SOME OPERATOR α-GEOMETRIC MEAN INEQUALITIES

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Abstract. In this paper, we refine an operator α-geometric mean inequality as follows: let Φ be a positive unital linear map and let A and B be positive operators. If $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$, then for each $\alpha \in [0,1]$,

$$(\Phi(A)\sharp_\alpha \Phi(B))^2 \leq \left(\frac{K(h)}{K^{2r}(h')}\right)^2 \Phi^2(A\sharp_\alpha B),$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h') = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r = \min\{\alpha, 1-\alpha\}$.
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1. Introduction

Throughout this paper, $\|\cdot\|$ is the operator norm and I denotes the identity operator. $A \geq 0$ $(A > 0)$ implies that A is positive (strictly positive) operator. Φ is a positive unital linear map if $\Phi(A) \geq 0$ with $A \geq 0$ and $\Phi(I) = I$. For $A, B > 0$ and $\alpha \in [0,1]$, the α-geometric mean $A\sharp_\alpha B$ is defined by

$$A\sharp_\alpha B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}},$$

when $\alpha = \frac{1}{2}$, $A\sharp_{\frac{1}{2}}B = A\sharp B$ is said to be the geometric mean.

Seo [1] gave the following α-geometric mean inequality: let Φ be a positive unital linear map. If $0 < m_1 \leq A, B \leq M_1$ for some numbers $m_1 \leq M_1$. Then for $\alpha \in [0,1]$,

$$\Phi(A)\sharp_\alpha\Phi(B) \leq K(m, M, \alpha)^{-1}\,\Phi(A\sharp_\alpha B),$$

where $m = \frac{m_1}{M_1}$, $M = \frac{M_1}{m_1}$ and the generalized Kantorovich constant $K(m, M, \alpha)$ ([2, Definition 2.2]) is defined by

$$K(m, M, \alpha) = \frac{mM^\alpha - Mm^\alpha}{(\alpha-1)(M-m)}\left(\frac{\alpha-1}{\alpha}\frac{M^\alpha - m^\alpha}{mM^\alpha - Mm^\alpha}\right)^\alpha$$

for any real number $\alpha \in R$.

Fu [3] squared operator α-geometric mean inequality: let Φ be a positive unital linear map. If $0 < m \leq A, B \leq M$ for some numbers $m \leq M$. Then for $\alpha \in [0,1]$

$$(\Phi(A)\sharp_\alpha \Phi(B))^2 \leq K^2(h)\,\Phi^2(A\sharp_\alpha B), \tag{1.1}$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

A great number of results on operator inequalities have been given in the literature, for example, see [4-8] and the references therein.

In this paper, we will get a stronger result than (1.1) and apply it to obtain an operator α-geometric mean inequality to the power of $2p$ $(p \geq 2)$.

2. Main results

In this section, the main results of this paper will be given. To do this, the following lemmas are necessary.

Lemma 1 ([9])**.** *Let $A, B > 0$. Then*

$$\|AB\| \leq \frac{1}{4}\|A+B\|^2. \tag{2.1}$$

Lemma 2 ([10])**.** *Let $A > 0$. Then for every positive unital linear map Φ,*

$$\Phi(A^{-1}) \geq \Phi^{-1}(A). \tag{2.2}$$

Lemma 3 ([11])**.** *Let $A, B > 0$. Then for $1 \leq r < \infty$,*

$$\|A^r + B^r\| \leq \|(A+B)^r\|. \tag{2.3}$$

Lemma 4 ([12])**.** *Let $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$. Then for each $\alpha \in [0,1]$,*

$$K^r(h')\,(A\sharp_\alpha B) \leq A\nabla_\alpha B, \tag{2.4}$$

where $K(h') = \frac{(h'+1)^2}{4h'}$, $h' = \frac{M'}{m'}$ and $r = \min\{\alpha, 1-\alpha\}$.

Lemma 5. *Let $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$. Then for each $\alpha \in [0,1]$,*

$$K^r(h')\left(A^{-1}\sharp_\alpha B^{-1}\right) \leq A^{-1}\nabla_\alpha B^{-1}, \tag{2.5}$$

where $K(h') = \frac{(h'+1)^2}{4h'}$, $h' = \frac{M'}{m'}$ and $r = \min\{\alpha, 1-\alpha\}$.

Proof. If $0 < m \le A \le m' < M' \le B \le M$, it follows that

$$0 < \frac{1}{M} \le B^{-1} \le \frac{1}{M'} < \frac{1}{m'} \le A^{-1} \le \frac{1}{m}.$$

By $h' = \frac{M'}{m'} = \frac{\frac{1}{m'}}{\frac{1}{M'}}$ and (2.4), we have

$$K^r\left(h'\right)\left(A^{-1}\sharp_\alpha B^{-1}\right) \le A^{-1}\nabla_\alpha B^{-1}.$$

If $0 < m \le B \le m' < M' \le A \le M$, similarly, (2.5) holds.

This completes the proof. □

Theorem 1. *Let Φ be a positive unital linear map and let A and B be positive operators. If $0 < m \le A \le m' < M' \le B \le M$ or $0 < m \le B \le m' < M' \le A \le M$, then for each $\alpha \in [0,1]$,*

$$(\Phi(A)\sharp_\alpha \Phi(B))^2 \le \left(\frac{K(h)}{K^{2r}(h')}\right)^2 \Phi^2(A\sharp_\alpha B), \tag{2.6}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $K(h') = \frac{(h'+1)^2}{4h'}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r = \min\{\alpha, 1-\alpha\}$.

Proof. The inequality (2.6) is equivalent to

$$\left\|(\Phi(A)\sharp_\alpha \Phi(B))\,\Phi^{-1}(A\sharp_\alpha B)\right\| \le \frac{K(h)}{K^{2r}(h')}.$$

It is easy to see that

$$(1-\alpha)\left(A + MmA^{-1}\right) \le (1-\alpha)(M+m) \tag{2.7}$$

and

$$\alpha\left(B + MmB^{-1}\right) \le \alpha(M+m). \tag{2.8}$$

Summing up inequalities (2.7) and (2.8), we get

$$A\nabla_\alpha B + Mm\left(A^{-1}\nabla_\alpha B^{-1}\right) \le M + m$$

and hence

$$\Phi(A\nabla_\alpha B) + Mm\Phi\left(A^{-1}\nabla_\alpha B^{-1}\right) \le M + m. \tag{2.9}$$

Compute

$$\begin{aligned}
&\left\|\Phi(A)\sharp_\alpha\Phi(B)\,MmK^{2r}\left(h'\right)\Phi^{-1}(A\sharp_\alpha B)\right\| \\
&\le \frac{1}{4}\left\|K^r\left(h'\right)\Phi(A)\sharp_\alpha\Phi(B) + MmK^r\left(h'\right)\Phi^{-1}(A\sharp_\alpha B)\right\|^2 \quad (\text{by}\,(2.1)) \\
&\le \frac{1}{4}\left\|K^r\left(h'\right)\Phi(A)\sharp_\alpha\Phi(B) + MmK^r\left(h'\right)\Phi\left(A^{-1}\sharp_\alpha B^{-1}\right)\right\|^2 \quad (\text{by}\,(2.2)) \\
&\le \frac{1}{4}\left\|\Phi(A)\nabla_\alpha\Phi(B) + Mm\Phi\left(A^{-1}\nabla_\alpha B^{-1}\right)\right\|^2 \quad (\text{by}\,(2.4),(2.5)) \\
&\le \frac{1}{4}\left\|\Phi(A\nabla_\alpha B) + Mm\Phi\left(A^{-1}\nabla_\alpha B^{-1}\right)\right\|^2 \\
&\le \frac{1}{4}(M+m)^2. \quad (\text{by}\,(2.9))
\end{aligned}$$

That is

$$\left\|\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)\Phi^{-1}\left(A\sharp_{\alpha}B\right)\right\| \leq \frac{\left(M+m\right)^{2}}{4MmK^{2r}\left(h'\right)} = \frac{K\left(h\right)}{K^{2r}\left(h'\right)}.$$

Thus, (2.6) holds.

This completes the proof. □

Remark 1. Since $h' > 1$, then

$$\frac{K\left(h\right)}{K^{2r}\left(h'\right)} < K\left(h\right).$$

Thus, inequality (2.6) is tighter than (1.1).

Theorem 2. *Let Φ be a positive unital linear map and let A and B be positive operators. If $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$ and $2 \leq p < \infty$, then for each $\alpha \in [0,1]$,*

$$\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{2p} \leq \frac{1}{16}\left(\frac{K^{2}\left(h\right)\left(M^{2}+m^{2}\right)^{2}}{K^{4r}\left(h'\right)M^{2}m^{2}}\right)^{p}\Phi^{2p}\left(A\sharp_{\alpha}B\right), \tag{2.10}$$

where $K\left(h\right) = \frac{(h+1)^{2}}{4h}$, $K\left(h'\right) = \frac{(h'+1)^{2}}{4h'}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ and $r = \min\left\{\alpha, 1-\alpha\right\}$.

Proof. The inequality (2.10) is equivalent to

$$\left\|\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{p}\Phi^{-p}\left(A\sharp_{\alpha}B\right)\right\| \leq \frac{1}{4}\left(\frac{K^{2}\left(h\right)\left(M^{2}+m^{2}\right)^{2}}{K^{4r}\left(h'\right)M^{2}m^{2}}\right)^{\frac{p}{2}}. \tag{2.11}$$

By the operator reverse monotonicity of inequality (2.6), we have

$$\Phi^{-2}\left(A\sharp_{\alpha}B\right) \leq \left(\frac{K\left(h\right)}{K^{2r}\left(h'\right)}\right)^{2}\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{-2}. \tag{2.12}$$

Since $0 < m \leq A, B \leq M$, it follows that

$$m \leq \Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right) \leq M$$

and hence

$$\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{2} + M^{2}m^{2}\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{-2} \leq M^{2}+m^{2}. \tag{2.13}$$

Compute

$$\begin{aligned}
&\left\|\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{p}M^{p}m^{p}\Phi^{-p}\left(A\sharp_{\alpha}B\right)\right\| \\
&\leq\frac{1}{4}\left\|\left(\frac{K\left(h\right)}{K^{2r}\left(h'\right)}\right)^{\frac{p}{2}}\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{p}+\left(\frac{M^{2}m^{2}}{\frac{K(h)}{K^{2r}(h')}}\right)^{\frac{p}{2}}\Phi^{-p}\left(A\sharp_{\alpha}B\right)\right\|^{2} \quad (\text{by}\,(2.1)) \\
&\leq\frac{1}{4}\left\|\frac{K\left(h\right)}{K^{2r}\left(h'\right)}\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{2}+\frac{M^{2}m^{2}}{\frac{K(h)}{K^{2r}(h')}}\Phi^{-2}\left(A\sharp_{\alpha}B\right)\right\|^{p} \quad (\text{by}\,(2.3)) \\
&\leq\frac{1}{4}\left\|\frac{K\left(h\right)}{K^{2r}\left(h'\right)}\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{2}+\frac{K\left(h\right)}{K^{2r}(h')}M^{2}m^{2}\left(\Phi(A)\sharp_{\alpha}\Phi(B)\right)^{-2}\right\|^{p} \quad (\text{by}\,(2.12)) \\
&\leq\frac{1}{4}\left(\frac{K(h)}{K^{2r}(h')}(M^{2}+m^{2})\right)^{p}.\ (\text{by}\,(2.13))
\end{aligned}$$

That is

$$\left\|\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{p}\Phi^{-p}\left(A\sharp_{\alpha}B\right)\right\|\leq\frac{1}{4}\left(\frac{K^{2}\left(h\right)\left(M^{2}+m^{2}\right)^{2}}{K^{4r}\left(h'\right)M^{2}m^{2}}\right)^{\frac{p}{2}}.$$

Thus, (2.10) holds.

This completes the proof. □

Lemma 6 ([13])**.** *For any bounded operator X,*

$$|X|\leq tI\Leftrightarrow\|X\|\leq t\Leftrightarrow\begin{bmatrix} tI & X \\ X^{*} & tI\end{bmatrix}\geq 0\quad(t\geq 0)\,. \tag{2.14}$$

Theorem 3. *Let Φ be a positive unital linear map and let A and B be positive operators. If $0<m\leq A\leq m'<M'\leq B\leq M$ or $0<m\leq B\leq m'<M'\leq A\leq M$ and $2\leq p<\infty$, then for each $\alpha\in[0,1]$,*

$$\begin{aligned}
&\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{p}\Phi^{-p}\left(A\sharp_{\alpha}B\right)+\Phi^{-p}\left(A\sharp_{\alpha}B\right)\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{p} \\
&\leq\frac{1}{2}\left(\frac{K^{2}\left(h\right)\left(M^{2}+m^{2}\right)^{2}}{K^{4r}\left(h'\right)M^{2}m^{2}}\right)^{\frac{p}{2}},
\end{aligned}\tag{2.15}$$

where $K\left(h\right)=\frac{(h+1)^{2}}{4h}$, $K\left(h'\right)=\frac{(h'+1)^{2}}{4h'}$, $h=\frac{M}{m}$, $h'=\frac{M'}{m'}$ and $r=\min\left\{\alpha,1-\alpha\right\}$.

Proof. Put $t=\frac{1}{2}(\frac{K^{2}(h)(M^{2}+m^{2})^{2}}{K^{4r}(h')M^{2}m^{2}})^{\frac{p}{2}}$, $X_{1}=\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{p}\Phi^{-p}\left(A\sharp_{\alpha}B\right)$, $X_{2}=\Phi^{-p}\left(A\sharp_{\alpha}B\right)\left(\Phi\left(A\right)\sharp_{\alpha}\Phi\left(B\right)\right)^{p}$ and $X=X_{1}+X_{2}$. By (2.11) and (2.14), we have

$$\begin{bmatrix} tI & X_{1} \\ X_{2} & tI\end{bmatrix}\geq 0 \tag{2.16}$$

and

$$\begin{bmatrix} tI & X_2 \\ X_1 & tI \end{bmatrix} \geq 0. \tag{2.17}$$

Summing up (2.16) and (2.17), we have

$$\begin{bmatrix} 2tI & X \\ X & 2tI \end{bmatrix} \geq 0.$$

Since X is self-adjoint, (2.15) follows from the maximal characterization of geometric mean.

This completes the proof. □

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