FINITE GROUPS WHOSE ALL PROPER SUBGROUPS ARE GPST-GROUPS

Pengfei Guo*
College of Mathematics and Statistics
Hainan Normal University
Haikou, 571158
P. R. China
guopf999@163.com

Yue Yang
College of Mathematics and Statistics
Hainan Normal University
Haikou, 571158
P. R. China
1441568839@qq.com

Abstract. A set $\mathcal{W} = \{W_1, \ldots, W_t\}$ of nilpotent Hall subgroups of G is a complete Wielandt set if $(|W_i|, |W_j|) = 1$ for all i, j. A finite group G is called a GPST-group if G has a complete Wielandt set $\mathcal{W}$ such that every member in $\mathcal{W}$ permutes all maximal subgroups of any non-cyclic subgroup S in $\mathcal{W}$. In this paper, we give a complete classification of those groups which are not GPST-groups but all of whose proper subgroups are GPST-groups, i.e., they are precisely minimal non-PST-groups.

Keywords: Wielandt set, GPST-group, supersoluble group, power automorphism, permutable subgroup.

1. Introduction

All groups considered in this paper are finite and our notation is standard.

Let Σ be an abstract group theoretical property, for example, nilpotency, supersolubility, solubility, etc. If all proper subgroups of a group G have the property Σ but G does not have it, then G is called a minimal non-Σ-group.

The structures of minimal non-Σ-groups have been studied for various classes of groups Σ, and many classical results about this topic have been obtained. For instance, Miller and Moreno [8], Schmidt [12], and Doerk [5] analyzed the structures of minimal non-abelian groups, minimal non-nilpotent groups, and minimal non-supersoluble groups, respectively. However, the complete classifications of minimal non-nilpotent groups and minimal non-supersoluble groups were given by Ballester-Bolinches, Esteban-Romero and Robinson [4], Ballester-Bolinches and Esteban-Romero [2], respectively.

*. Corresponding author

On the other hand, Robinson [10] characterized minimal non-T-groups (T-groups are groups in which normality is a transitive relation, i. e., if the normality of H in K and of K in G always imply that H is normal in G). A subgroup H of G is said to be s-permutable in G if H permutes all Sylow subgroups of G. Agrawal [1] studied PST-groups, i. e., groups in which Sylow permutability is a transitive relation. A group is a soluble PST-group if and only if it has an abelian Hall subgroup L of odd order such that G/L is nilpotent, and every element of G induces a power automorphism in L. Robinson [11] also gave a complete classification of minimal non-PST-groups.

The aim in the present work is to determine the structure of a kind of minimal non-Σ-groups. Guo and Skiba [6] called a set $\mathcal{W} = \{W_1, \dots, W_t\}$ of nilpotent Hall subgroups of G a complete Wielandt set if $(|W_i|, |W_j|) = 1$ for all i, j, and characterized the structure of a group G which has a complete Wielandt set $\mathcal{W}$ such that every member in $\mathcal{W}$ permutes all maximal subgroups of any non-cyclic subgroup S in $\mathcal{W}$. The specific result is as follows.

Theorem A [6, Theorem A]**.** A group G has a complete Wielandt set of subgroups $\mathcal{W}$ such that every member in $\mathcal{W}$ permutes all maximal subgroups of any non-cyclic subgroup S in $\mathcal{W}$ if and only if $G = D \rtimes M$ is a supersoluble group where $D = G^{\mathcal{N}}$ is a nilpotent Hall subgroup of G of odd order whose maximal subgroups are normal in G.

In view of the structure of a group described in Theorem A is very close to soluble PST-group but weaker than soluble PST-group, so Guo and Skiba [6] called this group a generalized PST-group or GPST-group for short.

In this paper, we give a complete classification of those groups which are not GPST-groups but all of whose proper subgroups are GPST-groups. Our main result is as follows:

Theorem 1.1. *Let p and q be distinct prime divisors of the order of a group G. Then G is a minimal non-GPST-group if and only if G is one of the following types:*

(I) *$G = P \rtimes Q$, where $P = \langle a, b\rangle$ is an elementary abelian p-group of order p^2, and $Q = \langle x\rangle$ is cyclic of order q^r. Define $a^x = a^i$, $b^x = b^{i^j}$, $p \equiv 1 (\mathrm{mod}\ q^f)$, and $r \geq 1$, where i is the least positive primitive q^f-th root of unity modulo p, $j = 1 + kq^{f-1}$, with $0 < k < q$ and $r \geq f$;*

(II) *$G = P \rtimes Q$, where $Q = \langle x\rangle$ is cyclic of order $q^r > 1$, with $q \nmid p - 1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle x^q\rangle$ in Q;*

(III) *$G = P \rtimes Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, $Q = \langle x\rangle$ is cyclic of order $q^r > 1$, x induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and x centralizes $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$;*

(IV) *$G = PQ$, where $P = \langle a_0, a_1, \dots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle x\rangle$ is cyclic of order q^r, q^f is the highest power of q dividing*

$p-1$ and $r > f \geq 1$. Define $a_j^x = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^x = a_0^i$, where i is a primitive q^f-th root of unity modulo p.

Coincidentally, by comparing with main result in [11], minimal non-GPST-groups are precisely minimal non-PST-groups. In addition, Ballester-Bolinches and Esteban-Romero [3] introduced an interesting definition. Let p be a prime. A group G is said to be a $\mathcal{Y}_p$-group if, for all p-subgroups H and S of G such that $H \leq S$, H is S-permutable in $N_G(S)$. They also gave that: a group G is a soluble PST-group if and only if G is a $\mathcal{Y}_p$-group for all primes p [3, Theorem 4]. Hence the classification of minimal non-$\mathcal{Y}_p$-group in [4, Theorem 2] may be regarded as a local approach to the classification of minimal non-PST-groups. Our result is given naturally.

Corollary 1.2. *Let G be a group. Then the following conditions are equivalent:*

(i) *G is a minimal non-PST-group.*

(ii) *G is a minimal non-GPST-group.*

(iii) *G is a minimal non-$\mathcal{Y}_p$-group for every prime divisor p of the order of G.*

2. Preliminary results

We collect some lemmas which will be frequently used in the sequel.

Lemma 2.1 ([7]). *Let $\{P_1, P_2, \ldots, P_r\}$ be a Sylow basis of a soluble group G. Then the following statements are equivalent:*

(a) *Every subgroup of P_i permutes every subgroup of P_j for $i \neq j$.*

(b) *The nilpotent residual $G^{\mathcal{N}}$ of G is an abelian Hall subgroup of G, and every element of G induces a power automorphism in $G^{\mathcal{N}}$.*

Lemma 2.2. *Let G be a minimal non-GPST-group. Then there exists a normal non-cyclic Sylow p-subgroup P of G and a non-normal cyclic Sylow q-subgroup Q of G with $p \neq q$ such that $|G| = p^a q^b$ for positive integers a and b.*

Proof. Since every proper subgroup of G is a GPST-group, G is supersoluble or minimal non-supersoluble by Theorem A. By a result of Doerk [5], G is soluble and G has a nontrivial normal Sylow p-subgroup $P = O_p(G) \neq 1$, for some prime p. Let $G = P \rtimes H$ and $\mathcal{W} = \{P, H_1, \ldots, H_t\}$ of nilpotent Hall subgroups of G be a complete Wielandt set, where H is a p'-group of G. If $t \geq 2$, then $PH_1, PH_2, \ldots, PH_t$ are GPST-groups. Thus $H_1, H_2, \ldots, H_t$ permute every maximal subgroup of P whether or not P is cyclic. Since the normality of P and the fact that H is a GPST-group again, G is a GPST-group, a contradiction. Hence $t = 1$. Similar arguments as above, if $|\pi(H)| \geq 2$, then G is a GPST-group, a contradiction. So $H = Q \in \mathrm{Syl}_q(G)$ with $q \neq p$ a prime. If Q is non-cyclic, then $\langle x, P\rangle \neq G$ for every element x of Q. The minimality of G implies that $\langle x, P\rangle$ is a GPST-group. By applying Theorem A, every maximal subgroup of P is normal in G. By induction again, every subgroup of

P permutes every subgroup of $\langle x\rangle$. By Lemma 2.1, P is abelian and x induces a power automorphism on P. So G is a GPST-group, a contradiction. This induces that $H = \langle x\rangle$, where $|x| = q^b > 1$. Clearly, P is non-cyclic by the definition of GPST-group. The proof of Lemma 2.2 is complete. □

Lemma 2.3 ([14],Lemma 5). *Suppose $G = P\langle x\rangle$, P is a normal p-subgroup of G and x is a q-element. If all maximal subgroups of Sylow subgroups of G are normal in G, then x induces a power automorphism on $P/\Phi(P)$.*

Lemma 2.4 ([9],13.4.3). *Let α be a power automorphism of an abelian group A. If A is a p-group of finite exponent, then there is a positive integer l such that $a^\alpha = a^l$ for all a in A. If α is nontrivial and has order prime to p, then α is fixed-point-free.*

Lemma 2.5 ([5]). *Let G be a minimal non-supersoluble group. Then*

(1) *G has a unique normal Sylow p-subgroup P;*

(2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, and $P/\Phi(P)$ is non-cyclic;*

(3) *If $p \neq 2$, then the exponent of P is p;*

(4) *If P is non-abelian and $p = 2$, then the exponent of P is 4;*

(5) *If P is abelian, then the exponent of P is p.*

3. The proof of Theorem 1.1

Proof. If G is a minimal non-GPST-group, then we may assume $G = PQ$ by Lemma 2.2, where P is a non-cyclic normal Sylow p-subgroup of G and $Q = \langle x\rangle$ is a non-normal Sylow q-subgroup of G of order q^r. Since all Sylow q-subgroups are conjugate in G, we only consider the case that Q acts on P. So we inverstigate the following two cases.

(1) Assume that G is supersoluble and $d(P) = k$, where $d(P)$ is the rank of P.

Let $1 \trianglelefteq \cdots \trianglelefteq R \trianglelefteq P \trianglelefteq \cdots \trianglelefteq G$ be an arbitrary chief series of G. By Maschke's Theorem [9, Theorem 8.1.2], there exists a normal subgroup N of G contained in P such that $P/\Phi(P) = R/\Phi(P) \times N/\Phi(P)$, where $|N/\Phi(P)| = p$. Clearly, $N \not\leq R$ and $1 \trianglelefteq N \trianglelefteq P \trianglelefteq G$ is a normal series of G. By applying Schreier's Refinement Theorem [9, Theorem 3.1.2], P has another maximal subgroup $K \neq R$ such that K is normal in G. Therefore, P has at least two maximal subgroups R and K which are normal in G.

Now we prove $k = 2$. If $k \geq 3$, then we can let $P/\Phi(P) = \langle \bar{a_1}\rangle \times \langle \bar{a_2}\rangle \times \cdots \times \langle \bar{a_k}\rangle$ where $a_1, a_2, \ldots, a_{k-1} \in R$, $a_2, a_3, \ldots, a_k \in K$. Since $R\langle x\rangle$ is a GPST-group, every maximal subgroup of R is normal in G. By Lemma 2.3, $(y\Phi(R))^x = y^l\Phi(R)$ for every $y \in R$, where l is a positive integer. Thus, $(y\Phi(P))^x = y^l\Phi(P)$ for every $y \in R$. Similarly, $(z\Phi(P))^x = z^m\Phi(P)$ for every $z \in K$, where m is a positive integer. Furthermore, $a_2^l\Phi(P) = (a_2\Phi(P))^x = a_2^m\Phi(P)$, and so $l \equiv m (\mathrm{mod}\ p)$. Hence, $(a_i\Phi(P))^x = a_i^l\Phi(P)$ for $i = 1, 2, \ldots, k$. It is easy to see that every maximal subgroup of P is normal in G. By Theorem A, G is a GPST-group. This contradiction implies $k = 2$.

Now we let $P/\Phi(P) = R/\Phi(P) \times K/\Phi(P) = \langle \bar{a_1} \rangle \times \langle \bar{a_2} \rangle$, where $a_1 \in R, a_2 \in K$, $\bar{a_1}^x = \bar{a_1}^{k_1}$ and $\bar{a_2}^x = \bar{a_2}^{k_2}$. If $k_1 = k_2$, then every maximal subgroup of P is normal in G, and so G is a GPST-group, a contradiction. Hence, $k_1 \neq k_2$. Furthermore, we have that P has only two maximal subgroups which are normal in G. Clearly, at least one action of which x acts on R and K is nontrivial. Without loss of generality, we may assume that x induces an automorphism α on R. Since every subgroup of $R\langle x \rangle$ is a GPST-group and by induction, it follows from Theorem A that every subgroup of R permutes every subgroup of $\langle x \rangle$. By Lemma 2.1, R is abelian and α is a power automorphism on R. By Lemma 2.4, α is fixed-point-free. Hence, we have either $K \cap R = 1$ if $K\langle x \rangle = K \times \langle x \rangle$ or $K\langle x \rangle \neq K \times \langle x \rangle$. If $K \cap R = 1$ and $K\langle x \rangle = K \times \langle x \rangle$, then $P = \langle a, b \rangle$ is an elementary abelian group of order p^2. We can easily have that G is of type (I) with $f = 1$ and $k = q - 1$.

If $K\langle x \rangle \neq K \times \langle x \rangle$, similar arguments as above, K is abelian, and x induces a power automorphism in K. Thus, $\Phi(P) = R \cap K \leq Z(P)$. If $|P : Z(P)| \leq p$, then P is abelian.

We prove that P is elementary abelian. Let $\Omega_1(P)$ be the group generated by all elements of order p in P and assume that $\Omega_1(P) \neq P$. Then $\langle \Omega_1(P), x \rangle \neq G$ and it is a GPST-group. Therefore x induces a power automorphism in $\Omega_1(P)$, i. e., there is a positive integer t, relatively prime to p, such that $a^x = a^t$ for all $a \in \Omega_1(P)$. Let β be the automorphism of P induced by x and let γ be the automorphism of P in which $a \mapsto a^t$. Then $\beta\gamma^{-1}$ is an automorphism of P fixing each element of order p and $\beta\gamma^{-1}$ has order equal to a power of p, say p^d. Obviously $\beta\gamma = \gamma\beta$, so $\beta^{p^d} = \gamma^{p^d} \in \langle \gamma \rangle$. But β has order prime to p, so $\beta \in \langle \gamma \rangle$ and β is a power automorphism of P, a contradiction.

Assume that $P = \langle a \rangle \times \langle b \rangle$ is elementary abelian. Let q^f be the order of the automorphism of P induced by x, $a^x = a^i$ and $b^x = b^s$, where i and s are two distinct primitive q^f-th roots of unity modulo p. Then $0 < f \leq r$ and $p \equiv 1 (\mathrm{mod}\ q^f)$. Since $P\langle x^q \rangle \neq G$, x^q induces a power automorphism in P and $i^q = s^q$. So i and s both have order q^f. Then $s = i^j$ for some integer $j \not\equiv 1 (\mathrm{mod}\ q^f)$. Now $i^q = s^q = s^{jq}$, so $j \equiv 1 (\mathrm{mod}\ q^{f-1})$, and we can assume that $j = 1 + kq^{f-1}$ where $0 < k < q$. Hence G is again of type (I).

If $|P : Z(P)| = p^2$, then $\Phi(P) = R \cap K = Z(P)$, and so P is minimal non-abelian and $|P'| = p$. Let $P_1 = \langle a, P' \rangle$ and $P_2 = \langle b, P' \rangle$. Then P_1Q and P_2Q are GPST-groups. By hypothesis, x induces power automorphisms in P_1 and P_2, say $g \mapsto g^{n_1}$ and $g \mapsto y^{n_2}$ respectively. By Lemma 2.4, these two power automorphisms are fixed-point-free. However, they must agree on P', so $n_1 \equiv n_2 (\mathrm{mod}\ p)$ and we can assume $n_1 = n_2$. Since $[a, b]^{n_1} = [a, b]^x = [a^{n_1}, b^{n_1}]$, $n_1^2 \equiv n_1 (\mathrm{mod}\ p)$ and $n_1 \equiv 1 (\mathrm{mod}\ p)$, a contradiction.

(2) Assume that G is minimal non-supersoluble.

Let M be a maximal subgroup of G such that $Q \leq M$. Then $M = P_3Q$, where P_3 is a Sylow p-subgroup of M. By $[P_3, Q] \leq P \cap P_3Q = P_3$, we have

$N_G(P_3) \geq P_3Q = M$. Since $N_P(P_3) > P_3$, P_3 is normal in G. By Lemma 2.5 and the maximality of M, $P_3 = \Phi(P)$ is the Sylow p-subgroup of M.

Case 1. If G is also a minimal non-nilpotent group, then by applying [4, Theorem 3], G is of either type (II) or type (III).

Case 2. If G is not a minimal non-nilpotent group and P is abelian, then by applying [2, Theorem 9, 10], we assume that $G = PQ$, where $P = \langle a_0, a_1, \ldots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle x\rangle$ is cyclic of order q^r, q^f is the highest power of q dividing $p-1$ and $r > f \geq 1$. Define $a_j^x = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^x = a_0^i$, where i is a primitive q^f-th root of unity modulo p.

For a maximal subgroup $P\langle x^q\rangle$ of G and any element a_k of P, by computation, $a_k^{x^q} = a_k^i$. So G is of type (IV).

Case 3. Assume that G is not a minimal non-nilpotent group and P is non-abelian. By applying [2, Theorem 9, 10], we may assume that $G = PQ$ such that $P = \langle a_0, a_1\rangle$ is an extraspecial group of order p^3 with exponent p, $Q = \langle x\rangle$ is a cyclic group of order 2^r with 2^f the largest power of 2 dividing $p-1$ and $r > f \geq 1$, and $a_0^x = a_1$ and $a_1^x = a_0^i y$, where $y \in \langle [a_0, a_1]\rangle$ and i is a primitive 2^f-th root of unity modulo p.

Since every subgroup of $P\langle x^2\rangle$ is a GPST-group, every subgroup of P permutes every subgroup of $\langle x^2\rangle$ by induction. If $P\langle x^2\rangle \neq P \times \langle x^2\rangle$, then P is abelian by Lemma 2.1, a contradiction. Hence $P\langle x^2\rangle = P \times \langle x^2\rangle$ and $a_0^{x^2} = a_1^x = a_0^i y = a_0$, which implies that $x = 1$ and $i \equiv 1 (\mathrm{mod}\ p)$, a contradiction. Therefore, G is not of the type as above.

Conversely, it is easy to check that all groups satisfied types (I)—(IV) are minimal non-GPST-groups. □

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