) \leq V\left(x\left(0\right)\right) e^{h_1\tau+h_2(T-\tau)}. \tag{2.4}$$

When $m=1$, let $t \in [T, T+\tau)$, by (2.1) and (2.4), we have

$$\begin{aligned} V\left(x\left(t\right)\right) &\leq V\left(x\left(T^{-}\right)\right) e^{h_1(t-T)} \\ &\leq V\left(x\left(0\right)\right) e^{h_1\tau+h_2(T-\tau)+h_1(t-T)}, \end{aligned}$$

so

$$V\left(x\left((T+\tau)^{-}\right)\right) \leq V\left(x\left(0\right)\right) e^{2h_1\tau+h_2(T-\tau)}. \tag{2.5}$$

Let $t \in [T+\tau, 2T)$, by (2.2) and (2.5), we have

$$\begin{aligned} V\left(x\left(t\right)\right) &\leq V\left(x\left((T+\tau)^{-}\right)\right) e^{h_2(t-T-\tau)} \\ &\leq V\left(x\left(0\right)\right) e^{2h_1\tau+h_2(T-\tau)+h_2(t-T-\tau)}. \end{aligned}$$

By induction, when $m=k,\ k=0,1,\cdots$, let $t \in [kT, kT+\tau)$, we have

$$V\left(x\left(t\right)\right) \leq V\left(x\left(0\right)\right) e^{kh_1\tau+kh_2(T-\tau)+h_1(t-kT)}, \tag{2.6}$$

so

$$V\left(x\left((kT+\tau)^{-}\right)\right) \leq V\left(x\left(0\right)\right) e^{(k+1)h_1\tau+kh_2(T-\tau)}. \tag{2.7}$$

Let $t \in [kT+\tau, (k+1)T)$, by (2.2) and (2.7), we have

$$\begin{aligned} V\left(x\left(t\right)\right) &\leq V\left(x\left((kT+\tau)^{-}\right)\right) e^{h_2(t-kT-\tau)} \\ &\leq V\left(x\left(0\right)\right) e^{(k+1)h_1\tau+kh_2(T-\tau)+h_2(t-kT-\tau)}. \end{aligned} \tag{2.8}$$

By (2.6), we have

$$\begin{aligned} V\left(x\left(t\right)\right) &\leq V\left(x\left(0\right)\right) e^{kh_1\tau+kh_2(T-\tau)} \\ &< V\left(x\left(0\right)\right) e^{\frac{t-\tau}{T}(h_1\tau+h_2(T-\tau))} \\ &< V\left(x\left(0\right)\right) e^{\frac{t-T}{T}(h_1\tau+h_2(T-\tau))}, \end{aligned} \tag{2.9}$$

where $t \in [kT, kT+\tau)$.

By (2.8), we know

Case 1. When $h_2>0$, we have

$$\begin{aligned} V\left(x\left(t\right)\right) &< V\left(x\left(0\right)\right) e^{(k+1)h_1\tau+(k+1)h_2(T-\tau)} \\ &< V\left(x\left(0\right)\right) e^{\frac{t}{T}(h_1\tau+h_2(T-\tau))} \\ &< V\left(x\left(0\right)\right) e^{\frac{t-T}{T}(h_1\tau+h_2(T-\tau))}. \end{aligned} \tag{2.10}$$

Case 2. When $h_2 \leq 0$, we have

$$\begin{aligned} V(x(t)) &\leq V(x(0))\, e^{(k+1)h_1\tau + kh_2(T-\tau)} \\ &< V(x(0))\, e^{kh_1\tau + kh_2(T-\tau)} \\ &< V(x(0))\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))}. \end{aligned} \tag{2.11}$$

So, for any $h_2 \in R$, by (2.10) and (2.11), we have

$$V(x(t)) < V(x(0))\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))}, \tag{2.12}$$

where $t \in [kT+\tau, (k+1)T)$.
For all $t > 0$, we can conclude from (2.9) and (2.12) that

$$V(x(t)) < V(x(0))\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))}. \tag{2.13}$$

By Lemma 2.2 and (2.13), we have

$$\begin{aligned} \lambda_{\min}(P)\, \|x(t)\|^2 &\leq V(x(t)) \\ &< V(x(0))\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))} \\ &\leq \|x(0)\|^2\, \lambda_{\max}(P)\, e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau))}. \end{aligned}$$

That is

$$\|x(t)\| < \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\, \|x(0)\|\, e^{\frac{t-T}{2T}(h_1\tau + h_2(T-\tau))}.$$

This completes the proof. □

Remark 2.1. [2, Theorem 1] and [3, Theorem 1] need to solve the linear matrix inequalities, however, Theorem 2.1 avoids solving them.

3. A numerical example

In this section, we study the control of Chua's oscillator via employing above theoretical result.

Example 3.1. The Chua's system [8] is given as follows:

$$\begin{cases} \dot{x}_1 = \alpha(x_2 - x_1 - g(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\eta x_2. \end{cases} \tag{3.1}$$

where α and η are two parameters,

$$g(x_1) = bx_1 + 0.5(a-b)(|x_1+1| - |x_1-1|),$$

where a and b are two given constants satisfying $a < b < 0$.

In order to use the above result, we rewrite system (3.1) as follows:

$$\dot{x}(t) = Ax + f(x),$$

where

$$A = \begin{bmatrix} -\alpha - \alpha b & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\eta & 0 \end{bmatrix}, f(x) = \begin{bmatrix} -0.5\alpha(a-b)(|x_1+1| - |x_1-1|) \\ 0 \\ 0 \end{bmatrix}.$$

By calculations, we have

$$\begin{aligned} ||f(x)||^2 &= 0.25\alpha^2(a-b)^2[(x_1+1)^2 \\ &+ (x_1-1)^2 - 2|(x_1+1)(x_1-1)|] \\ &= 0.5\alpha^2(a-b)^2(x_1^2+1-|x_1^2-1|) \\ &= \begin{cases} \alpha^2(a-b)^2, & x_1^2 > 1 \\ \alpha^2(a-b)^2x^2, & x_1^2 \leq 1 \end{cases} \\ &\leq \alpha^2(a-b)^2x_1^2 \\ &\leq \alpha^2(a-b)^2(x_1^2+x_2^2+x_3^2). \end{aligned}$$

So, we choose $l^2 = \alpha^2(a-b)^2$.

In the initial condition $x(0) = (22, -2, -15)^T$, the Chua's system exhibits chaotic phenomenon when

$$\alpha = 9.2156,\ \eta = 15.9946,\ a = -1.24905,\ b = -0.75735,$$

as shown in Figure 1.

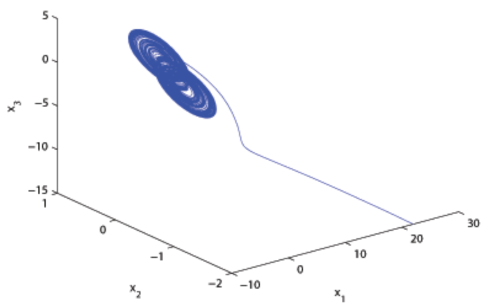

Figure 1: The chaotic phenomenon of (3.1) .

At the same time, for convenience of calculations, we choose $P = I$, $B_1 = diag(-12, -13, -14)$, $B_2 = diag(-12, -13, -12)$, $T = 1$ and $\tau = 0.5$. Small calculations show that $\beta_1 = -9.9296$, $\beta_2 = \beta_3 = 1$, $\beta_4 = -8.3946$, $l = 4.5313$, $h_1 = -0.8670$, $h_2 = 0.6680$ and $h_1\tau + h_2(T-\tau) = -0.0995 < 0$. Thus, in the initial condition $x(0) = (22, -2, -15)^T$, system (3.1) is exponentially stable by Theorem 2.1, as shown in Figure 2.

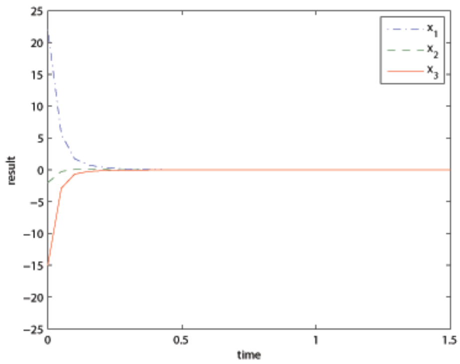

Figure 2: Time response curves of (3.1) via alternate control.

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