# ON THE CONFORMAL CURVATURE TENSOR OF $\epsilon$-KENMOTSU MANIFOLDS

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**Abstract.** The conformal curvature tensor under certain curvature conditions has been studied for an $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection. Finally, we give an example of a 3-dimensional $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection.

**Keywords:** $\epsilon$-Kenmotsu manifolds, semi-symmetric non-metric connection, $\eta$-Einstein manifold, conformal curvature tensor.

## 1. Introduction

In 1972, K. Kenmotsu [14] studied a class of contact Riemannian manifolds satisfying some special conditions. We call it Kenmotsu manifold. Kenmotsu manifolds have been studied by various authors such as J. B. Jun et al. [13], G. Pathak and U. C. De [17], M. M. Tripathi and N. Nakkar [19], A. Yildiz et al. [23] and many others. In 1993, A. Bejancu and K. L. Duggal [4] introduced the concept of ($\epsilon$)-Sasakian manifolds, which later on showed by X. Xufeng and C. Xiaoli [20] that the manifolds are real hypersurfaces of indefinite Kahlerian manifolds. An ($\epsilon$)-almost paracontact manifolds were introduced by M. M. Tripathi et al. [18], while the concept of ($\epsilon$)-Kenmotsu manifolds was introduced

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by U. C. De and A. Sarkar [8] who showed that the existence of new structure on indefinite metrics influences the curvatures.

In 1924, the idea of semi-symmetric linear connection on a differentiable manifold was introduced by A. Friedmann and J. A. Schouten [9]. In 1930, E. Bartolotti [3] gave a geometrical meaning of such a connection. In 1932, H. A. Hayden [12] introduced semi-symmetric metric connection in a Riemannian manifold and this was studied systematically by K. Yano [21].

A linear connection $\bar{\nabla}$ in a Riemannian manifold $M$ is said to be a semi-symmetric connection if the torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \tag{1.1}$$

satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y,$$

where $\eta$ is a non-zero 1-form associated with a vector field $\xi$ and is defined by $\eta(X) = g(X,\xi)$.

Further, a semi-symmetric connection is called a semi-symmetric non-metric connection [1], if

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) &= \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z)\\ &= -\eta(Y)g(X,Z) - \eta(Z)g(X,Y),\end{aligned} \tag{1.2}$$

where $X,Y,Z \in \chi(M)$ and $\chi(M)$ is the set of all differentiable vector fields on $M$.

Let $M$ be an $n$-dimensional $\epsilon$-Kenmotsu manifold and $\nabla$ be the Levi-Civita connection on $M$, the semi-symmetric non-metric connection $\bar{\nabla}$ on $M$ is given by [1]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X. \tag{1.3}$$

The semi-symmetric non-metric connection have been studied by several authors such as N. S. Agashe and M. R. Chafle [1], L. S. Das et al. [5], U. C. De and S. C. Biswas [6], U. C. De and D. Kamilya [7], S. K. Pandey et al. [16], Ajit Barman [2] and many others. Motivated by the above studies, in this paper we study certain curvature conditions of an $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection.

The paper is organized as follows : In Section 2, we give a brief introduction of an $\epsilon$-Kenmotsu manifold. In Section 3, we obtain the relation between the curvature tensor of an $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection and the Levi-Civita connection. In Section 4, we study quasi-conformally flat, $\xi$-conformally flat, pseudoconformally flat and $\phi$-conformally flat $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection and it is shown that in each case the manifold is an $\eta$-Einstein manifold. Section 5 is devoted to study $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection satisfying the curvature condition $\bar{S} \cdot \bar{C} = 0$.

**2. Preliminaries**

An $n$-dimensional smooth manifold $(M, g)$ is said to be an $\epsilon$-almost contact metric manifold [8], if it admits a $(1,1)$ tensor field $\phi$, a structure vector field $\xi$, a 1-form $\eta$ and an indefinite metric $g$ such that

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$g(\xi, \xi) = \epsilon, \tag{2.3}$$

$$\eta(X) = \epsilon g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y) \tag{2.5}$$

for all vector fields $X$, $Y$ on $M$, where $\epsilon$ is 1 or -1 according as $\xi$ is space like or time like vector field and rank $\phi$ is $(n-1)$. If

$$d\eta(X, Y) = g(X, \phi Y) \tag{2.6}$$

for every $X, Y \in \chi(M)$, then we say that $M(\phi, \xi, \eta, g, \epsilon)$ is an almost contact metric manifold. Also, we have

$$\phi\xi = 0, \;\; \eta(\phi X) = 0. \tag{2.7}$$

If an $\epsilon$-contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X, \tag{2.8}$$

where $\nabla$ denotes the Levi-Civita connection with respect to $g$, then $M$ is called an $\epsilon$-Kenmotsu manifold [8].
An $\epsilon$-almost contact metric manifold is an $\epsilon$-Kenmotsu manifold, if and only if

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi). \tag{2.9}$$

Further, in an $\epsilon$-Kenmotsu manifold, the following relations hold [8, 10, 11]:

$$(\nabla_X \eta)Y = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.10}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

$$R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi, \tag{2.12}$$

$$R(\xi, X)\xi = -R(X, \xi)\xi = X - \eta(X)\xi, \tag{2.13}$$

$$\eta(R(X, Y)Z) = \epsilon[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{2.14}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.15}$$

$$Q\xi = -\epsilon(n-1)\xi, \tag{2.16}$$

where $g(QX, Y) = S(X, Y)$.

$$S(\phi X, \phi Y) = S(X, Y) + \epsilon(n-1)\eta(X)\eta(Y). \tag{2.17}$$

We note that if $\epsilon = 1$ and the structure vector field $\xi$ is space like, then an $\epsilon$-Kenmotsu manifold is usual Kenmotsu manifold.

An $\epsilon$-Kenmotsu manifold $M$ is said to be an $\eta$-Einstein manifold if its Ricci tensor $S$ is of the form [22]

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.18}$$

where $a$ and $b$ are scalar functions of $\epsilon$.

## 3. Curvature tensor in an $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection

Let $M$ be an $n$-dimensional $\epsilon$-Kenmotsu manifold. The curvature tensor $\bar{R}$ with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{3.1}$$

By virtue of (1.1) and (3.1), we have

$$\begin{aligned}\bar{R}(X, Y)Z &= R(X, Y)Z + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &\quad + ((\nabla_X\eta)Z)Y - ((\nabla_Y\eta)Z)X,\end{aligned} \tag{3.2}$$

where

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the Riemannian curvature tensor of the connection $\nabla$. Using (2.10) in (3.2) we get

$$\begin{aligned}\bar{R}(X, Y)Z &= R(X, Y)Z + g(X, Z)Y - g(Y, Z)X \\ &\quad + (1+\epsilon)[\eta(Y)X - \eta(X)Y]\eta(Z).\end{aligned} \tag{3.3}$$

Now contracting $X$ in (3.3), we get

$$\bar{S}(Y, Z) = S(Y, Z) + (1-n)g(Y, Z) + (1+\epsilon)(n-1)\eta(Y)\eta(Z), \tag{3.4}$$

where $\bar{S}$ and $S$ are the Ricci tensors of the connections $\bar{\nabla}$ and $\nabla$, respectively on $M$.
This gives

$$\bar{Q}Y = QY + (1-n)Y + (1+\epsilon)(n-1)\eta(Y)\xi. \tag{3.5}$$

Contracting again $Y$ and $Z$ in (3.4), it follows that

$$\bar{r} = r + n(1-n) + (1+\epsilon)(n-1), \tag{3.6}$$

where $\bar{r}$ and $r$ are the scalar curvatures of the connections $\bar{\nabla}$ and $\nabla$, respectively on $M$.

**Lemma 3.1.** *Let $M$ be an $n$-dimensional $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection, then*

$$\bar{R}(X,Y)\xi = 0, \tag{3.7}$$

$$\bar{R}(\xi,X)Y = -(1+\epsilon)[g(X,Y)\xi - \eta(X)\eta(Y)\xi], \tag{3.8}$$

$$\bar{S}(Y,\xi) = 0, \tag{3.9}$$

$$\bar{Q}\xi = 0. \tag{3.10}$$

**Proof.** By replacing $Z = \xi$ in (3.3) and using (2.4) and (2.11), we get (3.7). (3.8) follows from (3.3) and (2.12). Taking $Z = \xi$ in (3.4) and using (2.2), (2.4) and (2.15), we get (3.9). From (3.5), (3.16) and (2.2), we get (3.10). □

**Definition 3.2.** *The conformal curvature tensor $\bar{C}$ of type $(1,3)$ in an $n$-dimensional $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ is given by* ([15], [22])

$$\begin{aligned}\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X \\ - g(X,Z)\bar{Q}Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{3.11}$$

*where $\bar{R}$, $\bar{S}$, $\bar{Q}$ and $\bar{r}$ are the Riemannain curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature with respect to the semi-symmetric non-metric connection, respectively. The Ricci tensor $\bar{S}$ and the Ricci operator $\bar{Q}$ are related by $g(\bar{Q}X,Y) = \bar{S}(X,Y)$.*

By using (3.3)-(3.6) in (3.11), we obtain

$$
\begin{aligned}
\bar{C}(X,Y)Z = C(X,Y)Z &+ \frac{2n+1+\epsilon}{n-2}(g(Y,Z)X - g(X,Z)Y) \\
&- \frac{1+\epsilon}{n-2}(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) \\
&- \frac{(1+\epsilon)(n-1)}{n-2}(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi),
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
C(X,Y)Z = R(X,Y)Z &- \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\
&- g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]
\end{aligned}
$$

is the conformal curvature tensor with respect to the Levi-Civita connection $\nabla$.

The equation (3.12) is the relation between the conformal curvature tensors with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ and the Levi-Civita connection $\nabla$.

## 4. Flatness conditions in $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection

**Definition 4.1.** *An $\epsilon$-Kenmotsu manifold $M$ is said to be:*
($i$) *quasi-conformally flat with respect to the semi-symmetric non-metric connection, if*

$$
g(\bar{C}(X,Y)Z, \phi W) = 0, \qquad X,Y,Z,W \in \chi(M);
\tag{4.1}
$$

($ii$) *$\xi$-conformally flat with respect to the semi-symmetric non-metric connection, if*

$$
\bar{C}(X,Y)\xi = 0, \qquad X,Y \in \chi(M);
\tag{4.2}
$$

($iii$) *pseudoconformally flat with respect to the semi-symmetric non-metric connection, if*

$$
g(\bar{C}(\phi X,Y)Z, \phi W) = 0, \qquad X,Y,Z,W \in \chi(M); \quad \textit{and}
\tag{4.3}
$$

($iv$) *$\phi$-conformally flat with respect to the semi-symmetric non-metric connection, if*

$$
\phi^2\bar{C}(\phi X, \phi Y)\phi Z = 0, \qquad X,Y,Z \in \chi(M).
\tag{4.4}
$$

Firstly, we consider quasi-conformally flat $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From the equations (3.11) and (4.1), we have

$$
\begin{aligned}
g(\bar{R}(X,Y)Z,\phi W) &= \frac{1}{(n-2)}[\bar{S}(Y,Z)g(X,\phi W)-\bar{S}(X,Z)g(Y,\phi W) \\
&\quad + g(Y,Z)g(\bar{Q}X,\phi W)-g(X,Z)g(\bar{Q}Y,\phi W)] \\
&\quad - \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(X,\phi W)-g(X,Z)g(Y,\phi W)]
\end{aligned} \tag{4.5}
$$

which by taking $Y = Z = \xi$ and using (2.4), (3.7), (3.9) and (3.10) reduces to

$$\bar{S}(X,\phi W) = \frac{\bar{r}}{(n-1)}g(X,\phi W). \tag{4.6}$$

Replacing $W$ by $\phi W$ and using (2.1), (4.6) yields

$$\bar{S}(X,W) = \frac{\bar{r}}{(n-1)}g(X,W) - \frac{\epsilon\bar{r}}{(n-1)}\eta(X)\eta(W). \tag{4.7}$$

In view of (3.4) and (3.6), (4.7) takes the form

$$S(X,W) = (\epsilon + \frac{r}{n-1})g(X,W) - (n + \frac{\epsilon r}{n-1})\eta(X)\eta(W). \tag{4.8}$$

Thus we have the following theorem:

**Theorem 4.2.** *An $n$-dimensional quasi-conformally flat $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an $\eta$-Einstein manifold with respect to the Levi-Civita connection.*

Secondly, we consider $\xi$-conformally flat $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From (3.11) and (4.2), we can write

$$
\begin{aligned}
& g[\bar{R}(X,Y)\xi - \frac{1}{(n-2)}(\bar{S}(Y,\xi)X - \bar{S}(X,\xi)Y + g(Y,\xi)\bar{Q}X - g(X,\xi)\bar{Q}Y) \\
& \qquad + \frac{\bar{r}}{(n-1)(n-2)}(g(Y,\xi)X - g(X,\xi)Y), W] = 0
\end{aligned} \tag{4.9}
$$

which by using (2.4), (3.7) and (3.9) reduces to

$$
\begin{aligned}
& \eta(X)\bar{S}(Y,W) - \eta(Y)\bar{S}(X,W) \\
& \quad + \frac{\bar{r}}{(n-1)}(\eta(Y)g(X,W) - \eta(X)g(Y,W)) = 0.
\end{aligned} \tag{4.10}
$$

Putting $Y = \xi$ and then using (2.2), (2.4) and (3.9) in (4.10), we get

$$\bar{S}(X,W) = \frac{\bar{r}}{(n-1)}g(X,W) - \frac{\epsilon\bar{r}}{(n-1)}\eta(X)\eta(W). \tag{4.11}$$

In view of (3.4) and (3.6), (4.11) takes the form

$$S(X,W) = (\epsilon + \frac{r}{n-1})g(X,W) - (n + \frac{\epsilon r}{n-1})\eta(X)\eta(W). \tag{4.12}$$

Thus we have the following theorem:

**Theorem 4.3.** *An n-dimensional $\xi$-conformally flat $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an $\eta$-Einstein manifold with respect to the Levi-Civita connection.*

Next, taking $Z = \xi$ in (3.12) and using (2.2) and (2.4), we have

$$\bar{C}(X,Y)\xi = C(X,Y)\xi + \frac{2n\epsilon}{n-2}(\eta(Y)X - \eta(X)Y). \tag{4.13}$$

Since $\eta(X)Y - \eta(Y)X = R(X,Y)\xi \neq 0$, in an $\epsilon$-Kenmotsu manifold, in general, then we have the following theorem:

**Theorem 4.4.** *In an $\epsilon$-Kenmotsu manifold $\xi$-conformally flatness with respect to the semi-symmetric non-metric connection and the Levi-Civita connection are not equivalent.*

Thirdly, we consider pseudoconformally flat $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From (3.11) and (4.3), we can write

$$\begin{aligned} g(\bar{R}(\phi X,Y)Z,\phi W) &= \frac{1}{(n-2)}[\bar{S}(Y,Z)g(\phi X,\phi W) \\ &- \bar{S}(\phi X,Z)g(Y,\phi W) + \bar{S}(\phi X,\phi W)g(Y,Z) - \bar{S}(Y,\phi W)g(\phi X,Z)] \\ &- \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)g(\phi X,\phi W) - g(\phi X,Z)g(Y,\phi W)]. \end{aligned} \tag{4.14}$$

In view of (3.3), (3.4) and (3.6), (4.14) takes the form

$$\begin{aligned} g(R(\phi X,Y)Z,\phi W) &= \frac{1}{(n-2)}[S(Y,Z)g(\phi X,\phi W) \\ &- 2(n-1)g(Y,Z)g(\phi X,\phi W) + (1+\epsilon)\eta(Y)\eta(Z)g(\phi X,\phi W) \\ &- S(\phi X,Z)g(Y,\phi W) + 2(n-1)g(\phi X,Z)g(Y,\phi W) \\ &+ g(Y,Z)S(\phi X,\phi W) - g(\phi X,Z)S(Y,\phi W)] \\ &- \frac{r-(n-1)(2n-3-\epsilon)}{(n-1)(n-2)}[g(Y,Z)g(\phi X,\phi W) - g(\phi X,Z)g(Y,\phi W)]. \end{aligned} \tag{4.15}$$

Let $\{e_1, e_2, \ldots.., e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields in $M$. Using that $\{\phi e_1, \phi e_2, \ldots.., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (4.15) and sum up with respect to $i$, then we have

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\phi e_i,Y)Z,\phi e_i) &= \frac{1}{(n-2)}\sum_{i=1}^{n-1}[S(Y,Z)g(\phi e_i,\phi e_i) \\ &- 2(n-1)g(Y,Z)g(\phi e_i,\phi e_i) + (1+\epsilon)\eta(Y)\eta(Z)g(\phi e_i,\phi e_i) \\ &- S(\phi e_i,Z)g(Y,\phi e_i) + 2(n-1)g(\phi e_i,Z)g(Y,\phi e_i) \\ &+ g(Y,Z)S(\phi e_i,\phi e_i) - g(\phi e_i,Z)S(Y,\phi e_i)] \\ &- \frac{r-(n-1)(2n-3-\epsilon)}{(n-1)(n-2)}\sum_{i=1}^{n-1}[g(Y,Z)g(\phi e_i,\phi e_i) - g(\phi e_i,Z)g(Y,\phi e_i)]. \end{aligned} \tag{4.16}$$

It is easy to verify that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, Y)Z, \phi e_i) = S(Y,Z) + g(Y,Z) - \epsilon\eta(Y)\eta(Z), \tag{4.17}$$

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{4.18}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r + n - 1, \tag{4.19}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y,Z) + \epsilon(n-1)\eta(Y)\eta(Z), \tag{4.20}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{4.21}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1, \tag{4.22}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, Z)g(Y, \phi e_i) = g(Y,Z) - \eta(Y)\eta(Z), \tag{4.23}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{4.24}$$

By virtue of (4.17), (4.19), (4.20), (4.22) and (4.23), (4.16) yields

$$S(Y,Z) = [\frac{r}{n-1} - n + 3 - \epsilon(n-2)]g(Y,Z) - [\frac{r}{n-1} - n + 2(1+\epsilon)]\eta(Y)\eta(Z). \tag{4.25}$$

Thus we have the following theorem:

**Theorem 4.5.** *An $n$-dimensional pseudoconformally flat $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an $\eta$-Einstein manifold with respect to the Levi-Civita connection.*

Lastly, we consider $\phi$-conformally flat $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From (4.4), we have

$$g(\bar{C}(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{4.26}$$

Using (3.11), (4.26) can be written as

$$\begin{aligned} g(\bar{R}(\phi X, \phi Y)\phi Z, \phi W) &= \frac{1}{(n-2)}[\bar{S}(\phi Y, \phi Z)g(\phi X, \phi W) \\ &- \bar{S}(\phi X, \phi Z)g(\phi Y, \phi W) + \bar{S}(\phi X, \phi W)g(\phi Y, \phi Z) - \bar{S}(\phi Y, \phi W)g(\phi X, \phi Z)] \\ &- \frac{\bar{r}}{(n-1)(n-2)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)]. \end{aligned} \tag{4.27}$$

In view of (3.3), (3.4) and (3.6), (4.27) takes the form

$$\begin{aligned} g(R(\phi X,\phi Y)\phi Z,\phi W) &= \frac{1}{(n-2)}[S(\phi Y,\phi Z)g(\phi X,\phi W) \\ &-S(\phi X,\phi Z)g(\phi Y,\phi W)+S(\phi X,\phi W)g(\phi Y,\phi Z)-S(\phi Y,\phi W)g(\phi X,\phi Z)] \\ &-\frac{r+(n-1)(1+\epsilon)}{(n-1)(n-2)}[g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \tag{4.28}$$

Let $\{e_1, e_2, ....., e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields in $M$. Using that $\{\phi e_1, \phi e_2, ....., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (4.28) and sum up with respect to $i$, then we have

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) &= \frac{1}{(n-2)}\sum_{i=1}^{n-1}[S(\phi Y,\phi Z)g(\phi e_i,\phi e_i) \\ &-S(\phi e_i,\phi Z)g(\phi Y,\phi e_i)+S(\phi e_i,\phi e_i)g(\phi Y,\phi Z)-S(\phi Y,\phi e_i)g(\phi e_i,\phi Z)]. \\ &-\frac{r+(n-1)(1+\epsilon)}{(n-2)(n-2)}\sum_{i=1}^{n-1}[g(\phi Y,\phi Z)g(\phi e_i,\phi e_i)-g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)]. \end{aligned} \tag{4.29}$$

By virtue of (4.18), (4.19), (4.21), (4.22) and (4.24), (4.29) can be written as

$$\begin{aligned} S(\phi Y,\phi Z)+g(\phi Y,\phi Z) &= \frac{1}{(n-2)}[(n-3)S(\phi Y,\phi Z) \\ &+(r+n-1)g(\phi Y,\phi Z)]-\frac{r+(n-1)(1+\epsilon)}{(n-1)}g(\phi Y,\phi Z) \end{aligned} \tag{4.30}$$

from which it follows that

$$S(\phi Y,\phi Z)=(n-2)[\frac{r+n-1}{n-2}-\frac{r+(n-1)(1+\epsilon)}{n-1}-1]g(\phi Y,\phi Z). \tag{4.31}$$

Using (2.5) and (2.17), (4.31) yields

$$\begin{aligned} S(Y,Z) &= [\frac{r}{n-1}-(n-2)\epsilon-n+3]g(Y,Z) \\ &-\epsilon[\frac{r}{n-1}-(n-2)\epsilon+2]\eta(Y)\eta(Z). \end{aligned} \tag{4.32}$$

Thus we have the following theorem:

**Theorem 4.6.** *An $n$-dimensional $\phi$-conformally flat $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an $\eta$-Einstein manifold with respect to the Levi-Civita connection.*

## 5. $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection satisfying the condition $\bar{S}\cdot\bar{C}=0$

In this section we investigate $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection satisfying the condition $\bar{S}\cdot\bar{C}=0$, where $\bar{S}$ is the Ricci tensor with respect to the semi-symmetric non-metric connection of type $(0,2)$. Let the manifold satisfies the condition

$$(\bar{S}(X,Y)\cdot\bar{C})(U,V)W=0, \tag{5.1}$$

where $X,Y,U,V,W\in\chi(M)$. The above equation implies

$$\begin{aligned}&(X_{\wedge\bar{S}}Y)\bar{C}(U,V)W+\bar{C}((X_{\wedge\bar{S}}Y)U,V)W\\&+\bar{C}(U,(X_{\wedge\bar{S}}Y)V)W+\bar{C}(U,V)(X_{\wedge\bar{S}}Y)W=0,\end{aligned} \tag{5.2}$$

where the endomorphism $X\wedge_{\bar{S}}Y$ is defined by

$$(X\wedge_{\bar{S}}Y)Z=\bar{S}(Y,Z)X-\bar{S}(X,Z)Y. \tag{5.3}$$

Therefore in view of (5.3), (5.2) takes the form

$$\begin{aligned}&\bar{S}(Y,\bar{C}(U,V)W)X-\bar{S}(X,\bar{C}(U,V)W)Y+\bar{S}(Y,U)\bar{C}(X,V)W\\&-\bar{S}(X,U)\bar{C}(Y,V)W+\bar{S}(Y,V)\bar{C}(U,X)W-\bar{S}(X,V)\bar{C}(U,Y)W\\&+\bar{S}(Y,W)\bar{C}(U,V)X-\bar{S}(X,W)\bar{C}(U,V)Y=0.\end{aligned} \tag{5.4}$$

Taking $X=\xi$ in (5.4) and using (3.9), we obtain

$$\begin{aligned}&\bar{S}(Y,\bar{C}(U,V)W)\xi+\bar{S}(Y,U)\bar{C}(\xi,V)W\\&+\bar{S}(Y,V)\bar{C}(U,\xi)W+\bar{S}(Y,W)\bar{C}(U,V)\xi=0\end{aligned} \tag{5.5}$$

which by taking $U=W=\xi$ and then using (3.9) reduces to

$$\bar{S}(Y,\bar{C}(\xi,V)\xi)\xi+\bar{S}(Y,V)\bar{C}(\xi,\xi)\xi=0. \tag{5.6}$$

Using (3.11) in (5.6), we obtain

$$\bar{S}^2(Y,V)=\frac{\bar{r}}{n-1}\bar{S}(Y,V). \tag{5.7}$$

In view of (3.4)-(3.6) and (3.9), (5.7) yields

$$\begin{aligned}S^2(Y,V)=&(\frac{r}{n-1}+n-1+\epsilon)S(X,Y)-(r+n\epsilon-\epsilon)g(Y,V)\\&+(1+\epsilon)(r+n^2-n)\eta(Y)\eta(V).\end{aligned} \tag{5.8}$$

Thus we can state the following theorem:

**Theorem 5.1.** *If an $n$-dimensional $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ satisfying the condition $\bar{S}\cdot\bar{C}=0$. Then*

$$S^2(Y,V)=(\frac{r}{n-1}+n-1+\epsilon)S(X,Y)-(r+n\epsilon-\epsilon)g(Y,V)$$
$$+(1+\epsilon)(r+n^2-n)\eta(Y)\eta(V).$$

**Example.** We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$, where $(x, y, z)$ are standard coordinates of $R^3$. Let $e_1$, $e_2$ and $e_3$ be the vector fields on $M$ given by

$$e_1 = \epsilon z\frac{\partial}{\partial x}, \quad e_2 = \epsilon z\frac{\partial}{\partial y}, \quad e_3 = -\epsilon z\frac{\partial}{\partial z} = \xi$$

which are linearly independent at each point of $M$ and hence form a basis of $M$. Define an indefinite metric $g$ on $M^3$ as

$$g(e_1, e_1) = \quad g(e_2, e_2) = \quad g(e_3, e_3) = \epsilon,$$

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let $\eta$ be the 1-form on $M$ defined as $\eta(X) = \epsilon g(X, e_3)$ for all $X \in \chi(M)$, and let $\phi$ be the $(1, 1)$ tensor field on $M$ defined as

$$\phi e_1 = -e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.$$

By applying linearity of $\phi$ and $g$, we have

$$\eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\phi X) = 0,$$

$$g(X, \xi) = \epsilon\eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$. Then for $\xi = e_3$, the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost contact metric structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to the indefinite metric $g$. Then we have

$$[e_1, e_2] = 0, \quad [e_2, e_3] = \epsilon e_2, \quad [e_1, e_3] = \epsilon e_1.$$

The Riemannian connection $\nabla$ of the indefinite metric $g$ is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ & + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = -\epsilon e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = \epsilon e_1, \quad \nabla_{e_2} e_1 = 0, \tag{5.9}$$

$$\nabla_{e_2} e_2 = -\epsilon e_3, \quad \nabla_{e_2} e_3 = \epsilon e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

Thus from (5.9), it follows that the manifold satisfies $\nabla_X \xi = \epsilon(X - \eta(X)\xi)$ for $\xi = e_3$. Hence the manifold is an indefinite Kenmotsu manifold.

From the equation (5.9) and the expression of curvature tensor $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, it can be easily verified that

$$R(e_1, e_2)e_1 = e_2, \quad R(e_1, e_3)e_1 = e_3, \quad R(e_2, e_3)e_1 = 0, \tag{5.10}$$

$$R(e_1,e_2)e_2=-e_1,\qquad R(e_1,e_3)e_2=0,\qquad R(e_2,e_3)e_2=e_3,$$

$$R(e_1,e_2)e_3=0,\qquad R(e_1,e_3)e_3=-e_1,\quad R(e_2,e_3)e_3=-e_2.$$

By using (1.3) in (5.9), we obtain

$$\bar\nabla_{e_1}e_1=-\epsilon e_3,\ \ \bar\nabla_{e_2}e_1=0,\ \ \bar\nabla_{e_3}e_1=0,\ \ \bar\nabla_{e_1}e_2=0,\ \ \bar\nabla_{e_2}e_2=-\epsilon e_3, \tag{5.11}$$

$$\bar\nabla_{e_3}e_2=0,\ \ \bar\nabla_{e_1}e_3=(1+\epsilon)e_1,\ \ \bar\nabla_{e_2}e_3=(1+\epsilon)e_2,\ \ \bar\nabla_{e_3}e_3=e_3.$$

From (3.3) and (5.10), we can easily obtain the following components of the curvature tensor with respect to the semi-symmetric non-metric connection as

$$\bar R(e_1,e_2)e_1=(1+\epsilon)e_2,\quad \bar R(e_1,e_3)e_1=(1+\epsilon)e_3,\quad \bar R(e_2,e_3)e_1=0, \tag{5.12}$$

$$\bar R(e_1,e_2)e_2=-(1+\epsilon)e_1,\quad \bar R(e_1,e_3)e_2=0,\quad \bar R(e_2,e_3)e_2=(1+\epsilon)e_3,$$

$$\bar R(e_1,e_2)e_3=0,\quad \bar R(e_1,e_3)e_3=0,\quad \bar R(e_2,e_3)e_3=0.$$

By using the above expressions, we get the Ricci tensors and the scalar curvatures as follows:

$$S(e_1,e_1)=S(e_2,e_2)=S(e_3,e_3)=-2\epsilon,$$

$$\bar S(e_1,e_1)=\bar S(e_2,e_2)=-(1+\epsilon),\quad \bar S(e_3,e_3)=-2(1+\epsilon),$$

$$r=-6\epsilon,\qquad \bar r=-4(1+\epsilon).$$

By using the equations (1.1) and (5.11), the torsion tensors are given by

$$\overline{T}(e_1,e_1)=0,\quad \overline{T}(e_1,e_3)=e_1,\quad \overline{T}(e_1,e_2)=0,$$

$$\overline{T}(e_2,e_2)=0,\quad \overline{T}(e_2,e_3)=e_2,\quad \overline{T}(e_3,e_3)=0.$$

In view of (1.2), we find

$$(\bar\nabla_{e_1}g)(e_1,e_3)=-\epsilon,\quad (\bar\nabla_{e_2}g)(e_2,e_3)=-\epsilon,\quad (\bar\nabla_{e_3}g)(e_1,e_2)=0.$$

Hence $M$ is a 3-dimensional $\epsilon$-Kenmotsu manifold with respect to the semi symmetric non-metric connection.

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