GLOBAL DYNAMICS OF AN SIVS EPIDEMIC MODEL WITH BILINEAR INCIDENCE RATE

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Abstract. An SIS type epidemic model with variable population size is considered. The model includes a temporary vaccination program to prevent individuals from infection and to eradicate the disease. If $\mathcal{R}_0 < 1$, the disease-free equilibrium is locally and globally asymptotically stable i.e. the disease will be wiped out from population. When $\mathcal{R}_0 > 1$, the endemic equilibrium is locally asymptotically stable employing a result in stability of the second additive compound matrix. In addition, by using a geometric approach it is shown that this equilibrium is also globally asymptotically stable. So in this case, the disease will persist in population permanently. Also, a briefly discussion is made on the minimum amount of vaccination which is necessary to eradicate the disease. Finally, some numerical examples are given to confirm the obtained results.

Keywords: SIS epidemic model, vaccination, asymptotic stability, compound matrix method, geometric approach.

1. Introduction

The spread and control of infectious diseases in a population have been an important issue in recent years. The behavior of Population can be studied by using mathematical models and computer simulation. Many epidemic models have been introduced by authors for various type of diseases. The susceptible-infected-susceptible (SIS) epidemic models are one of the well known type of epidemic models. This type of models is appropriate for some infections in which individuals don't obtain permanent immunity after recovery. Vaccination is known as an efficient strategy to give immunity to the individuals and thus may also be included in the SIS epidemic models by considering a separate compartment for vaccinated individuals in the model formulation. These models may be deterministic [13,16] or stochastic [23,6], with constant [13] or variable [16,17] population size, and with standard [16,17] or bilinear incidence [10,5]. The organization of this paper is as the following: The formulation of the model and some basic properties such that, the boundedness of solutions, the basic reproduction number, and the equilibria of the model, will be given in the next section. The asymptotic stabilities of the disease-free equilibrium and the

endemic state are studied in sections 3 and 4, respectively. A brief discussion on the effect of vaccination is done in section 5. Eventually after section 6 for some numerical examples, we summarize the results presented in the paper.

2. Model description

We consider the following deterministic SIS epidemic model with vaccination and bilinear incidence established by Li and Ma [10]:

$$\begin{cases} S' = (1-q)\Lambda - \beta SI - (\mu+\varphi)S + \gamma I + \theta V, \\ I' = \beta SI - (\mu+\gamma+\alpha)I, \\ V' = q\Lambda + \varphi S - (\mu+\theta)V. \end{cases} \tag{2.1}$$

Variables $S(t), I(t)$ and $V(t)$ denote respectively the number of susceptible, infectious and vaccinated individuals at time time t. The parameters of the model are as follows:

Λ: Number of new individuals added into the population per unit of time,
q: Fraction of new individuals that are vaccinated,
β: Contact rate,
γ: Recovery rate,
μ: Natural death rate,
α: Disease-related death rate,
φ: Vaccination rate for susceptible individuals,
θ: Rate of loosing immunity in vaccinated individuals.

All parameter values are assumed to be non-negative and Λ and μ are positive. The force of infection βSI represents the number of new infected individuals per unit time and is of bilinear form.

From system (2.1) it can be seen that the total population size N is not constant and is expressed by the following equation:

$$N' = \Lambda - \mu N - \alpha I. \tag{2.2}$$

We see that

$$N' \leq \Lambda - \mu N,$$

and thus

$$\limsup_{t\to\infty} N \leq \frac{\Lambda}{\mu}.$$

Therefore total population N and as a result I, S and V are also bounded. We can see that the feasible region $\Gamma = \{(S,I,V) \in \mathbb{R}^3_+ : \ S+I+V \leq \frac{\Lambda}{\mu}\}$ is a positively invariant set of system (2.1) i.e. the solutions remain in Γ with initial vector $(S(0), I(0), V(0))$ in Γ. System (2.1) has two equilibria: A disease-free equilibrium when $I = 0$;

$$E^0 = (S^0, I^0, V^0) = \left(\frac{\Lambda[\mu(1-q)+\theta]}{\mu(\mu+\theta+\varphi)}, 0, \frac{\Lambda(\mu q+\varphi)}{\mu(\mu+\theta+\varphi)} \right),$$

and an endemic equilibrium when $I > 0$;

$$E^* = (S^*, I^*, V^*),$$

with

$$\begin{aligned} S^* &= \frac{\mu+\gamma+\alpha}{\beta}, \\ I^* &= \frac{\beta\Lambda[\mu(1-q)+\theta] - \mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}{\beta(\mu+\alpha)(\mu+\theta)}, \\ V^* &= \frac{q\Lambda + \varphi(\mu+\gamma+\alpha)/\beta}{\mu+\theta}. \end{aligned}$$

We use the next generation matrix method developed in [21], to find the basic reproduction number of the model. Let $\mathbf{y} = I$, where $\mathbf{y}$ indicates all infected states such as exposed and infectious individuals. Thus the second equation in system (2.1) can be written as

$$\frac{d\mathbf{y}}{dt} = \mathcal{F} - \mathcal{W},$$

with $\mathcal{F} = \beta SI$ and $\mathcal{W} = (\mu+\gamma+\alpha)I$.

Also let

$$F = \frac{\partial \mathcal{F}}{\partial \mathbf{y}}\Big|_{E^0} = \beta S^0 \quad \text{and} \quad W = \frac{\partial \mathcal{W}}{\partial \mathbf{y}}\Big|_{E^0} = \mu+\gamma+\alpha.$$

Therefore the basic reproduction number of the model is obtained by

$$\mathcal{R}_0 = \rho(FW^{-1}) = \frac{\beta S^0}{\mu+\gamma+\alpha} = \frac{\beta\Lambda[\mu(1-q)+\theta]}{\mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}. \tag{2.3}$$

Notice that

$$I^* = \frac{\mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}{\beta(\mu+\alpha)(\mu+\theta)}(\mathcal{R}_0 - 1),$$

and so the endemic equilibrium E^* exists if $\mathcal{R}_0 > 1$. Hence we can state the following:

Lemma 2.1. *When $\mathcal{R}_0 \leq 1$ system (2.1) has only the disease-free equilibrium E^0 and if $\mathcal{R}_0 > 1$ it also has a unique endemic equilibrium E^*.*

3. Stability of the disease-free equilibrium

In this section we consider the local and global stability of the disease-free equilibrium. Firstly it can be seen easily that eigenvalues of the Jacobian matrix of system (2.1) at E^0 are

$$\begin{aligned} \lambda_1 &= (\mu+\gamma+\alpha)(\mathcal{R}_0 - 1), \\ \lambda_2 &= -\mu, \\ \lambda_3 &= -(\mu+\varphi+\theta), \end{aligned}$$

thus we can obtain the following theorem:

Theorem 3.1. *The disease-free equilibrium E^0 is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$.*

Our next task is to prove the global stability of E^0. This task has been discussed also by many authors especially [3,11,15,12] and various methods have been established.

Theorem 3.2. *The disease-free equilibrium E^0 of system (2.1) is globally asymptotically stable if $\mathcal{R}_0 < 1$.*

Proof. Consider the Lyapunov function $L(t) = I(t)$. Now, we consider two below cases:

Case(a): If $\frac{\beta\Lambda}{\mu(\mu+\gamma+\alpha)} \leq 1$, we see that $\mathcal{R}_0 < 1$ and we have

$$\beta S - (\mu + \gamma + \alpha) \leq \beta \left(\frac{\Lambda}{\mu}\right) - (\mu + \gamma + \alpha) \leq 0. \tag{3.1}$$

Case (b): If $\frac{\beta\Lambda}{\mu(\mu+\gamma+\alpha)} > 1$ and $\mathcal{R}_0 \leq 1$. $\mathcal{R}_0 \leq 1$ implies $\frac{\beta\Lambda}{\mu(\mu+\gamma+\alpha)} \leq \frac{\mu+\varphi+\theta}{\mu(1-q)+\theta}$ and therefore $S^0 \leq \frac{\mu+\gamma+\alpha}{\beta}$. So the region

$$\Omega = \left\{(S, I, V) \in \Gamma : S > \frac{\mu + \gamma + \alpha}{\beta}, I > 0\right\}$$

is not an invariant set because contains no equilibrium when $\mathcal{R}_0 \leq 1$. Hence, any solution with initial value in Ω lies in $\Gamma\backslash\Omega$ after a finite period of time. For any $(S, I, V) \in \Gamma\backslash\Omega$ we have $S \leq \frac{\mu+\gamma+\alpha}{\beta}$ and thus

$$\beta S - (\mu + \gamma + \alpha) \leq 0. \tag{3.2}$$

From (3.1) and (3.2) we see

$$L' = I[\beta S - (\mu + \gamma + \alpha)] \leq 0. \tag{3.3}$$

Therefore, from the LaSalle's invariance principle [14], we find that every solution of the model (2.1) with initial values in Γ, ultimately approaches E^0, when $\mathcal{R}_0 \leq 1$. □

4. Stability of the endemic equilibrium

In this section we study the local and global asymptotic stability of the endemic state E^*. We firstly attempt to explore the uniform persistence of (2.1) when $\mathcal{R}_0 > 1$. Suppose that (X, d) is a locally compact metric space and H is a closed subset of X with boundary ∂H and interior $\overset{\circ}{H}$. We state the following definition about uniform persistence which can be found in [8]:

Definition 4.1. A semi-dynamical system $\Phi_t(x) : H \times \mathbb{R}^+ \to H$ defined on H is said to be uniform persistence if there exists some $\eta > 0$ such that

$$\liminf_{t\to\infty} \mathrm{d}(\Phi_t(x), \partial H) > \eta,$$

for all $x \in H$.

Definition 4.2. A subset Σ of H is said to be a uniform repeller if and only if there exists an $\eta > 0$ such that for all $x \in H\backslash\Sigma$, $\liminf_{t\to\infty} \mathrm{d}(\Phi_t(x), \Sigma) > \eta$.

The definitions (4.1) and (4.2) state that a semi-dynamical system defined on a closed subset of a locally compact metric space is uniform persistence if the boundary of such subset is uniform repeller. The following result about persistence has been proved by Fonda in [7]:

Lemma 4.3. *Let F be a compact subset of X such that $X\backslash F$ is positively invariant. F is uniform repeller if and only if there exists a neighborhood U of F and a continuous function $P : X \longrightarrow \mathbb{R}^+$ satisfying:*
(i) $P(x) = 0$ if and only if $x \in F$,
(ii) For any $x \in U$, there exists a T_x such that $P(\Phi_{T_x}(x)) > P(x)$.

Now, consider the dynamical system (2.1) on positively invariant region Γ. Also, we let

$$H = \Gamma = \{(S, I, V) \in \mathbb{R}^3_+ : \ S + I + V \leq \frac{\Lambda}{\mu}\},$$

$$\Sigma = \partial\Gamma = \{(S, I, V) \in \Gamma : \ I = 0\},$$

and hence

$$H\backslash\Sigma = \overset{\circ}{\Gamma} = \{(S, I, V) \in \Gamma : \ I > 0\}.$$

Thus to show uniform persistence of system (2.1), we must show that Σ is uniform repeller when $\mathcal{R}_0 > 1$.

Theorem 4.4. *System* (2.1) *is uniform persistence if $\mathcal{R}_0 > 1$.*

Proof. Obviously, Σ is a compact set and $H\backslash\Sigma$ is positively invariant when $\mathcal{R}_0 > 1$. Let define $P : \Gamma \longrightarrow \mathbb{R}_+$ by $P(S, I, V) = I$, and $U = \{(S, I, V) \in \Gamma : P(S, I, V) < \zeta\}$, where $\zeta > 0$ is chosen so small that

$$\frac{\beta\Lambda[\mu(1-q)+\theta]}{(\mu+\gamma+\alpha)[\mu(\mu+\varphi+\theta)+2(\mu+\theta)\beta\zeta]} > 1. \tag{4.1}$$

The condition (i) is clearly satisfied. Assume that the condition (ii) doesn't hold, i.e. there exists a $\widetilde{x} \in U$ such that $P(\Phi_t(\widetilde{x})) < P(\widetilde{x}) < \zeta$ for all $t > 0$. This implies $I < \zeta$ and thus we see from system (2.1),

$$\begin{cases} S' \geq (1-q)\Lambda - \beta\zeta S - (\mu+\varphi)S + \theta V, \\ V' = q\Lambda + \varphi S - (\mu+\theta)V. \end{cases} \tag{4.2}$$

Hence, $\liminf_{t\to\infty} S(t;\widetilde{x}) \geq \frac{\Lambda[\mu(1-q)+\theta]}{\mu(\mu+\varphi+\theta)+(\mu+\theta)\beta\zeta}$, where $S(t;\widetilde{x})$ denotes the solution $S(t)$ with initial value $\widetilde{x}$. Thus there exists some T_S such that for any $t > T_S$

$$S(t;\widetilde{x}) \geq \frac{\Lambda[\mu(1-q)+\theta]}{\mu(\mu+\varphi+\theta)+2(\mu+\theta)\beta\zeta}. \tag{4.3}$$

Now consider the function $W(t) = I(t)$. We see

$$\begin{aligned} W' = I' &= I[\beta S - (\mu+\gamma+\alpha)] \\ &\geq I\left[\beta\frac{\Lambda[\mu(1-q)+\theta]}{\mu(\mu+\varphi+\theta)+2(\mu+\theta)\beta\zeta} - (\mu+\gamma+\alpha)\right] \\ &= I(\mu+\gamma+\alpha)\left[\frac{\beta\Lambda[\mu(1-q)+\theta]}{(\mu+\gamma+\alpha)[\mu(\mu+\varphi+\theta)+2(\mu+\theta)\beta\zeta]} - 1\right]. \end{aligned} \tag{4.4}$$

Therefore from (4.1) and (4.4) it is concluded that $W(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. But this result contradicts the boundedness of $W(t)$. Hence the condition (ii) must be also satisfied and Σ is uniformly repeller. This completes the proof. □

In the following, we consider the stability of the endemic equilibrium employing the approach used in [1] and properties of compound matrices. The (local asymptotic) stability of the E^* is equivalent to stability of the corresponding Jacobian matrix of system (2.1) at E^*:

$$J^* = J(E^*) = \begin{pmatrix} -\beta I^* - (\mu+\varphi) & -(\mu+\alpha) & \theta \\ \beta I^* & 0 & 0 \\ \varphi & 0 & -(\mu+\theta) \end{pmatrix}.$$

We can easily see that $tr(J^*) < 0$ and $det(J^*) < 0$. Assume that λ_1, λ_2 and λ_3 are eigenvalues of J^* such that $\Re(\lambda_1) \leq \Re(\lambda_2) \leq \Re(\lambda_3)$, where $\Re(.)$ denotes the real part of a complex number. Then, $\lambda_1\lambda_2\lambda_3 < 0$ yields either $\Re(\lambda_j) < 0$ for $j = 1, 2, 3$ or $\Re(\lambda_1) < 0 < \Re(\lambda_2) \leq \Re(\lambda_3)$.

On the other hand, the second additive compound matrix $J^{[2]}(E^*)$ of the Jacobian matrix J^* (for example, by a strict formula in appendix of [19]) can be calculated as

$$J^{[2]}(E^*) = \begin{pmatrix} -\beta I^* - (\mu+\varphi) & 0 & -\theta \\ 0 & -\beta I^* - (2\mu+\varphi+\theta) & -(\mu+\alpha) \\ \varphi & \beta I^* & -(\mu+\theta) \end{pmatrix}$$

Notice that eigenvalues of $J^{[2]}(E^*)$ are $\lambda_1+\lambda_2$, $\lambda_1+\lambda_3$ and $\lambda_2+\lambda_3$. Besides, $tr(J^*) = \lambda_1+\lambda_2+\lambda_3 < 0$ implies $\Re(\lambda_1+\lambda_2) < 0$ and $\Re(\lambda_1+\lambda_3) < 0$. One can show that $det(J^{[2]}(E^*)) < 0$, and thus we must have $\Re(\lambda_2+\lambda_3) < 0$. All eigenvalues of $J^{[2]}(E^*)$ have negative real part and as a result it is stable. Any n by n real matrix M is stable if and only if $(-1)^n det(M) > 0$ and $M^{[2]}$ is stable [19], thus the Jacobian matrix J^* is stable and we have the following theorem:

Theorem 4.5. *The endemic equilibrium E^* of system (2.1) is locally asymptotically stable if $\mathcal{R}_0 > 1$.*

To analyze the global stability of E^* we use a geometric approach developed by Li and Muldowney [18]. This method has been used to deal with the global stability of endemic equilibrium in many epidemic models [4,1,20,22,2]. Here we briefly explain the method:

Consider the autonomous equation $x' = f(x)$, where $f : D \longrightarrow \mathbb{R}^n$ is a C^1 function on an open set $D \subset \mathbb{R}^n$. The solution of the differential equation with initial value x_0 is denoted by $x(t; x_0)$. Assume that two following conditions hold:

($\mathcal{G}_1$) The system has a unique equilibrium $\overline{x}$ in D;

($\mathcal{G}_2$) There exists a compact absorbing set $K \subset D$.

Let $Q(x)$ be a $M \times M$, $M = \binom{n}{2}$, matrix-valued function which is C^1 on D. Moreover, assume that $Q^{-1}(x)$ exists and is continuous for $x \in K$. Define a quantity $\overline{q}_2$ as

$$\overline{q}_2 = \limsup_{t\to\infty} \sup_{x_0\in K} \frac{1}{t} \int_0^t \psi\left(B(x(r; x_0))\right) dr,$$

where,

$$B = Q_f Q^{-1} + QJ^{[2]}Q^{-1}.$$

Here, $J^{[2]}$ is the second additive compound matrix of the Jacobian matrix J, Q_f is obtained by $(q_{ij})_f = \left(\frac{\partial q_{ij}}{\partial x}\right)^{\top} .f(x)$, and $\psi(B)$ is the *Lozinskiĭ* measure of B with respect to a vector norm $|.|$ in $\mathbb{R}^M$, defined by $\psi(B) = \lim_{h\to 0^+} \frac{|I+hB|-1}{h}$. The following result has been proved in [18].

Lemma 4.6. *Suppose that D is simply connected and conditions $\mathcal{G}_1$ and $\mathcal{G}_2$ are satisfied. The unique equilibrium $\overline{x}$ of system $x' = f(x)$ is globally asymptotically stable in D if $\overline{q}_2 < 0$.*

We are now in position to prove the global stability of the endemic equilibrium.

Theorem 4.7. *The endemic equilibrium E^* of system (2.1) is globally asymptotic stable if $\mathcal{R}_0 > 1$ and the following condition holds:*

$$\mu > \max\Big\{\theta - \varphi, \gamma - \theta + \varphi, \gamma - \theta + \alpha\Big\}.$$

Proof. The uniform persistence of system (2.1) in the bounded region Γ, stated in Theorem 4.4, implies that there exists a compact set K in $\overset{\circ}{\Gamma}$ which is absorbing for solutions of system(2.1) (see [9]).

The Jacobian matrix of system (2.1) is

$$J = J(S, I, V) = \begin{pmatrix} -\beta I - (\mu + \varphi) & -\beta S + \gamma & \theta \\ \beta I & \beta S - (\mu + \gamma + \alpha) & 0 \\ \varphi & 0 & -(\mu + \theta) \end{pmatrix},$$

and its corresponding second additive compound matrix is

$$J^{[2]} = \begin{pmatrix} -\beta I + \beta S & & \\ -(2\mu+\varphi+\gamma+\alpha) & 0 & -\theta \\ 0 & -\beta I-(2\mu+\varphi+\theta) & -\beta S+\gamma \\ -\varphi & \beta I & \beta S-(2\mu+\gamma+\alpha+\theta) \end{pmatrix}.$$

Set the function $Q = Q(S, I, V) = \frac{S}{I}\mathbb{I}_3$, where $\mathbb{I}_3$ is the identity matrix. We obtain $Q_f = \left(\frac{S'}{I} - \frac{SI'}{I^2}\right)\mathbb{I}_3$ and thus $Q_f Q^{-1} = \left(\frac{S'}{S} - \frac{I'}{I}\right)\mathbb{I}_3$. On the other hand obviously $QJ^{[2]}Q^{-1} = J^{[2]}$. Therefore the matrix $B = Q_f Q^{-1} + QJ^{[2]}Q^{-1}$ can be written in block form as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

in which,

$B_{11} = \frac{S'}{S} - \frac{I'}{I} - \beta I + \beta S - (2\mu+\varphi+\gamma+\alpha)$,
$B_{12} = (0-\theta), \qquad B_{21} = (0-\varphi)^\top$,
$B_{22} = \begin{pmatrix} \frac{S'}{S} - \frac{I'}{I} - \beta I - (2\mu+\varphi+\theta) & -\beta S+\gamma \\ \beta I & \frac{S'}{S} - \frac{I'}{I} + \beta S - (2\mu+\gamma+\alpha+\theta) \end{pmatrix}$.

From [18], if we select the norm $\Big|(u,v,w)\Big| = \max\Big\{\Big|u\Big|, \Big|v\Big|+\Big|w\Big|\Big\}$ for $(u,v,w) \in \mathbb{R}^3$, then we have

$$\psi(B) = \sup\{g_1, g_2\},$$

where $g_1 = \psi(B_{11}) + \Big|B_{12}\Big|$ and $g_2 = \psi(B_{22}) + \Big|B_{21}\Big|$ and ψ denotes the Lozinskiĭ measure with respect to the defined norm. Thus, we have $\psi(B_{11}) = \frac{S'}{S} - \frac{I'}{I} - \beta I + \beta S - (2\mu+\varphi+\gamma+\alpha), \mid B_{12} \mid = \theta, \mid B_{21} \mid = \varphi$ and

$$\begin{aligned}\psi(B_{22}) &= \max\left\{\frac{S'}{S} - \frac{I'}{I} - (2\mu+\varphi+\theta), \frac{S'}{S} - \frac{I'}{I} - (2\mu+\gamma+\alpha+\theta)\right\} \\ &= \frac{S'}{S} - \frac{I'}{I} - (2\mu+\theta) + \max\{-\varphi, -\alpha\}\end{aligned}$$

From second equation of system (2.1) we have $\beta S = \frac{I'}{I} + (\mu+\gamma+\alpha)$, thus

$$\begin{aligned}g_1 &= \frac{S'}{S} - \frac{I'}{I} - \beta I + \beta S - (2\mu+\varphi+\gamma+\alpha) + \theta \\ &= \frac{S'}{S} - \beta I - \mu - \varphi + \theta \le \frac{S'}{S} - \mu - \varphi + \theta,\end{aligned}$$

and

$$\begin{aligned} g_2 &= \frac{S'}{S} - \frac{I'}{I} - (2\mu + \theta) + \max\{-\varphi, -\alpha\} + \varphi \\ &= \frac{S'}{S} - \beta S - \mu - \theta + \gamma + \alpha + \varphi + \max\{-\varphi, -\alpha\} \\ &= \frac{S'}{S} - \beta S - \mu - \theta + \gamma + \max\{\alpha, \varphi\}. \\ &\leq \frac{S'}{S} - \mu - \theta + \gamma + \max\{\alpha, \varphi\}. \end{aligned}$$

Therefore

$$\begin{aligned} \psi(B) = \sup\{g_1, g_2\} &\leq \frac{S'}{S} - \mu + \max\{-\varphi + \theta, -\theta + \gamma + \varphi, -\theta + \gamma + \alpha\} \\ &= \frac{S'}{S} - \mu - \min\{\varphi - \theta, \theta - \gamma - \varphi, \theta - \gamma - \alpha\}. \end{aligned}$$

By the assumption in the state of theorem we get

$$\psi(B) = \sup\{g_1, g_2\} \leq \frac{S'}{S} - \eta,$$

in which $\eta = \beta + \mu + \min\{\varphi - \theta, \theta - \gamma - \varphi, \theta - \gamma - \alpha\} > 0$.

Therefore for any solution of system (2.1) with $(S(0), I(0), V(0)) \in K$, we have

$$\frac{1}{t}\int_0^t \psi(B)dr \leq \frac{1}{t}\int_0^t \left(\frac{S'}{S} - \eta\right) dr = \frac{1}{t}\ln\frac{S(t)}{S(0)} - \eta,$$

which implies

$$\overline{q}_2 = \limsup_{t\to\infty} \sup \frac{1}{t}\int_0^t \psi(B)dr < -\frac{1}{2}\eta < 0,$$

and according to the Lemma 4.6, E^* is globally asymptotically stable. □

5. The impact of vaccination

In this section we discuss briefly the model in absence of vaccination. If we consider model (2.1) without vaccination i.e. we omit the compartment V, we have an SIS epidemic model and the vaccination-free basic reproduction number becomes

$$\widetilde{\mathcal{R}}_0 = \frac{\beta\Lambda}{\mu(\mu + \gamma + \alpha)},$$

and therefore

$$\mathcal{R}_0 = \left(1 - \frac{\mu q + \varphi}{\mu + \theta + \varphi}\right)\widetilde{\mathcal{R}}_0.$$

This shows that $\mathcal{R}_0 \leq \widetilde{\mathcal{R}}_0$ and thus disease will extinct sooner in present of vaccination. For under study population if $\widetilde{\mathcal{R}}_0 < 1$ the disease will die out,

but when $\widetilde{\mathcal{R}}_0 > 1$ it will persist in population and thus vaccination must be performed such that $\mathcal{R}_0 \leq 1$. To this be satisfied we must have

$$\varphi + \mu q \widetilde{\mathcal{R}}_0 \geq (\mu + \theta)(\widetilde{\mathcal{R}}_0 - 1).$$

This shows that to eradicate the disease at least a fraction

$$q^*_{vac} = \frac{1}{\mu \widetilde{\mathcal{R}}_0} \left[(\mu + \theta)(\widetilde{\mathcal{R}}_0 - 1) - \varphi \right]. \tag{5.1}$$

of new members must be vaccinated. From (5.1) it can be concluded that greater values for φ and less values for θ yield less values for q^*_{vac}. This shows that increase in the proportion of susceptible individuals who protected by vaccination (φ), and also in the period time of loosing immunity $\left(\frac{1}{\theta}\right)$, accelerates the disease eradication. however, it must be noted that if $q^*_{vac} > 1$, the vaccination can not overcome the disease even if all new members are vaccinated.

6. An example

In this section, using numerical approach the theoretical results obtained in preceding sections will be examined. First, a bifurcation diagram of compartment $I(t)$ in terms of various values of parameter q is presented. Then, some solutions of the model in two cases $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$ are given.

Example 6.1. Let values of parameters in model (2.1) are as $\Lambda = 0.2, \beta = 0.6, \mu = 0.1, \varphi = 0.2, \alpha = 0.2, \gamma = 0.3, \theta = 0.2$, and $q \in (0, 1)$. Assume also that the unit of population size is one million individuals and initial values are $S(0) = 0.8, I(0) = 0.4$ and $V(0) = 0.5$.

A bifurcation diagram of number of infected individuals (variable $I(t)$) in terms of fraction of vaccinated new members (parameter q) is presented in Figure 1. For values $q < 0.5$ obviously final values of $I(t)$ have positive values and the endemic equilibrium E^* is stable, while for $q > 0.5$ the disease-free equilibrium E^0 is stable. This can be also concluded from relation (5.1) where $q^*_{vac} = 0.5$ is optimal value for which the disease will be extinct from population.

Now let $q = 0.2$ and $q = 0.8$. For these values for q and same values for other parameters as before, we have $\mathcal{R}_0 = 1.12 > 1$ and $\mathcal{R}_0 = 0.88 < 1$, respectively. Then, according to Theorem 4.6 and Theorem 3.2 disease persists and will be wiped out in these cases, respectively. Solutions of the model for these two values of parameter q are shown in Figure 2.

7. Summary

In this paper, we studied an SIS epidemic model that includes a vaccination program. The vaccination consist of new members and susceptible individuals. Although vaccination effect is perfect i.e. no vaccinated individual becomes infectious, its immunity is lost gradually. The basic reproduction number $\mathcal{R}_0$, and

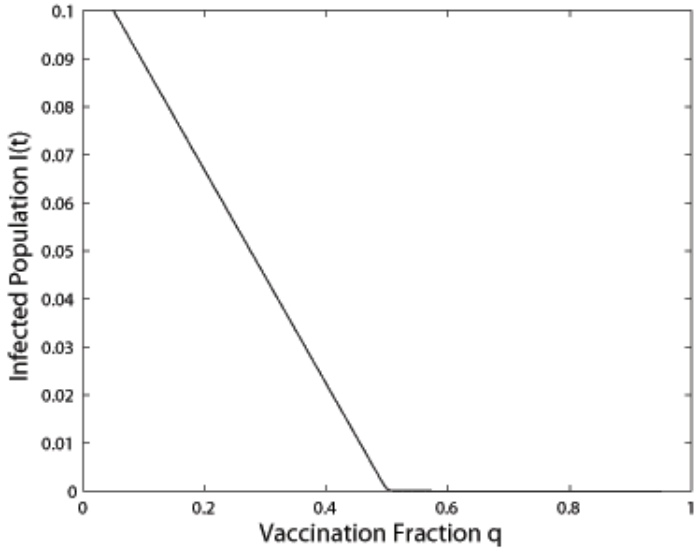

Figure 1: A diagram for final values of infected population $I(t)$ in terms of various values of fraction q.

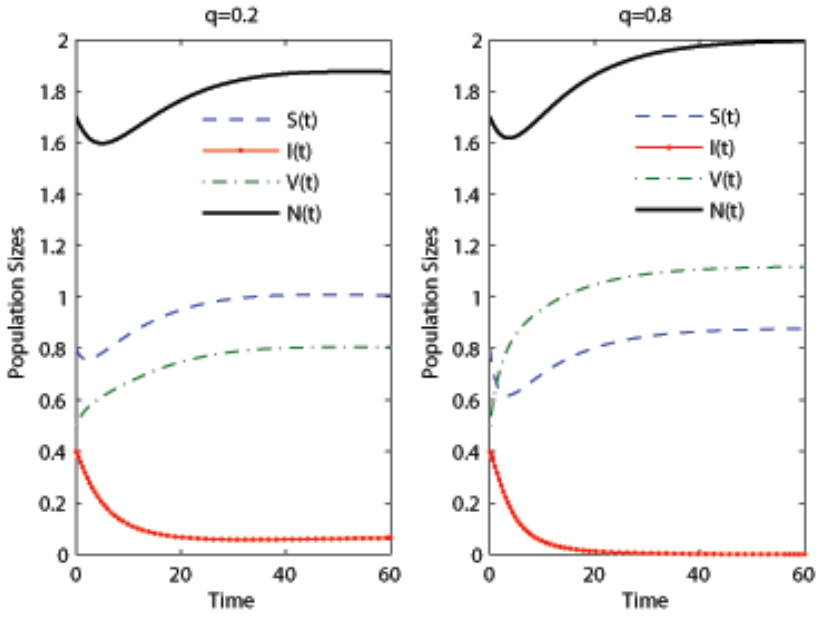

Figure 2: Solutions of model (2.1) for values $q = 0.2$ and $q = 0.8$ and other parameter values as in Example 6.1.

equilibria of the model were found. The dynamics of the model were determined by threshold $\mathcal{R}_0$; if $\mathcal{R}_0 < 1$, it was proved that the disease-free equilibrium is locally as well as globally asymptotically stable. In this case the disease dies out and disease extinction occurs. while the disease will persist in population permanently if $\mathcal{R}_0 > 1$. Indeed it was proved that the endemic equilibrium is locally asymptotically stable by means of the second additive compound matrix method and is globally asymptotically stable using a geometric approach. Impact of vaccination was briefly discussed and an optimal fraction of new members who must be vaccinated to disease dies out, was found. Finally, the theoretical results were discussed also numerically in some examples.

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