ON THE DOUBLE FROBENIUS GROUP OF THE FORM $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$

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Abstract. Let G be a finite group. Let $\overline{H} = NH$ be a Frobenius group with kernel N and complement H. If G admits $\overline{H}$ as a group of automorphisms such that $C_G(N) = \{1_G\}$ and GN is also a Frobenius group with kernel G and complement N, then $\overline{G} = GNH$ is called a double Frobenius group (or 2-Frobenius group). The group $\overline{G} = GNH$ is a product of subgroups $G \leq \overline{G}$, $N \leq \overline{G}$, $H \leq \overline{G}$ with $G \trianglelefteq \overline{G}$, $GN \trianglelefteq \overline{G}$ and $\overline{G} = G{:}NH = GN{:}H$. In this article we shall construct a double Frobenius group of the form $\overline{G} = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$, where $G \cong 2^{2r}$, $N \cong \mathbb{Z}_{2^r-1}$ and $H \cong \mathbb{Z}_2$, where $r \in \mathbb{N}, r \geq 2$. The construction is a general one that gives examples of double Frobenius groups for particular values of n. In addition to the general construction of the group $\overline{G} = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$, we calculate in general the conjugacy classes, Fischer matrices and character table of the group. One example $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$, (the case $r = 2$) is demonstrated.

Keywords: double Frobenius group, Fischer matrices, character table

1. Introduction

The case where a Frobenius group $\overline{H} = NH$ acts by automorphisms on a group G has received some study in recent years. In this situation various properties (parameters) of G are found to be close to the corresponding properties of $C_G(H)$ and H. These properties include the order, rank, exponent, nilpotency class and Fitting height of G. Khukhro in [7], Khukhro and Makarenko in [8] and Khukhro, Makarenko and Shumyatsky in [6] have obtained some results in this regard. We are interested in the case where a Frobenius group $\overline{H}$ with kernel N and complement H acts as a group of automorphisms on a group G such that $C_G(N) = \{1_G\}$ and GN is also a Frobenius group. In recent years graphs associated with finite groups have received much attention. In particular the *prime graph* or Gruenberg-Kegel graph has been the subject of most attention in interest and research. Both Frobenius and double Frobenius groups appear in the study of the prime graphs of finite groups. As a result of one of the key classification theorems of the prime graphs of finite groups (the Gruenberg-Kegel Theorem), we have that: if G is a finite solvable group (Frobenius groups may be solvable, double Frobenius groups are always solvable) with a disconnected prime graph, then G is either Frobenius or double Frobenius.

2. Preliminary results

We will briefly describe here some results on Frobenius and double Frobenius groups, but first include some material on the method of coset analysis and Fischer matrices.

2.1 Coset analysis

We will briefly discuss the method of *coset analysis* which is used to determine the conjugacy classes of group extensions $\overline{G} = N.G$ where N is an abelian normal subgroup of $\overline{G}$. The technique is used for both split and non-split extensions. The technique was first used by Moori in [11]. More details can be found in [12] and [11].

Let $\overline{G} = N.G$, where $N \lhd \overline{G}$ and $\overline{G}/N \cong G$, be a finite group extension.

1. For each $g \in G$ let $\overline{g} \in \overline{G}$ map to g under the natural epimorphism $\pi : \overline{G} \to G$ and let $g_1 = N\overline{g}_1, g_2 = N\overline{g}_2, \ldots, g_r = N\overline{g}_r$ be representatives for the conjugacy classes of $G \cong \overline{G}/N$. Therefore, $\overline{g}_i \in \overline{G},\ \forall i$, and by convention we take $\overline{g}_1 = 1_{\overline{G}}$.

2. The method of coset analysis constructs for each conjugacy class $[g_i]_G,\ 1 \leq i \leq r$, a number of conjugacy classes of $\overline{G}$. For each $1 \leq i \leq r$, we let $g_{i1}, g_{i2}, \ldots, g_{ic(g_i)}$ be the corresponding representatives of these classes. That is, each conjugacy class of $\overline{G}$ corresponds uniquely to a conjugacy class of G.

3. We use the notation $U = \pi(\overline{U})$ for any subset $\overline{U} \subseteq \overline{G}$. Therefore, $\pi^{-1}([g_i]_G) = \bigcup_{j=1}^{c(g_i)} [g_{ij}]_{\overline{G}}$ for any $1 \leq i \leq r$. We also assume that $\pi(g_{ij}) = g_i$ and by convention we take $g_{11} = 1_{\overline{G}}$.

4. For fixed $i \in \{1, 2, \ldots, r\}$, act N by conjugation on the coset $N\overline{g}_i$ and let the resulting orbits be $Q_{i1}, Q_{i2}, \ldots, Q_{ik_i}$. If N is abelian (for both a split and non-split extensions), then
$$|Q_{i1}| = |Q_{i2}| = \ldots = |Q_{ik_i}| = \frac{|N|}{k}.$$

5. Act $\overline{G}$ on $Q_{i1}, Q_{i2}, \ldots, Q_{ik_i}$ and suppose f_{ij} orbits fuse together to form a new orbit Δ_{ij}, and let the total number of new resulting orbits in this

action be $c(g_i)$ (that is $1 \leq j \leq c(g_i)$). Then $\overline{G}$ has a conjugacy class $[g_{ij}]_{\overline{G}}$ that contains Δ_{ij} and $|[g_{ij}]_{\overline{G}}| = |[g_i]| \times |\Delta_{ij}|$. We repeat steps 4 and 5 above for all $i \in \{1, 2, \ldots, r\}$.

2.2 Fischer matrices

If $\overline{G} = N.G$ is a group extension, then $\overline{G}$ acts on the classes of N and on the $Irr(N)$. By Brauer's Theorem (see [5], theorem 6.32 and corollary 6.33) the number of orbits of both these actions are the same. Let $\theta_1, \theta_2, \ldots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$ and let $\overline{H}_k$ and H_k denote the corresponding inertia and inertia factor groups of θ_k. Bernd Fischer showed that the character table of $\overline{G}$ can be constructed by using the character tables of the inertia factor groups H_k together with some matrices called Fischer-Clifford matrices (see [4]). We define here the Fischer matrices which are used in the construction of the character table of any group extension $\overline{G} = N.G,\ N \lhd \overline{G}$

1. For each $[g_i]_G$ (conjugacy class of G), there corresponds a Fischer matrix which we denote by M_i.

2. $[g_{ij}]_{\overline{G}} \bigcap \overline{H}_k = \bigcup_{n=1}^{c(g_{ijk})} [g_{ijkn}]_{\overline{H}_k}$, where $g_{ijkn} \in \overline{H}_k$ and by $c(g_{ijkn})$ we mean the number of $\overline{H}_k$-conjugacy classes that form a partition for $[g_{ij}]_{\overline{G}}$. Since $g_{11} = 1_{\overline{G}}$, we have $g_{11k1} = 1_{\overline{G}}$ and thus $c(g_{11k1}) = 1$ for all $1 \leq k \leq t$.

3. $[g_i]_G \bigcap H_k = \bigcup_{m=1}^{c(g_{ik})} [g_{ikm}]_{H_k}$, where $g_{ikm} \in H_k$ and by $c(g_{ik})$ we mean the number of H_k-conjugacy classes that form a partition for $[g_i]_G$. Since $g_1 = 1_G$, we have $g_{1k1} = 1_G$ and thus $c(g_{1k1}) = 1$ for all $1 \leq k \leq t$. Also, $\pi(g_{ijkn}) = g_{ikm}$ for some $m = f(j, n)$.

4. The top of the columns of the Fischer matrix M_i are labeled by the representatives of $[g_{ij}]_{\overline{G}}$, $1 \leq j \leq c(g_i)$ obtained by coset analysis and below each g_{ij} we put $|C_{\overline{G}}(g_{ij})|$.

5. The bottom of the columns of M_i are labeled by some weights m_{ij} defined by
$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N| \frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}.$$

6. To label the rows of M_i we define the set J_i to be
$$J_i = \{(k, m)|\ 1 \leq k \leq t,\ 1 \leq m \leq c(g_{ik})\}.$$

7. Then each row of M_i is indexed by a pair $(k, m) \in J_i$.

8. For a fixed $1 \leq k \leq t$, we let M_{ik} be a submatrix of M_i with rows corresponding to the pairs $(k, 1), (k, 2), \ldots, (k, r_k)$.

9. Let
$$a_{ij}^{(k,m)} := \sum_{n=1}^{c(g_{ijk})} \frac{|C_{\overline{G}}(g_{ij})|}{|C_{\overline{H}_k}(g_{ijkn})|}\widetilde{\psi_k}(g_{ijkn})$$
for which $\pi(g_{ijkn}) = g_{ikm}$) and where $\widetilde{\psi_k}$ is the extension of θ_k to $\overline{H}_k$.

10. For each i, corresponding to the conjugacy class $[g_i]_G$, we define the Fischer matrix $M_i = \big(a_{ij}^{(k,m)}\big)$, where $1 \leq k \leq t,\ 1 \leq m \leq c(g_{ik}),\ 1 \leq j \leq c(g_i)$.

11. The Fischer matrix M_i is now given by
$$M_i = \big(a_{ij}^{(k,m)}\big) = \begin{pmatrix} M_{i1} \\ \hline M_{i2} \\ \hline \vdots \\ \vdots \\ \hline M_{it} \end{pmatrix}$$

12. In the Fischer matrix M_i, each M_{ik} is the submatrix corresponding to the inertia group $\overline{H}_k$ and it's inertia factor H_k.

The Fischer matrices have numerous properties that help in their computations. For details on the Fischer matrices and their properties, see [1], [2], [13], [15], [17], [14] and [18].

2.3 Frobenius group

Definition 2.1. *We define a group $\overline{H} = N{:}H$ to be a Frobenius group if it has a proper subgroup $H \neq \{1_{\overline{H}}\}$ such that $H \cap H^x = \{1_{\overline{H}}\}$ for all $x \in \overline{H} - H$. The subgroup H is called the complement and the non-trivial subgroup N is called the kernel.*

2.4 Properties of Frobenius Groups and their characters

We list here some properties of Frobenius groups and their ordinary characters. Let $\overline{H} = NH$ be a Frobenius group with kernel N and complement H.

1. The kernel N is unique and nilpotent and $N \lhd \overline{H}$.

2. $C_{\overline{H}}(n) \leq N\ \forall\ 1_{\overline{H}} \neq n \in N$ and $C_{\overline{H}}(h) \leq H\ \forall\ 1_{\overline{H}} \neq h \in H$.

3. $|H|\big|\big(|N| - 1\big)$.

4. The action of H on N is fixed point free. Hence, the lengths of the non-trivial orbits of the action of H on N is $|H|$ by the orbit stabilizer theorem. Therefore $|N| = 1 + m|H|$ where m is the number of non-trivial orbits of the action of H on N.

5. The Sylow p-subgroups of H are generalized quaternion or cyclic if $p = 2$ and cyclic if $p \neq 2$.

6. $\overline{H}$ has a trivial center.

7. $c(\overline{H}) = \frac{c(N)-1}{|H|} + c(H)$ where $c(\overline{H})$, $c(N)$ and $c(H)$ are the number of conjugacy classes of $\overline{H}$, N and H respectively.

8. If $\phi_1 \neq \phi \in Irr(N)$ where ϕ_1 is the trivial character of N, then $\phi^{\overline{H}} \in Irr(\overline{H})$.

9. If $\psi \in Irr(\overline{H})$, then either $N \subset ker\psi$ or $\psi = \phi^{\overline{H}}$ for some irreducible character $\phi \neq \phi_1$ of N, thus the irreducible characters of $\overline{H}$ are of two types; those with kernel containing N and those induced from irreducible characters of N.

10. If $\psi \in Irr(\overline{H})$, such that $ker\psi \not\supset N$ and ρ is the regular representation of H, then $\psi|_H = n\rho$ where $n \in \mathbb{N}$ and we have that $|H| \big| \phi(1_{\overline{H}})$. Furthermore, if $\phi_1 \neq \phi \in Irr(N)$ is a linear character, then $|H| = \phi(1_{\overline{H}})$.

11. If $\phi_1 \neq \phi \in Irr(N)$, then ϕ has inertia group $I_{\overline{H}}(\phi) = N$.

12. $\overline{H}$ has $\alpha = \frac{c(N)-1}{|H|}$ distinct irreducible characters of the form $\phi^{\overline{H}}, \phi_1 \neq \phi \in Irr(N)$ and hence, $c(H) = n$ irreducible characters $\psi \in Irr(\overline{H})$, with $N \subset ker\psi$

Proofs of the results mentioned here can be found in [16].

2.5 Double Frobenius group

Definition 2.2. *Let G be a finite group. Let $\overline{H} = NH$ be a Frobenius group with kernel N and complement H. If G admits $\overline{H}$ as a Frobenius group of automorphisms with $C_G(N) = \{1_G\}$ such that GN is also a Frobenius group with kernel G and complement N, then $\overline{G} = GNH$ is called a double Frobenius group.*

2.6 Properties of double Frobenius groups

We list here some properties of double Frobenius groups. Let $\overline{G} = GNH$ be a double Frobenius group with Frobenius subgroups GN and NH.

1. $G \trianglelefteq \overline{G}$ and $GN \trianglelefteq \overline{G}$.

2. Double Frobenius groups are solvable.

3. In the double Frobenius group $\overline{G} = GNH$, N is cyclic and of odd order and H is cyclic.

4. The center of a double Frobenius group is trivial.

Before describing our construction we mention the following two results which we make reference to.

Lemma 2.3 ([10])**.** *Let $(G,+)$ be an elementary abelian group of order p^n for some prime p. There is a cyclic fixed point free automorphism group of order k on G if and only if $k \mid p^n - 1$.*

Proof. See [10][Corollary 5.4]. □

Proposition 2.4 ([10])**.** *Let Φ be a fixed point free automorphism group on the additive group $(N,+)$. Then the semi-direct product $G = \Phi{:}N$ is a Frobenius group with complement Φ and kernel N.*

Proof. See [10][Proposition 7.3]. □

3. Constructing the group $\overline{G} = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$

From our definition, a double Frobenius group is the result of a group action of a Frobenius group $\overline{H}$ on a finite group G. Our construction therefore begins here. We look for a Frobenius group $\overline{H}$. Frobenius groups play an important role in finite group theory as point stabilizers of Zassenhaus groups (doubly transitive permutation groups in which some non-identity element fixes two points but none fixes three). They also appear frequently as maximal subgroups of the finite simple groups.

We aim to construct double Frobenius groups using $PSL(n,q)$ with $n = 2$ and q even. For q even, $PSL(2,q)$ has maximal subgroups which are Dihedral groups of order $2(q-1)$ or $2(q+1)$, (see King [9]).

Since q is even, $q-1$ is odd and hence, the maximal subgroup of order $2(q-1)$ is a Frobenius group due to the fact that the Dihedral group D_{2m} is Frobenius if m is odd. Let $q = 2^r,\ 2 \leq r \in \mathbb{N}$, then the Frobenius group $D_{2(q-1)}$ has the form $\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2$. Now for q even, $PSL(2,q) \cong SL(2,q) \leq GL(2,q)$. The natural action of $GL(2,q)$ on the elementary abelian group of order q^2 implies that $q^2{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2) \leq q^2{:}GL(2,q)$.

Therefore, $(2^r)^2{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2) \leq (2^r)^2{:}GL(2,2^r)$. Since $2^r - 1$ divides $2^{2r} - 1$ (because $2^{2r} - 1 = 2^r - 1 \times 2^r + 1$), if $2^{2r}{:}\mathbb{Z}_{2^r-1}$ is a Frobenius group, then by Proposition 2.4 and Lemma 2.3, $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$ is a double Frobenius group.

4. Fischer matrices and character table of $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$

In this section we shall determine the conjugacy classes, Fischer matrices and character table of the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$. We give a general description of the conjugacy classes, Fischer matrices and character table of the group.

Let $\overline{G} = GNH$ be a double Frobenius group with GN and NH Frobenius groups. Consider now the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$. Let $\overline{G} = GNH = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$. Then, $G \cong 2^{2r},\ \ N \cong \mathbb{Z}_{2^r-1}$ and $H \cong \mathbb{Z}_2$.

5. The group $\overline{H} = \mathbb{Z}_{2^r-1}:\mathbb{Z}_2$

We will first determine the conjugacy classes of the Frobenius group $\overline{H} = \mathbb{Z}_{2^r-1}:\mathbb{Z}_2$.

5.1 Conjugacy classes of $\overline{H}$

We know that $\mathbb{Z}_{2^r-1}:\mathbb{Z}_2$ is a Frobenius group with kernel $N = \langle a\rangle \cong \mathbb{Z}_{2^r-1}$ and complement $H = \langle b\rangle \cong \mathbb{Z}_2$. Since H acts on N with fixed point free action, we have that the number of non-trivial orbits of H on N is given by $\alpha = \frac{|N|-1}{|H|}$ and the length of each orbit is given by $|H|$. Therefore, here the orbits of $\mathbb{Z}_2$ on $\mathbb{Z}_{2^r-1}$ have lengths 1 and 2. Using the method of coset analysis, briefly described in Section 2.1, we analyse the coset Nh for each $h \in H$ and find the values of k where k is the order of the stabilizer in N of h. The values of k can be determined from the action of H on N. Since this action is fixed point free, $k = 2^r - 1$ for $h = 1_H$ and $k = 1$ for $h = b$.

For $h = 1_H,\ \ k = 2^r - 1,\ f_1 = 1$ and $f_i = 2\ \ \forall i \in \{2, 3, \ldots, \alpha + 1\}$.

For $h = 1_H,\ \ k = 2^r - 1,\ \ f_1 = 1:$

$$\left|C_{\overline{H}}(x)\right| = \frac{(2^r-1)\times 2}{1} = |\overline{H}|.$$

So for $f_1 = 1$, we have the identity class of $\overline{H}$.

For $h = 1_H,\ \ k = 2^r - 1\ \ , f_i = 2:$

$$\left|C_{\overline{H}}(x)\right| = \frac{(2^r-1)\times 2}{2} = |N|.$$

So for $f_i = 2$, we have a class of $\overline{H}$ containing x with $\circ(x) = 2^r - 1$. The size of the conjugacy class is

$$\left|[x]_{\overline{H}}\right| = \frac{|\overline{H}|}{\left|C_{\overline{H}}(x)\right|} = \frac{(2^r-1)\times 2}{2^r-1} = 2.$$

Note that there are $\frac{2^r-2}{2} = 2^{r-1} - 1$ such classes in NH.

For $h = b$ we have $k = 1,\ \ f = 1:$

$$\left|C_{\overline{H}}(x)\right| = \frac{1\times 2}{1} = 2.$$

Therefore,

$$\left|[x]_{\overline{H}}\right| = \frac{|\overline{H}|}{\left|C_{\overline{H}}(x)\right|} = \frac{(2^r-1)\times 2}{2} = 2^r - 1.$$

So, for the coset Nb there is a unique involutary class of $\overline{H}$ containing h.

From the above we deduce that there are $1+(2^{r-1}-1)+1 = 2^{r-1}+1$ classes in $\overline{H}$. This number can be confirmed by (6) in section 2.4, since $\overline{H} = \mathbb{Z}_{2^r-1}:\mathbb{Z}_2$ and we have that

$$c(\overline{H}) = c(H) + \alpha = c(H) + \frac{c(N)-1}{|H|} = 2 + \frac{2^r-2}{2} = 2^{r-1} + 1.$$

The full list of conjugacy classes based on coset analysis is given in Table 1.

Table 1: Conjugacy Classes of $\overline{H}$

H	$\overline{H} = NH$	$\circ(\overline{h})$	$\lvert C_{NH}(\overline{h})\rvert$
1_H	1	1	$(2^r-1)2$
	$(2^r-1)A_1$	c_1	2^r-1
	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$
	$(2^r-1)A_\alpha$	c_α	2^r-1
b	$2A$	2	2

Note: Table notes
1. $(2^r-1)A_i$ is the conjugacy class containing elements of order 2^r-1.
2. $(2A)$ is the conjugacy class containing elements of order 2 using the notation of the ATLAS.
3. Also here $H = \langle b\rangle$ and c_i for $1 = 1, 2, ..., \alpha$ divides the order of $N = 2^r-1$.

5.2 Character Table of $\overline{H}$

We obtain here a general description of the character table of the Frobenius group $\overline{H}$. To construct the character table we will use the results (7-11) listed in section 2.4 above and the following:

1. The irreducible characters of $\overline{H}$ with kernel containing N are χ_1 and χ_2 of degree 1 and those induced from non-trivial irreducible characters of N are $\chi_3, \chi_4, \ldots, \chi_{2+\alpha}$ of degree 2 (see table below).
2. $$\psi^{\overline{H}}(a_j) = \left|C_{\overline{H}}(a_j)\right| \sum_{i=1}^{m} \frac{\phi(x_i)}{\left|C_N(x_i)\right|},$$
   where $\phi \in Irr(N)$, $[a_j]$ is the conjugacy class of $\overline{H}$ containing a_j and $x_1, x_2, \ldots, x_m$ are class representatives for the classes of N that fuse to $[a_j]$. Since $\left|C_{\overline{H}}(a_j)\right| = 2^r-1$ and $\left|C_N(x_i)\right| = |N| = 2^r-1$, $\phi^{\overline{H}}(a_j) = \sum_{i=1}^{m} \phi(x_i)$. The sum on the right hand side are the orbit sums of the action of H on N.
3. We know that $\mathbb{Z}_2$ acts fixed point free on $\mathbb{Z}_{2^r-1}$, the number of non-trivial orbits is given by $\alpha = \frac{|N|-1}{|H|} = 2^{r-1}-1$ and the length of each orbit is given by $|H| = 2$. Also, since $\overline{H}$ is dihedral, the action of $\mathbb{Z}_2$ on $\mathbb{Z}_{2^r-1}$ is given by $ba^ib^{-1} = a^{-i}$.

   The orbits are: $\Theta_j = \{a^i, a^{-i}\}$ for $j = 1, 2, \ldots, \alpha$ and $i = 1, 2, \ldots, \alpha$.
4. Now let $p_j = \sum_{i=1}^{2} \phi(x_i) = \left[c^j(i,1)\right]$, for $j = 2, 3, \ldots, \alpha$, then $p_j = \phi(x_i) + \phi(x_i^{-1})$.

So $p_j = \phi(x_i) + \overline{\phi(x_i)} = 2t$, where t is the real part of $exp(\frac{2\pi i}{2^r-1})$.

The following table gives the partial character table of $\overline{H}$, namely the values of the induced characters of degree 2, $\chi_3, \chi_4, \ldots \chi_{2+\alpha}$, on classes $\big((2^r-1)A_i\ ,\ i = 1, 2, \ldots, \alpha\big)$, where $p_j = \sum_{j=1}^{2} t_j = 2t_j$ and $t_j = \exp\big(\frac{2\pi i}{2^r-1}\big)$.

	$(2^r-1)A_1$	$(2^r-1)A_2$	$(2^r-1)A_3$	$\ldots\ldots\ldots$	$(2^r-1)A_\alpha$
χ_3	p_1	p_2	p_3	$\ldots\ldots\ldots$	p_α
χ_4	p_2	p_3	p_4	$\ldots\ldots\ldots$	$p_{\alpha-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\chi_{2+\alpha}$	p_α	$p_{\alpha-1}$	$p_{\alpha-2}$	$\ldots\ldots\ldots$	p_1

We now produce the character tables of $\mathbb{Z}_2$, $\mathbb{Z}_{2^r-1}$ and the Frobenius group $\overline{H}$ in Tables 2, 3 and 4 respectively.

Table 2: Character Table of $\mathbb{Z}_2$

Classes	e	b
χ_1	1	1
χ_2	1	-1

Table 3: Character Table of $\mathbb{Z}_{2^r-1}$

Classes	e	a	a^2	$\ldots$	$\ldots\ a^{2^r-2}$
ϕ_1	1	1	1	1	1 1
ϕ_2	1	p	p^2	$\ldots$	$\ldots\ p^{2^r-2}$
$\vdots$					$\vdots$
$\vdots$					$\vdots$
ϕ_{2^r-1}	1	p^{2^r-2}	p^{2^r-3}	$\ldots$	$\ldots\ p$

Note:Table note: $p = e^{\frac{2\pi i}{2^r-1}}$

Remark 5.1. For $n \in PSL(2,q) \cong SL(2,q)$, q even $(q = 2^r)$, let $n = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \in \mathbb{F}_q^*$ and $\circ(\lambda) = q-1$. Then there exists an involution $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq b = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ for $x, y, z, t \in \mathbb{F}_q$ such that $bnb^{-1} = n^{-1}$ and $\langle n,\ b\rangle \cong \mathbb{Z}^{2^r-1} : \mathbb{Z}_2$.

For $b = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, the following conditions apply:

Table 4: Character Table of $\overline{H} = NH$

(g)	(1)	$(2^r-1)A_1 \ldots\ldots\ldots (2^r-1)A_\alpha$	$(2A)$
$\lvert C_{\overline{H}}(g)\rvert$	$(2^r-1)2$	$2^r-1 \ldots\ldots\ldots\ldots 2^r-1$	2
χ_1	1	$1 \ldots\ldots\ldots\ldots 1$	1
χ_2	1	$1 \ldots\ldots\ldots\ldots 1$	-1
χ_3	2	$p_1 \ldots\ldots\ldots\ldots p_\alpha$	0
χ_4	2	$p_2 \ldots\ldots\ldots\ldots p_{\alpha-1}$	0
$\vdots$	$\vdots$	$\vdots \ldots\ldots\ldots\ldots \vdots$	$\vdots$
$\chi_{2+\alpha}$	2	$p_\alpha \ldots\ldots\ldots\ldots p_1$	0

Note: Table note: $p_j = \sum_{j=1}^{2} t_j = 2t_j$ where $t_j = \exp\left(\frac{2\pi i}{2^r-1}\right)$.

- $xt - yz = 1 \quad \ldots\ldots\ldots$ (1)
- $b^2 = \begin{pmatrix} x^2+yz & xy+yt \\ xz+zt & yz+t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \ldots\ldots\ldots$ (2)
- $bnb^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \quad \ldots\ldots\ldots$ (3)

From equation (2) above , we have that:

- $x^2 + yz = 1 \quad \ldots\ldots\ldots$ (4)
- $xy + yt = 0 \quad \ldots\ldots\ldots$ (5)
- $xz + zt = 0 \quad \ldots\ldots\ldots$ (6)
- $zy + t^2 = 1 \quad \ldots\ldots\ldots$ (7)

From equation (5) above, we have that $y(x+t) = 0$ and that $y = 0$ or $x + t = 0$. Similarly from equation (6), we have $z(x+t) = 0$ and $z = 0$ or $x+t = 0$. From these two equations we have the following six cases to consider.

- Case 1: $y = 0$ and $x + t = 0$.
- Case 2: $z = 0$ and $x + t = 0$.
- Case 3: $y = z = 0$.
- Case 4: $y = 0$.
- Case 5: $z = 0$.
- Case 6: $x + t = 0$.

We now consider each case:

1. Case 1: $y = 0$ and $x + t = 0$. If $x + t = 0$ then $x = -t = t$ since in prime field $\mathbb{F}_2, -1 \equiv 1$. Now $y = 0$ implies by equation (4) that $x^2 = 1$ and $x = 1$. This gives us $b = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$.

   Now $bnb^{-1} = n^{-1}$ implies that $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$.

   So, $\begin{pmatrix} \lambda & 0 \\ z\lambda + z\lambda^{-1} & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$. Therefore, we must have $z\lambda + z\lambda^{-1} = 0$ and

   $z(\lambda + \lambda^{-1}) = 0$ implies $z = 0$ since $\lambda + \lambda^{-1} = 0$ implies that $\circ(\lambda) = 2$ which we can not have.

   Hence, this gives $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

2. Case 2: $z = 0$ and $x + t = 0$. Just as in Case 1 above we arrive at the conclusion that $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

3. Case 3: If $y = z = 0$, then as in Cases 1 and 2 above, we get $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

4. Case 4: If $y = 0$ then equation (4) implies that $x = 1$. So b has the form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. This case is similar to Case 1 above and we get: $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

5. Case 5: $z = 0$ gives us a similar conclusion as in Case 4 above where we get $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction.

6. Case 6: If $x + t = 0$, then $x = t$. There are two cases to consider here: (i)$x = t \neq 0$ and (ii)$x = t = 0$.

   If $x = t \neq 0$, then b has the form $\begin{pmatrix} x & y \\ z & x \end{pmatrix}$. Then for $n = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\lambda \neq 0$, $bnb^{-1} = n^{-1}$ gives:

   $$\begin{pmatrix} x & y \\ z & x \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ z & x \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

   So,

   $$\begin{pmatrix} x^2\lambda + zy\lambda^{-1} & xy\lambda + xy\lambda^{-1} \\ xz\lambda + xz\lambda^{-1} & yz\lambda + x^2\lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

So, this gives us the following: $xy\lambda + xy\lambda^{-1} = 0 \quad \ldots\ldots\ldots$ (8). Then $xy(\lambda+\lambda^{-1}) = 0$ and $\lambda+\lambda^{-1} = 0$ or $xy = 0$. If $\lambda+\lambda^{-1} = 0$, then $\lambda = \lambda^{-1}$ and $\circ(\lambda) = 2$ which is not possible since λ is an element in a cyclic group of odd order. Therefore, $xy = 0$ and since $x \neq 0$, $y = 0$. Similarly using the equation $xz\lambda + xz\lambda^{-1} = 0$, we arrive at the conclusion $z = 0$. So for this first case $x = t \neq 0$ implies that $y = z = 0$ and equation (1) gives $x = 1$. Hence, $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction. For the second case, if $x + t = 0$ and $x = t = 0$, then equations (1) and (4) implies that $yz = 1$. Thus, b has the form $\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$.

Then $z = y^{-1}$ and $b = \begin{pmatrix} 0 & \lambda^i \\ \lambda^{-i} & 0 \end{pmatrix}$ for $\lambda^i \in \mathbb{F}_q^*$.

Before describing the conjugacy classes of the double Frobenius group 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$), we make the following Note.

Note 5.2. 1. $|PSL(2,q)| = q^3 - q$ if q is even.

2. If q is even, then $PSL(2,q) \cong SL(2,q)$.

3. The group $SL(2,q)$, $q = 2^t$, $t \geq 1$ has $q+1$ distinct conjugacy classes. These classes are described in the Table 5.

Table 5: Conjugacy Classes of $SL(2,q)$, q even.

Class	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
Rep. of Class	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & r+r^q \end{pmatrix}$
No. of Classes	1	1	$\frac{q-2}{2}$	$\frac{q}{2}$
$\lvert C_{SL(2,2^t)}(g)\rvert$	$q^3 - q$	q	$q-1$	$q+1$
$\lvert C_g\rvert$	1	q^2-1	$q(q+1)$	$q(q-1)$

Note: Table notes;
1. $\alpha \in \mathbb{F}_q^*$, $\alpha = \epsilon^k$, $k \neq 0$
2. $\mathbb{F}_{q^2}^* = \langle\theta\rangle$ and $r = \theta^{(q-1)j}$ for $j = 1, 2, \ldots, \frac{q}{2}$.

6. Conjugacy classes of 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$)

We determine now the conjugacy classes of the double Frobenius group $\overline{G} = GNH = G{:}NH$ where $N = \langle a\rangle \cong \mathbb{Z}_{2^r-1}$ and $H = \langle b\rangle \cong \mathbb{Z}_2$.

To determine the conjugacy classes of $\overline{G}$, we consider the cosets $\overline{h}G$ where $\overline{h} \in NH = \overline{H}$.

<u>The coset $1G$:</u>

Now for $\overline{h} = 1_{\overline{H}}$, the identity of $\overline{H}$, $\overline{h}$ fixes all elements of G so $k = 2^{2r}$. We now act the centralizer of $\overline{h} = 1_{\overline{H}}, C_{\overline{H}}(\overline{h}) = \overline{H}$ on G.

Now let $nh \in \overline{H} = NH$ for $1_{\overline{H}} \neq n \in N,\ 1_{\overline{H}} \neq h \in H$. Then for $g \in G$, $g^{nh} = nhgh^{-1}n^{-1} = n(hgh^{-1})n^{-1} = n(g^h)n^{-1} = (g^h)^n$.

Therefore to act $\overline{H}$ on G, we act $h \in H$ first and then act $n \in N$.

Now $\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2 \leq PSL(2,2^r) \cong SL(2,2^r) \leq GL(2,2^r)$ and $H = \langle b \rangle$ where b is an involution in $PSL(2,2^r)$. The group $SL(2,q)$, q even, $(q = 2^r\ , r \geq 1)$ has $q+1$ distinct conjugacy classes. Of these $q+1$ classes there is only one class of involutions. The size of this class is $q^2 - 1$ and for any involution $b \in SL(2,q)$, b has the form $\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$, where $x, z \in \mathbb{F}_q^*$. See the Note 5.2.

Now $G \cong V_2(\mathbb{F}_q)$, the vector space of dimension two over the field of $q = 2^r$ elements. So $G = \{0,\ \lambda^i e_1,\ \lambda^j e_2,\ (\lambda^i e_1 + \lambda^j e_2)\}$ for $i,\ j = \{0,1,2,\ldots,q-2\}$, where $\lambda^i,\ \lambda^j \in \mathbb{F}_q^*$, and $\{e_1,\ e_2\}$ is a basis of G, with $e_1^2 = 1,\ e_2^2 = 1$. So in the action of $\overline{H}$ on G, we look at the action of $b = \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$ on the elements of $G \cong 2^{2r} = \{0,\ \lambda^i e_1,\ \lambda^j e_2,\ (\lambda^i e_1 + \lambda^j e_2)\}$ followed by the action of N on these orbits (the orbits of b on G).

Now H acts on the 2^{2r} elements of G fixing (including the identity) 2^r elements and permuting the remaining $2^{2r} - 2^r$ elements in orbits of length two. So the elements of G are now in $\frac{2^{2r}-2^r}{2}$ orbits of length two plus the 2^r fixed points. Acting N on these 2^r fixed points and the $\frac{2^{2r}-2^r}{2}$ orbits gives the following:

1. Each of the $(2^r - 1)$ (identity excluded) fixed points fuses with $\frac{2^r-2}{2} = 2^{r-1} - 1$ of the two cycles to form an orbit of size $(2^r - 1)$. There are $(2^r - 1)$ of these orbits.
2. The remaining $2^{2r} - \{(2^r-1)(2^r-1)+1\} = 2(2^r) - 2$ elements fuse to form an orbit of size $2(2^r - 1)$.

So the action of $C_{\overline{H}}(\overline{h}) = \overline{H}$ on G, gives the following: one orbit of length one, one orbit of length $2(2^r-1)$ and (2^r-1) orbits of length (2^r-1).

Therefore we have: $k = 2^{2r}$ and $f_1 = 1,\ f_i = 2^r - 1$ for $i = 2,3,\ldots,2^r-1$ and $f_{2^r+1} = 2(2^r-1)$.

<u>$k = 2^{2r},\ f_1 = 1$:</u>

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_1} = \frac{2^{2r} \times (2^r-1) \times 2}{1} = |\overline{G}|,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^{2r} \times (2^r-1) \times 2}{2^{2r} \times (2^r-1) \times 2} = 1.$$

So for $f_1 = 1$ we have the identity class of $\overline{G}$.

$\underline{k = 2^{2r},\ f_i = 2^r - 1}$ for $i = 2, 3, \ldots, 2^r - 1$.

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_i} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^r - 1} = 2 \times 2^{2r},$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2 \times 2^{2r}} = 2^r - 1.$$

$\underline{k = 2^{2r},\ f_i = 2(2^r - 1):}$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_i} = \frac{2^{2r} \times (2^r - 1) \times 2}{2 \times (2^r - 1)} = 2^{2r},$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^{2r}} = 2 \times (2^r - 1).$$

Therefore the identity coset $1G$ produces the following conjugacy classes of $\overline{G} = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$: the identity conjugacy class, one class of size $2(2^r - 1)$ and $(2^r - 1)$ classes of size $(2^r - 1)$. The order of the non -identity elements in all the classes from the identity coset is two.

The coset bG **:**

When G acts on the coset bG, it partitions the coset into 2^r orbits of size 2^r. The 2^r orbits of the action of G on the coset bG consists of the orbit containing b, and $(2^r - 1)$ remaining orbits each containing $b\lambda^i e_1$ for $i = 0, 1, \ldots, q-2$ where $q = 2^r$. The orbit containing b also contains $b\lambda^i(e_1 + e_2)$ for $i = 0, 1, \ldots, q - 2$. Each of the orbits containing $b\lambda^i e_1$ also contain $b\lambda^i e_2$ for $i = 0, 1, \ldots, q - 2$ and a two cycle $\{(\lambda^i e_1 + \lambda^j e_2)\ ,\ (\lambda^j e_1 + \lambda^i e_2)\}$ for $i \neq j$ and $i,\ j = 0, 1, \ldots, q - 2$.

We now act the centralizer of $b \in H$, $C_{\overline{H}}(b)$ on these 2^r orbits. Since $C_{\overline{H}}(b) = \langle b \rangle$, the action of the centralizer is just the action of $\langle b \rangle$.

When b acts on G, it fixes zero and each $\lambda^i(e_1 + e_2)$ for $i = 0, 1, \ldots, q - 2$ and permutes the remaining $2^{2r} - 2^r$ elements of G into $\frac{2^{2r}-2^r}{2}$ orbits of length two. These $\frac{2^{2r}-2^r}{2}$ orbits are of the form $(\lambda^i e_1\ ,\ \lambda^i e_2)$ for $i = 0, 1, \ldots, q - 2$ and $\{(\lambda^i e_1 + \lambda^j e_2)\ ,\ (\lambda^j e_1 + \lambda^i e_2)\}$ for $i \neq j$ and $i,\ j = 0, 1, \ldots, q - 2$.

This implies that when $\langle b \rangle = C_{\overline{H}}(b)$ acts on the 2^r orbits, it permutes the elements in each of the 2^r orbits.

$\underline{k = 2^r,\ f_i = 1}$ for $i = 1, 2, \ldots, 2^r$:

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(b)|}{f_i} = \frac{2^r \times 2}{1} = 2 \times 2^r,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2 \times 2^r} = 2^r \times (2^r - 1).$$

Therefore, the coset bG produces 2^r conjugacy classes of $\overline{G}$ each of size $2^r(2^r - 1)$. We determine now the orders of the elements in these 2^r conjugacy classes. Using the result for element orders in the group $\overline{G} = G{:}\overline{H}$, where G is an elementary 2-group, we have that $\omega = gg^b$ for $g \in G,\ b \in H$. Now if $\omega = 1$, then $g^{-1} = g = g^b$ since $\circ(g) = 2$. Therefore, if b fixes $g \in G$, then $\circ(\overline{g}) = 2$ for $gb = \overline{g} \in \overline{G}$. If $\omega \neq 1$, then $\circ(\overline{g}) = 4$. Since the conjugacy class containing b is the class that has the fixed points, from the discussion above the order of the elements in this class is two (as would be expected since the class has the element b in it). The order of the elements in the remaining $(2^r - 1)$ conjugacy classes is four.

The coset a_iG : for $i = 1, 2, \ldots, m$ where m is the number of non-trivial orbits of the action of H on N. Now the action of G on the coset a_iG for $i = 1, 2, \ldots, m$ produces a single orbit of length 2^{2r}. The centralizer of $a_i \in N$, $C_{\overline{H}}(a_i) = \langle a_i \rangle$ acts fixed point free on the orbit permuting the elements in the orbit. Therefore, for each $a_i \in N$ for $i = 1, 2, \ldots, m$, there is a single conjugacy class of $\overline{G}$. There are m such classes.

$\underline{k = 1,\ f = 1:}$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(a)|}{f_i} = \frac{1 \times (2^r - 1)}{1} = 2^r - 1,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^{2r} \times (2^r - 1) \times 2}{2^r - 1} = 2 \times 2^{2r}.$$

The coset a_iG, produces a single conjugacy class of $\overline{G}$ of size $2(2^{2r})$. The order of the elements in the class is $2^r - 1$.

Note 6.1. From the discussion above we can determine the number of conjugacy classes of $\overline{G}$. The classes produced by the identity coset are the identity class, $(2^r - 1)$ classes of size $(2^r - 1)$ and one class of size $2(2^r - 1)$. The coset bG produces 2^r classes of size $2^r(2^r - 1)$. Each of the cosets a_iG for $i = 1, 2, \ldots, m$ where $m = \frac{|N|-1}{H} = \frac{2^r-2}{2} = 2^{r-1} - 1$ is the number of non-trivial orbits of H on N produces a single conjugacy class of size $2(2^{2r})$. There are m such classes.

$$c(\overline{G}) = \big(1 + 2^r - 1 + 1\big) + \big(2^r\big) + \big(2^{r-1} - 1\big) = 2 \times 2^r + 2^{r-1} = 2^{r+1} + 2^{r-1}.$$

The full list of conjugacy classes based on coset analysis is given in Table 6.

The following Proposition and Remark will be used to construct the Fischer matrices of the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$.

Proposition 6.2 ([15])**.** *If G is elementary abelian and $M = Im(\phi_g)$ where ϕ_g is an endomorphism of G defined by $\phi_g : x \mapsto xgx^{-1}g^{-1}$ for $x \in G$, then $[G : M] = k$ where k is the number of elements of G fixed by a class representative $\overline{h} \in \overline{H}$ where $\overline{G} = G : \overline{H}$.*

Proof. The orbits $Q_1, Q_2, \ldots, Q_k$ of G acting on $\overline{h}G$ are the same as the orbits $D_1, D_2, \ldots, D_k$ of M acting on $\overline{h}G$ by left multiplication. Also the orbits $D_1, D_2, \ldots, D_k$ can be identified with the elements of G/M. Then it follows that $G/M = [G : M] = k$. □

Table 6: Conjugacy Classes of $\overline{G} = G$:(NH)

$\overline{H} = NH$	$\overline{G} = G$:(NH)	$\circ(\overline{g})$	$\lvert[g]\rvert$	$\lvert C_{\overline{G}}(\overline{g})\rvert$
1	1	1	1	$2^{2r}(2^r-1)2$
	$(2A)_1$	2	2^r-1	$2(2^{2r})$
	$(2A)_2$	2	2^r-1	$2(2^{2r})$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$(2A)_{2^r-1}$	2	2^r-1	$2(2^{2r})$
	$(2A)_{2^r+1}$	2	$2(2^r-1)$	2^{2r}
b	$(2B)$	2	$2^r(2^r-1)$	$2(2^r)$
	$(4A)_1$	4	$2^r(2^r-1)$	$2(2^r)$
	$(4A)_2$	4	$2^r(2^r-1)$	$2(2^r)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$(4A)_{2^r-1}$	4	$2^r(2^r-1)$	$2(2^r)$
a_1	$(2^r-1)A_1$	c_1	$2(2^{2r})$	2^r-1
a_2	$(2^r-1)A_2$	c_2	$2(2^{2r})$	2^r-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	$(2^r-1)A_m$	c_α	$2(2^{2r})$	2^r-1

Remark 6.3. If G is an elementary abelian p-group, then from coset analysis for the group $\overline{G} = G$:$\overline{H}$, we obtain $k = p^m$ for $0 \leq m \leq n$, where $|G| = p^n$ and k is the number of elements of G fixed by a class representative $\overline{h}$ of $\overline{H}$. Suppose for some class representative $\overline{h} \in \overline{H}$, we have the orbits $Q_1, Q_2, \ldots, Q_k$ of the action of G on $\overline{h}G$. Then for $h \in C_{\overline{H}}(\overline{h})$, suppose that acting h on the orbits $Q_1, Q_2, \ldots, Q_k$, we get $f_1 = f_2 = \ldots = f_k = 1$ and that the entries of the first column of $M(\overline{h})$ are 1. Then in this case, the Fischer matrix $M(\overline{h})$ coincides with the character table of the abelian group G/M of order $k = p^m$.

7. Fischer matrices of 2^{2r}:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$)

In this section we will give a general description of the number of Fischer matrices and their form for the double Frobenius group $\overline{G} = 2^{2r}$:($\mathbb{Z}_{2^r-1}$:$\mathbb{Z}_2$). For each conjugacy class of $\overline{H}$ there is a corresponding Fischer matrix. Therefore there are $2^{r-1}+1$ such matrices. The action of $\overline{H}$ on G produces 2^r+1 orbits. The Fischer matrix $M(1_{\overline{H}})$ corresponding to the identity coset is therefore a $\big((2^r+1)\times(2^r+1)\big)$ matrix. Since the action of $\overline{H}$ of G has 2^r+1 orbits, by Brauer's Lemma the action of $\overline{H}$ on $Irr(G)$ has 2^r+1 orbits also. The lengths of the orbits are 1, (2^r-1) and $2(2^r-1)$. The number of orbits of each length is 1, (2^r-1) and 1 respectively. We can show that the orbits of the action of $\overline{H}$ on $Irr(G)$ also has lengths 1, (2^r-1) and $2(2^r-1)$ and that the number of orbits is also 1, (2^r-1) and 1 respectively. Now when $\overline{H}$ acts on $Irr(G)$, the possibilities for orbit lengths are: 1, 2, (2^r-1) and $2(2^r-1)$. Let the number of orbits of length one be a, the number of orbits of length two be b, the number of orbits of length 2^r-1 be c and the number of orbits of length $2(2^r-1)$ be d. Then

$$a+b+c+d=2^r+1 \;\cdots\cdots(1)$$

and

$$a+2b+(2^r-1)c+2(2^r-1)d=2^{2r} \;\cdots\cdots(2).$$

where $a,b,c,d\in\mathbb{N}$.

We find values for a,b,c,d. Note first that we can assume that $r\geq 2$ in equations 1 and 2 above since $r=0$ and $r=1$ give trivial cases for the double Frobenius group $2^{2r}:(\mathbb{Z}_{2^r-1}:\mathbb{Z}_2)$.

We know that $a\geq 1$ since the action of $\overline{H}$ on $Irr(G)$ fixes the identity character. We claim that $a=1$. So suppose that $a>1$. Then equation (2) implies that $(a-1)+2b+c(2^r-1)+2d(2^r-1)=2^{2r}-1 \;\cdots\cdots(3)$. So, $(a-1)+2b+c(2^r-1)+2d(2^r-1)=(2^r-1)(2^r+1) \;\cdots\cdots(4)$. Since 2^r-1 divides the right hand side of equation (4), it must divide the left hand side also. Therefore, 2^r-1 divides $(a-1)+2b$. So $(2^r-1)\alpha=a-1+b$ for some $\alpha\in\mathbb{N}$. From this equation we get $a+b=\alpha 2^r-\alpha+1$. Substituting this into equation (1) above gives $c+d=2^r(1-\alpha)+\alpha \;\cdots\cdots(5)$. Since $r\geq 2$, and $a,b,c,d\in\mathbb{N}$, equation (5) will only be true if $\alpha=0$ or $\alpha=1$.

$\underline{\alpha=0}$**:**

Then $a+b=1$ and since $a\neq 0$, $b=0$ and $a=1$. This is a contradiction since $a>1$.

$\underline{\alpha=1}$**:**

Then $a+b=2^r$ and $c+d=1$. Therefore there are two cases to consider:(i) $c=0$ and $d=1$ (ii) $c=1$ and $d=0$.

If $c=0$ and $d=1$, then equation (2) implies that $a+2b+2(2^r-1)=2^{2r}$ and hence $2^r+b+2(2^r-1)=2^{2r}$ after substituting for $a=2^r-b$. This gives $2^{2r}=b-2+2^r+2.2^r \;\cdots\cdots(6)$. Now 2^r divides the left hand side of equation (6) and must divide the right hand side also. Therefore, 2^r divides $b-2$ and $b=2^r\gamma+2$ for some $\gamma\in\mathbb{N}$. This is a contradiction since $b<2^r$.

If $d=0$ and $c=1$, then equation(2) implies that $a+2b+2^r-1=2^{2r}$ and hence $2^r+b+2^r-1=2^{2r}$ after substituting for $a=2^r-b$. This gives $2^{2r}=b-1+2.2^r \;\cdots\cdots(7)$. Now 2^r divides the left hand side of equation(7) and must divide the right hand side also. Therefore, 2^r divides $b-1$ and $b = 2^r\delta + 1$ for some $\delta \in \mathbb{N}$. This is a contradiction since $b < 2^r$.

Therefore, $a = 1$ as claimed. With $a = 1$, from equation (1) we now have that $b+c+d = 2^r$ and from equation (2) we have that $2b+(2^r-1)c+2(2^r-1)d = (2^r-1)(2^r+1)$. Since 2^r-1 divides the right hand side of this equation, it must divide the left hand side. This implies that 2^r-1 divides $2b$. Since 2^r-1 is odd, 2^r-1 must divide b. Thus, we have $(2^r-1)\epsilon = b$ for some $\epsilon \in \mathbb{N}$. From equation (1), $a+b+c+d = 2^r+1$ which implies that $b+c+d = 2^r$ and hence that $b \le 2^r$. So, $(2^r-1)\epsilon \le 2^r$ and since $r \ge 2$, this inequality is true only if $\epsilon = 0$ or $\epsilon = 1$.

If $\epsilon = 1$, then $b = 2^r - 1$ and equation (1) now implies that $c + d = 1$. There are two cases to consider.

$\underline{c = 0\ , d = 1}$**:**

Equation (2) now implies that $2(2^r-1)+0+2(2^r-1) = (2^r+1)(2^r-1)$, which gives us $4 = 2^r + 1$ which is false.

$\underline{c = 1\ , d = 0}$**:**

Equation (2) now implies that $2(2^r-1)+2^r-1 = (2^r+1)(2^r-1)$ and $3 = 2^r+1$ which is false for $r \ge 2$.

Thus we must have $\epsilon = 0$ and hence $b = 0$.

With $a = 1$ and $b = 0$, equation (1) and equation (2) now give $c + d = 2^r$ and $c + 2d = 2^r + 1$ respectively. Solving gives us $d = 1$ and $c = 2^r - 1$.

Therefore $a = 1 = d$, $b = 0$ and $c = 2^r$. Therefore, when $\overline{H}$ acts on $Irr(G)$, there is one orbit of length one, 2^r-1 orbits of length 2^r-1 and one orbit of length $2(2^r-1)$. This is the same number of orbits and orbit lengths as the action of $\overline{H}$ on G.

7.1 The Inertia groups and inertia factor groups of $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$

Using the results of the section above, we can give a general description of the inertia groups and inertia factor groups for the double Frobenius group $2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)$.

Now denote the inertia groups by $\overline{P}$ and the inertia factor groups by P. Since the action of $\overline{H}$ on the $Irr(G)$ has orbit lengths $1,\ (2^r-1),\ 2(2^r-1)$, the inertia groups are:
$\overline{P}_1 = \overline{G} = 2^{2r}{:}(\mathbb{Z}_{2^r-1}{:}\mathbb{Z}_2)\ ,\ \overline{P}_2 = \overline{P}_3 = \cdots = \overline{P}_{2^r} = G{:}H = 2^{2r}{:}\mathbb{Z}_2\ ,\ \overline{P}_{2^r+1} = G = 2^{2r}$.

The inertia factor groups are:

$$P_1 = \overline{H} = \mathbb{Z}_{2^r-1} : \mathbb{Z}_2\ ,\ P_2 = P_3 = \cdots = P_{2^r} = H = \mathbb{Z}_2\ ,\ P_{2^r+1} = \{1_{\overline{H}}\}.$$

7.2 Fischer matrices

There are (2^r+1) Fischer matrices, namely, $M(1_{\overline{H}}),\ M(b)$ and $M(a_i)$ for $i = 1, 2, \cdots\cdots, m$ where $m = 2^{r-1} - 1$ is the number of non-trivial orbits of the action of H on N.

$\underline{M(1_{\overline{H}}):}$

This is a $\big((2^r+1)\times(2^r+1)\big)$ matrix. The first row of the matrix is a row of 1's. The first column of the matrix consists of the (2^r+1) entries: $\big(1,\ 2^r-1,\ 2^r-1,\ \cdots\cdots\cdots, 2^r-1,\ 2(2^r-1)\big)^t$. There are (2^r-1) entries of (2^r-1). The last column of the matrix consists of the entries $\big(1,\ -1,\ -1,\cdots\cdots\cdots, 2^r-2\big)^t$. There are (2^r-1) entries of -1. The last row of the matrix consists of the entries $\big(2(2^r-1),\ -2,\ -2,\cdots\cdots\cdots, 2^r-2\big)$. The remainder of the matrix is a $\big(2^r-1\times 2^r-1\big)$ block whose rows are just a permutation of the entries $\big(2^r-1,\ -1,\ -1,\cdots\cdots\cdots,-1\big)$. This is the block denoted by $X\ X\ X$ in the Fischer matrix shown below.

$$M(1_{\overline{H}}) = \left(\begin{array}{|c|cccc|c|}
1 & 1 & 1 & \ldots\ldots & 1 & 1 \\ \hline
2^r-1 & & & & & -1 \\
2^r-1 & & & & & -1 \\
\cdots & & & & & \cdots \\
\cdots & X & X & X & X & \cdots \\
\cdots & & & & & \cdots \\
2^r-1 & & & & & -1 \\ \hline
2(2^r-1) & -2 & -2 & \ldots\ldots & -2 & 2^r-2
\end{array}\right)$$

$\underline{M(b):}$

This is a $\big(2^r\times 2^r\big)$ matrix. By Remark 6.3, the Fischer matrix corresponding to $b\in\overline{H}$ coincides with the character table of the elementary abelian group of order $k=2^m$ where k is number of fixed points of the action of b on G.

$$M(b) = \left(\begin{array}{c} Character\ Table\ of \\ elementary\ abelian\ group \\ of\ order\ 2^m \end{array}\right)$$

$\underline{M(a_i):}$

These are just singleton matrices with entry 1 of which there are $m=2^{r-1}-1$ in number.

8. Example-the group $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$

In this section we apply the theory developed in section 4 to the group $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2$. Let $\overline{G}=GNH$ where $G\cong 2^{2r}$, $N\cong\mathbb{Z}_{2^r-1}$, $H\cong\mathbb{Z}_2$ and $NH\cong\overline{H}$. Let $r=2$, then $G\cong 2^4$, $N\cong\mathbb{Z}_3=\langle a\rangle$, $H\cong\mathbb{Z}_2=\langle b\rangle$ and $NH\cong\overline{H}\cong\mathbb{Z}_3{:}\mathbb{Z}_2$. We know that $\mathbb{Z}_3{:}\mathbb{Z}_2\leq PSL(2,4)\cong SL(2,4)$. We note also that $PSL(2,4)$ has a single class of involutions and a single class of elements of order three. Now $o(b)=2,\ b=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $a=\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix}$ where $\langle\lambda\rangle=\mathbb{F}_4^*,\ \circ(a)=3$.

$G = \{0,\ e_1,\ e_2,\ (e_1+e_2),\ \lambda e_1,\ \lambda e_2,\ \lambda^2 e_1,\ \lambda^2 e_2,\ \lambda(e_1+e_2),\ \lambda^2(e_1+e_2),\ \lambda e_1+e_2,\ \lambda^2 e_1+e_2,\ e_1+\lambda e_2,\ e_1+\lambda^2 e_2,\ \lambda^2 e_1+\lambda e_2,\ \lambda e_1+\lambda^2 e_2\}$ and $GF(4)=\{0,\ 1,\ \lambda,\ \lambda^2\}$.

Now, we have that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $be_1 = e_2$ and $be_2 = e_1$.

8.1 Conjugacy classes of $\overline{G} = 2^4$:($\mathbb{Z}_3$:$\mathbb{Z}_2$)

To calculate the conjugacy classes of $\overline{G}$, we need the conjugacy classes of the group $\overline{H} = \mathbb{Z}_3$:$\mathbb{Z}_2$. The conjugacy classes of $\overline{H}$ are represented in Table 7.

Conjugacy classes of : $\overline{H} = \mathbb{Z}_3$:$\mathbb{Z}_2$.

Table 7: Conjugacy Classes of $\mathbb{Z}_3$:$\mathbb{Z}_2$

classes of $\mathbb{Z}_3$:$\mathbb{Z}_2$	[1]	[b]	[a]
$\lvert C_G(g)\rvert$	6	2	3
$\circ(g)$	1	2	3
$\lvert[g]\rvert$	1	3	2

To calculate the conjugacy classes of $\overline{G}$ we use the method of coset analysis.

Conjugacy classes of : $\overline{G} = 2^4$:($\mathbb{Z}_3$:$\mathbb{Z}_2$).

To determine the conjugacy classes of $\overline{G} = 2^4$:($\mathbb{Z}_3$:$\mathbb{Z}_2$), we consider the cosets $\overline{h}G$ where $\overline{h} \in \overline{H} = \mathbb{Z}_3$:$\mathbb{Z}_2$.

The coset $1G$: Now for $\overline{h} = 1_{\overline{H}}$, the identity of $\overline{H}$, $\overline{h}$ fixes all elements of G so $k = 2^4$. We now act the centralizer of $\overline{h} = 1_{\overline{H}}$, $C_{\overline{H}}(1_{\overline{H}}) = \overline{H}$ on G.

To act $\overline{H}$ on G, we first act $b \in \mathbb{Z}_2$ and then act $a \in \mathbb{Z}_3$.

The action of b on G:

In this action, b fixes $\{0,\ (e_1+e_2),\ \lambda(e_1+e_2),\ \lambda^2(e_1+e_2)\}$ and permutes the remaining 12 elements in the following 2 cycles:

$\{e_1,\ e_2\},\ \{\lambda e_1,\ \lambda e_2\},\ \{\lambda^2 e_1,\ \lambda^2 e_2\},\ \{(\lambda e_1+e_2),\ (e_1+\lambda e_2)\},\ \{(\lambda^2 e_1+e_2),\ (e_1+\lambda^2 e_2\},\ \{(\lambda^2 e_1+\lambda e_2),\ (\lambda e_1+\lambda^2 e_2\}$.

Now acting $a \in \mathbb{Z}_3$ on the 4 fixed points of b and the six 2 cycles we get:

a fixes the zero vector of G and when it acts on the orbits of b, each fixed point $\lambda^i(e_1+e_2)$ for $i = 0,\ 1,\ 2$ fuses with a 2 cycle to form an orbit of size three as follows:

$$\Theta_1 = \{(e_1+e_2),\ (\lambda^2 e_1+\lambda e_2),\ (\lambda e_1+\lambda^2 e_2\},$$

$$\Theta_2 = \{\lambda(e_1+e_2),\ (\lambda^2 e_1+e_2),\ (e_1+\lambda^2 e_2)\},$$

$$\Theta_3 = \{\lambda^2(e_1+e_2),\ (\lambda e_1+e_2),\ (e_1+\lambda e_2)\}.$$

The remaining orbits come together under the action of a to form the orbit Θ_4 of size six.

$$\Theta_4 = \{e_1,\ e_2,\ \lambda e_1,\ \lambda e_2,\ \lambda^2 e_1,\ \lambda^2 e_2\}.$$

Thus, the identity coset produces 5 orbits (conjugacy classes) of $\overline{G}$, viz, the singleton orbit containing the identity, three orbits of size three containing the remaining three fixed points, one in each orbit, and an orbit of size six.

Therefore we have:$k = 2^4$ and $f_1 = 1,\ f_2 = 3,\ f_3 = 3,\ f_4 = 3,\ f_5 = 6.$

$\underline{k = 2^4,\ f_1 = 1:}$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_1} = \frac{2^4 \times 6}{1} = |\overline{G}|,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{2^4 \times 6} = 1.$$

So for $f_1 = 1$ we have the identity class of $\overline{G}$.

$\underline{k = 2^4,\ f_i = 3:}$ for $i = 2, 3, 4$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_i} = \frac{2^4 \times 6}{3} = 32,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{32} = 3.$$

This will give us three conjugacy classes of $\overline{G}$ of size three. The order of the elements in all three classes is two.

$\underline{k = 2^4,\ f_5 = 6:}$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(1_{\overline{H}})|}{f_5} = \frac{2^4 \times 6}{6} = 16,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{16} = 6.$$

This gives us a fifth conjugacy class from the identity coset of $\overline{G}$ of size six. The order of the elements in this class is two.

The coset bG :

First we act G on the coset bG. The action of G on the coset bG partitions the coset into four orbits of size four. The orbits are:

$$\Delta_1 = \{b,\ b(e_1+e_2),\ b\lambda(e_1+e_2),\ b\lambda^2(e_1+e_2)\},$$

$$\Delta_2 = \{be_1,\ be_2,\ b(\lambda e_1+\lambda^2 e_2),\ b(\lambda^2 e_1+\lambda e_2)\},$$

$$\Delta_3 = \{b\lambda e_1,\ b\lambda e_2,\ b(\lambda^2 e_1 + e_2),\ b(e_1 + \lambda^2 e_2)\},$$

$$\Delta_4 = \{b\lambda^2 e_1,\ b\lambda^2 e_2,\ b(e_1 + \lambda e_2),\ b(\lambda e_1 + e_2)\}.$$

We also note that in the orbit Δ_1 the g entry of the element bg where $g \in G$ is the fixed point of the action of b on G, and in the orbits Δ_i for $i = 2, 3, 4$, the g entries of the element bg where $g \in G$ are the entries of the two cycles of the action of b on G. See the action of b on G above.

Next, we act the centralizer of b on the four orbits Δ_i for $i = 1, 2, 3, 4$. Now $C_{\overline{H}}(b) = \langle b \rangle$. Therefore the action of the centralizer is the same as the action of b. From the comment above, when b acts on the four orbits Δ_i for $i = 1, 2, 3, 4$, it permutes the elements in each orbit. Therefore for the coset bG we have $k = 4$ and $f_i = 1$ for $i = 1, 2, 3, 4$.

$\underline{k = 4,\ f_i = 1}$ for $i = 1, 2, 3, 4:$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(b)|}{f_i} = \frac{4 \times 2}{1} = 8,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{8} = 12.$$

Therefore, the coset bG produces four conjugacy classes of $\overline{G}$ each of size twelve. The order of the elements in the first of these classes is two (the class containing b) and the order of the elements in the remaining three classes is four.

The coset aG :

Finally we act G on the coset aG. When G acts on the coset aG, it simply permutes the 16 elements in the coset producing an orbit of length sixteen. Next we act the centralizer of a on this orbit. But $C_{\overline{H}}(a) = \langle a \rangle$. Therefore the action of the centralizer is the same as the action of $\langle a \rangle$. When $\langle a \rangle$ acts on the orbit of length sixteen it permutes the elements in the orbit. Therefore, here $k = 1,\ f = 1$.

$\underline{k = 1,\ f = 1:}$

$$\left|C_{\overline{G}}(x)\right| = \frac{k \times |C_{\overline{H}}(a)|}{f_i} = \frac{1 \times 3}{1} = 3,$$

$$\left|[x]_{\overline{G}}\right| = \frac{|\overline{G}|}{\left|C_{\overline{G}}(x)\right|} = \frac{2^4 \times 6}{3} = 32.$$

The coset aG, produces a single conjugacy class of $\overline{G}$ of size thirty two. The order of the elements in the class is three.

The full list of the conjugacy classes of $\overline{G}$ is described in Table 8.

Table 8: Conjugacy Classes of $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$

$\overline{H} = NH$	$\overline{G} = G : (NH)$	$\circ(\overline{g})$	$\lvert C_{\overline{G}}(\overline{g})\rvert$	*power map*(π^2)	*power map*(π^3)
1	$1A$	1	96	$(1A)$	$(1A)$
	$(2A)$	2	32	$(1A)$	$(2A)$
	$(2B)$	2	32	$(1A)$	$(2B)$
	$(2C)$	2	32	$(1A)$	$(2C)$
	$(2D)$	2	16	$(1A)$	$(2D)$
b	$(2E)$	2	8	$(1A)$	$(2E)$
	$(4A)$	4	8	$(2A)$	$(4A)$
	$(4B)$	4	8	$(2B)$	$(4B)$
	$(4C)$	4	8	$(2C)$	$(4C)$
a	$(3A)$	3	3	$(3A)$	$(1A)$

8.2 Fischer matrices of $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$.

We construct the Fischer matrices of $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$, for each conjugacy class of $\overline{H} = \mathbb{Z}_3 : \mathbb{Z}_2$. From the previous sections we know that there are three conjugacy classes of $\overline{H}$ and therefore three Fischer matrices of $\overline{G}$. For the Fischer matrix corresponding to the identity class of $\mathbb{Z}_3{:}\mathbb{Z}_2$ we look at the action of $\overline{H}$ on $G = 2^4$. There are five orbits of lengths 1, 3, 3, 3, 6. The Fischer matrix corresponding to the identity class is $M(1_{\overline{H}})$ which is a (5×5) matrix. Since the action of $\overline{H}$ on G has five orbits of lengths 1, 3, 3, 3 and 6, we know that the action of $\overline{H}$ on $Irr(G)$ also produces five orbits of lengths 1, 3, 3, 3 and 6 as described in Section 7.

From Section 7.1 we have the following inertia and inertia factor groups. The inertia groups are:

$$\overline{P}_1 = \overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)\ ,\ \overline{P}_2 = \overline{P}_3 = \overline{P}_4 = G{:}H = 2^4{:}\mathbb{Z}_2\ ,\ \overline{P}_5 = G = 2^4.$$

The corresponding inertia factor groups are:

$$P_1 = NH = \mathbb{Z}_3{:}\mathbb{Z}_2\ ,\ P_2 = P_3 = P_4 = H = \mathbb{Z}_2\ ,\ P_5 = \{1_{\overline{H}}\}.$$

The Fischer matrices are constructed using the theory of the Fischer Clifford matrices. We refer the reader to [1], [2], [13], [15], [17] and [18] for details.

$\underline{M(1_{\overline{H}}):}$

$$M(1_{\overline{H}}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & -1 & -1 \\ 3 & -1 & 3 & -1 & -1 \\ 3 & -1 & -1 & 3 & -1 \\ 6 & -2 & -2 & -2 & 2 \end{pmatrix}.$$

The matrix corresponding to $b \in \mathbb{Z}_2$ is a 4×4 matrix $M(b)$ given by:

$$M(b) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Finally the third Fischer matrix is a (1×1) matrix with the singleton entry 1. This matrix $M(a)$ is given by :$M(a) = (1)$.

8.3 Character table of $\overline{G} = 2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$

We can now construct the character table of $\overline{G}$ using the Fischer matrices above and the character tables of the inertia factor groups $P_1 = \overline{H} = \mathbb{Z}_3{:}\mathbb{Z}_2$ and $P_2 = P_3 = P_4 = H = \mathbb{Z}_2$.

We divide the character table of $\overline{G}$ into blocks as shown in the matrix below. Each block A_i, B_i, C_i for $i = 1, 2, 3, 4, 5$ corresponds to an inertia group $\overline{P}_i$. Also the A_i blocks for $i = 1, 2, 3, 4, 5$ come from the conjugacy classes produced by the identity coset $1G$, the B_i blocks for $i = 1, 2, 3, 4, 5$ come from the conjugacy classes produced by the coset bG and the C_i blocks for $i = 1, 2, 3, 4, 5$ come the conjugacy classes produced by the coset aG.

$$\left(\begin{array}{|c|c|c|} A_1 & B_1 & C_1 \\ \hline A_2 & B_2 & C_2 \\ \hline A_3 & B_3 & C_3 \\ \hline A_4 & B_4 & C_4 \\ \hline A_5 & B_5 & C_5 \end{array}\right)$$

First we need the character tables of $\overline{H} = \mathbb{Z}_3{:}\mathbb{Z}_2$ and $H = \mathbb{Z}_2$

Character table $\overline{H}$:

$(\overline{h})$	(1)	(b)	(a)
$\lvert C_{\overline{H}}(\overline{h})\rvert$	6	2	3
$\circ(\overline{h})$	1	2	3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Character table H :

(h)	(1)	(b)
$\lvert C_H(h)\rvert$	2	2
$\circ(h)$	1	2
χ_1	1	1
χ_2	1	-1

We now calculate the characters of $\overline{G}$, which fall into five blocks ($A_i,\ for\ i = 1, 2, 3, 4, 5$) with inertia groups $\overline{P}_1 = \overline{G},\ \overline{P}_2 = G{:}H,\ \overline{P}_3 = G{:}H,\ \overline{P}_4 = G{:}H,\ \overline{P}_5 = G$ by using the Fischer matrices and inertia factor groups $P_1 = \overline{H},\ P_2 = \mathbb{Z}_2,\ P_3 = \mathbb{Z}_2,\ P_4 = \mathbb{Z}_2,\ P_5 = \{1_{\overline{G}}\}$.

We complete the character table of $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$ by multiplying rows of $M(g)$ for $g \in \{1_{\overline{H}},\ b,\ a\}$ with sections of the character tables of the inertia factor groups corresponding to each $g \in \{1_{\overline{H}},\ b,\ a\}$.

The first block of table above A_1 is the block corresponding to conjugacy classes from the identity $1_{\overline{H}}$. To obtain this block, we multiply the first column $C_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ = $1st$ column of $\overline{H}$ by $M_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ = $1st$ row of $M(1_{\overline{H}})$.

We get:

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

For the A_2 block, we multiply $C_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = $1st$ column of $\mathbb{Z}_2$ by $M_2 = \begin{pmatrix} 3 & 3 & -1 & -1 & -1 \end{pmatrix}$ = $2nd$ row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 & -1 & -1 \\ 3 & 3 & -1 & -1 & -1 \end{pmatrix}.$$

For the A_3 block, we multiply $C_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = $1st$ column of $\mathbb{Z}_2$ by $M_3 = \begin{pmatrix} 3 & -1 & 3 & -1 & -1 \end{pmatrix}$ = $3rd$ row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 3 & -1 & -1 \\ 3 & -1 & 3 & -1 & -1 \end{pmatrix}.$$

For the A_4 block, we multiply $C_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = $1st$ column of $\mathbb{Z}_2$ by $M_4 = \begin{pmatrix} 3 & -1 & -1 & 3 & -1 \end{pmatrix}$ = $4th$ row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 & 3 & -1 \\ 3 & -1 & -1 & 3 & -1 \end{pmatrix}.$$

For the A_5 block, we multiply $C_5 = \begin{pmatrix} 1 \end{pmatrix}$ = $1st$ column of $\{1_{\overline{G}}\}$ by $M_5 = \begin{pmatrix} 6 & -2 & -2 & -2 & 2 \end{pmatrix}$ = $5th$ row of $M(1_{\overline{H}})$. We get:

$$\begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 6 & -2 & -2 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -2 & -2 & -2 & 2 \end{pmatrix}.$$

For the next block, the B_i block for $i = 1, 2, 3, 4, 5$ of the character table of $\overline{G}$, we use the Fischer matrix $M(b)$. To complete the B_1 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$=$2nd$ column of $\overline{H}$ by $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$= $1st$ row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the B_2 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$=$2nd$ column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}$= $2nd$ row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

For the B_3 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$=$2nd$ column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}$= $3rd$ row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

For the B_4 block of the table, we multiply $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$=$2nd$ column of $\mathbb{Z}_2$ by $\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}$= $4th$ row of $M(b)$. We get:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

For the B_5 block of the table, we will have a row of zeros since $P_5 \bigcap [b] = \emptyset$ and hence, $M_5(b)$ will not exist.

To complete the C_i block for $i = 1, 2, 3, 4, 5$ of the character table of $\overline{G}$, we use the Fischer matrix $M(a) = \{1_{\overline{G}}\}$. To complete the C_1 block of the table, we multiply $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$=$3rd$ column of $\overline{H}$ by $\begin{pmatrix} 1 \end{pmatrix}$= $1st$ row of $M(a)$. We get:

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

For the C_i blocks for $i = 2, 3, 4, 5$, we have zeros since $P_i \bigcap [a] = \emptyset$ for $i = 2, 3, 4, 5$ and therefore M_i for $i = 2, 3, 4, 5$ does not exist.

Character table of $2^4{:}(\mathbb{Z}_3{:}\mathbb{Z}_2)$:

$(\overline{g})$	$(1A)$	$(2A)$	$(2B)$	$(2C)$	$(2D)$	$(2E)$	$(4A)$	$(4B)$	$(4C)$	$(3A)$
$\lvert[(\overline{g})]\rvert$	1	3	3	3	6	12	12	12	12	32
$\lvert C_{\overline{G}}(\overline{g})\rvert$	96	32	32	32	16	8	8	8	8	3
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1	-1	-1	1
χ_3	2	2	2	2	2	0	0	0	0	-1
χ_4	3	3	-1	-1	-1	1	1	-1	-1	0
χ_5	3	3	-1	-1	-1	-1	-1	1	1	0
χ_6	3	-1	3	-1	-1	1	-1	1	-1	0
χ_7	3	-1	3	-1	-1	-1	1	-1	1	0
χ_8	3	-1	-1	3	-1	1	-1	-1	1	0
χ_9	3	-1	-1	3	-1	-1	1	1	-1	0
χ_{10}	6	-2	-2	-2	2	0	0	0	0	0

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