CONFORMAL ANTI-INVARIANT SUBMERSIONS FROM KENMOTSU MANIFOLDS ONTO RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we define conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds. Further we obtain some results on such submersions from Kenmotsu manifolds into Riemannian manifolds admitting vertical or horizontal structural vector fields. Among the results we find necessary and sufficient conditions for conformal anti-invariant submersions to be totally geodesic. Moreover, we derive decomposition theorems by using the existence of conformal anti-invariant submersions. Finally, we give some examples of conformal anti-invariant submersions such that characteristic vector field ξ is horizontal or vertical.
Keywords: Riemannian submersion, conformal submersion, anti-invariant submersion, conformal anti-invariant submersion.

1. Introduction

The theory of Riemannian submersion between Riemannian manifolds was introduced by O'Neill [13] in 1966 and Gray [9] in 1967. It was useful if one should study such submersions between manifolds with differentiable structures. When Watson was studying almost Hermitian submersions between almost Hermitian manifolds [16] in 1976, he found that the base manifold and each fiber have the same kind of structure as the total space in most of the cases. We note that almost Hermitian submersions have been extended to the almost contact manifolds [6] in 1985. We know that Riemannian submersions are related with mathematical physics and have their applications in the Yang-Mills theory [15], supergravity and superstring theories ([12], [13]), Kaluza-Klein theory ([5], [11]) etc.

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On the other hand, as a generalization of Riemannian submersion, horizontally conformal submersion are introduced [3] and defined as follows: Consider two Riemannian manifolds M of dimension m and N of dimension n with Riemannian metrics g_M and g_N respectivly. A smooth map $f:(M,g_M)\to(N,g_N)$ between Riemannian manifolds is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_M(U,V)=g_N(f_*U,f_*V), \tag{1.1}$$

for every $U,V\in\Gamma(\ker f_*)^\perp$. It is known that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda=1$. Let f is a smooth map between Riemannian manifolds and $x\in M$. Then, f is called horizontally weakly conformal map at x if either (i) $f_{*x}=0$ or (ii) f_{*x} maps the horizontal space $\mathcal{H}=(\ker f_*)^\perp$ conformally onto $T_{f(x)}N$, *i.e.*, f_{*x} is surjective and f_* satisfies the equation (1.1) for U,V vectors tangent to $\mathcal{H}_x$. We call the point x a critical point if it satisfies the type (i) and we call the point x regular point if it satisfies the type (ii). The square root $\lambda(x)=\sqrt{\wedge(x)}$ is called dilation, where number $\wedge(x)$ is called the square dilation which is necessarily non-negative . If horizontally weakly conformal map f is said to be horizontally homothetic, then the gradient of their dilation λ is vertical, i.e., $\mathcal{H}(grad\lambda)=0$ at regular points. A horizontally weakly conformal map f is called horizontally conformal submersion if f has no critical points [3]. Thus, it follows that a Riemannian submersion is a horizontally conformal submersion with dilation identically one. The horizontal conformal maps were introduced independently by Fuglede in 1978 [8] and by Ishihara in 1979 [12]. From the above argument, one can determine that the notion of horizontal conformal maps is a generalization of the idea of Riemannian submersions.

We denote the kernel space of f_* by $\ker f_*$ and consider the orthogonal complementary space $\mathcal{H}=(\ker f_*)^\perp$ to $\ker f_*$. Then the tangent bundle of M has the following decomposition

$$TM=(\ker f_*)\oplus(\ker f_*)^\perp. \tag{1.2}$$

We also denote the *range* of f_* by $rangef_*$ and consider the orthogonal complementary space $(rangef_*)^\perp$ to $rangef_*$ in the tangent bundle TN of N. Thus the tangent bundle TN of N has the following decomposition

$$TN=(rangef_*)\oplus(rangef_*)^\perp. \tag{1.3}$$

We know that Riemannian submersions are very special maps comparing with conformal submersions. Although conformal maps do not preserve distance between points contrary to isometries, they preserve angles between vector fields. This property enables one to transfer certain properties of a manifold to another manifold by deforming such properties. The concept of conformal anti-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds was studied by Akyol and Sahin [1].

In this paper, we study conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds. The paper is organized as follows. In the second section, we collect main notions and formulae which need for this paper.

In section 3, we introduce conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds, investigates the geometry of leaves of the horizontal distribution and the vertical distribution. In this section we also find necessary and sufficient conditions for a conformal anti-invariant submersion to be totally geodesic.

In section 4, we consider conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds such that the characteristic vector field ξ is horizontal vector field. Finally in section 5, we give some examples of conformal anti-invariant submersions such that the characteristic vector field ξ is vertical or horizontal.

2. Preliminaries

Let $\widetilde{M}$ be an almost contact metric manifold. So there exist on $\widetilde{M}$, a $(1,1)$ tensor field ϕ, a vector field ξ, a $1-$form η and g is Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \eta(\xi) = 1, \tag{2.2}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(\phi X, Y) = -g(X, \phi Y), \tag{2.3}$$

for any vector fields X and Y on $\widetilde{M}$ and I is the identity tensor field [2]. An almost contact metric manifold $\widetilde{M}$ is also denoted by $(\widetilde{M}, \phi, \xi, \eta, g)$.

An almost contact metric manifold $\widetilde{M}$ is called a Kenmotsu manifold if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

for X and Y on $\widetilde{M}$, where ∇ is the Riemannian connection of the Riemannian metric g. From above equation, we have

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.5}$$

Definition 1 ([3]). *Let M and N are two Riemannian manifolds with Riemannian metrics g_M and g_N, respectively. If f is a differentiable map from (M, g_M) to (N, g_N), then f is called semi-conformal or horizontally weakly conformal at p if either*

(i) $df_p = 0$, or

(ii) df_p maps the horizontal space $\mathcal{H}_p = (\ker(df_p))^{\perp}$ conformally onto $T_{f(p)}N$ i.e., df_p is surjective and there exists a number $\Lambda(p) \neq 0$ such that

$$g_N(df_*U, df_*V) = \Lambda(p) g_M(U, V), \qquad (U, V \in \mathcal{H}_p), \tag{2.6}$$

where $p \in M$.

Watson introduced the fundamental tensors of a submersion in [13]. It is known that the fundamental tensor play similar role to that of the second fundamental form of an immersion. Defined O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, for vector fields E, F on M by

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.7}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.8}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections [7], and ∇ is Levi-Civita connection on M. On the other hand, from equations (2.7) and (2.8), we have

$$\nabla_X Y = \mathcal{T}_X Y + \widehat{\nabla}_X Y, \tag{2.9}$$

$$\nabla_X U = \mathcal{H}\nabla_X U + \mathcal{T}_X U, \tag{2.10}$$

$$\nabla_U X = \mathcal{A}_U X + \mathcal{V}\nabla_U X, \tag{2.11}$$

$$\nabla_U V = \mathcal{H}\nabla_U V + \mathcal{A}_U V, \tag{2.12}$$

for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$, where $\mathcal{V}\nabla_X Y = \widehat{\nabla}_X Y$. If U is basic, then $\mathcal{A}_X U = \mathcal{H}\nabla_X U$.

It is seen that for $p \in M$, $X \in \mathcal{V}_p$ and $U \in \mathcal{H}_p$ the linear operators $\mathcal{A}_U$, $\mathcal{T}_X : T_pM \to T_pM$, are skew-symmetric, that is

$$g(\mathcal{A}_U E, F) = -g(E, \mathcal{A}_U F) \text{ and } g(\mathcal{T}_X E, F) = -g(E, \mathcal{T}_X F), \tag{2.13}$$

for each $E, F \in T_pM$. We have also defined the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}|_{\mathcal{V}\times\mathcal{V}}$ is precisely the second fundamental form of the fibers of f. Since $\mathcal{T}_V$ is skew-symmetric we get: f has totally geodesic fibers if and only if $\mathcal{T} \equiv 0$. For the special case when f is horizontally conformal we have the following:

Proposition 1 ([10], (2.1.2)). *Let f be a horizontal conformal submersion between Riemannian manifolds (M, g_M) and (N, g_N) with dilation λ and U, V be horizontal vectors, then*

$$\mathcal{A}_U V = \frac{1}{2}\{\mathcal{V}[U, V] - \lambda^2 g_M(U, V) grad_{\mathcal{V}}(\frac{1}{\lambda^2})\}. \tag{2.14}$$

We know that the skew-symmetric part of $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ measures the obstruction integrability of the horizontal distribution $\mathcal{H}$.

Let $f : (M, g_M) \to (N, g_N)$ is a smooth map between Riemannian manifolds. Then the differential of f_* of f can be observed a section of the bundle $Hom(TM, f^{-1}TN) \to M$, where $f^{-1}TN$ is the pullback bundle which has fibers $(f^{-1}TN)_p = T_{f(p)}N$ has a connection ∇ induced from the Riemannian connection and ∇^M the pullback connection. Then the second fundamental form of f is given by

$$(\nabla f_*)(U, V) = \nabla^f_U f_*(V) - f_*(\nabla^M_U V), \tag{2.15}$$

for vector fields $U, V \in \Gamma(TM)$, where ∇^f is the pullback connection. We know that the second fundamental form is symmetric.

Next, we find necessary and sufficient condition for conformal anti-invariant submersion to be totally geodesic. We recall that a differentiable map f between two Riemannian manifolds is called totally geodesic if $(\nabla f_*)(V, W) = 0$, for all $V, W \in \Gamma(TM)$.

A geometric clarification of a totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

We know that the followings from [14]. Let $B = M \times N$ be a manifold with Riemannian metric g_B and assume that the canonical foliations D_M and D_N intersect perpendicularly everywhere. Then g_B is the metric tensor of

(i) a twisted product $M \times_F N$ if and only if D_M is totally geodesic foliation and D_N is totally umbilical foliation,

(ii) a warped product $M \times_F N$ if and only if D_M is totally geodesic foliation and D_N is a spheric foliation, i.e., it is umbilical and its mean curvature vector field is parallel,

We note in this case, from [4] we have $\nabla_X U = X(\ln F)U$, for $X \in \Gamma(TM)$ and $U \in \Gamma(TN)$, where ∇ is the Riemannian connection on $M \times N$.

(iii) a usual product of Riemannian manifolds if and only if D_M and D_N are totally geodesic foliations.

Next, we explain a decomposition theorem related to the concept of twisted product manifold. However, we first define the adjoint map of a map. Let $f : (M, g_M) \to (N, g_N)$ be a map between Riemannian manifolds (M, g_M) and (N, g_N). Then the adjoint map ${}^*f_*$ of f_* is characterized $g_M(X, {}^*f_{*p}Y) = g_N(f_{*p}X, Y)$ by $X \in T_pM$, $Y \in T_{f(p)}N$ and $p \in M$. Considering f^h_* at each $p \in M$ as a linear transformation

$$f^h_{*p} : ((\ker f_*)^\perp_{(p)}, g_{M(p)((\ker f_*)^\perp_p)}) \to (range f_{*(q)}, g_{N(q)(range f_*)(q)}),$$

we will denote the adjoint $f^h_{*(p)}$ by ${}^*f^h_{*(p)}$. Let $f^h_{*(p)}$ be the adjoint of $f^h_{*(p)}$: $(T_pM, g_{M(p)}) \to (T_{(q)}N, g_{N(q)})$. The linear transformation $({}^*f_{*p})^h : (range f_{*(p)}) \to (\ker f_*)^\perp_{(p)}$ defined $({}^*f_{*(p)})^h Y = {}^* f^h_{*(p)} Y$, where $Y \in (ranrge f_{*(p)}), q = f(p)$, is an isomorphism and $(f^h_{*(p)})^{-1} = ({}^*f_{*p})^h = {}^* f^h_{*(p)}$.

Lastly, we recollection the subsequent lemma from [3].

Lemma 1. *Let (M, g_M) and (N, g_N) are two Riemannian manifolds. If $f : M \to N$ horizontally conformal submersion between Riemannian manifolds, then for any horizontal vector fields U, V and vertical vector fields X, Y, we have*

$$\begin{aligned}(i)\nabla df(U,V) &= U(ln\lambda)df(V) + V(ln\lambda)df(U) - g_M(U,V)df(gradln\lambda);\\ (ii)\nabla df(X,Y) &= -df(\mathcal{A}^{\mathcal{V}}_X Y);\\ (iii)\nabla df(U,X) &= -df(\nabla^M_U X) = df((\mathcal{A}^{\mathcal{H}})^*_U X).\end{aligned}$$

where $(\mathcal{A}^{\mathcal{H}})^*_X$ is the adjoint of $(\mathcal{A}^{\mathcal{H}}_X)$ characterized by $\langle(\mathcal{A}^{\mathcal{H}})^*_U E, F\rangle = \langle E, \mathcal{A}^{\mathcal{H}}_U F\rangle$, $(E, F \in \Gamma(TM))$.

3. Conformal anti-invariant submersions

In this section we are going to introduce and study conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds such that the characteristic vector field ξ is vertical vector field.

Definition 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. A horizontally conformal submersion $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ with dilation λ is a called conformal anti-invariant submersion if the distribution $\ker f_*$ is anti-invariant with respect to ϕ i.e., $\phi(\ker f_*) \subseteq (\ker f_*)^\perp$. We have $\phi(\ker f_*)^\perp \cap \ker f_* \neq \{0\}$. We denote the complementary orthonormal distribution to $\phi(\ker f_*)$ of μ in $(\ker f_*)^\perp$. Then we have*

$$(\ker f_*)^\perp = \phi(\ker f_*) \oplus \mu. \tag{3.1}$$

For any $U \in \Gamma(\ker f_*)^\perp$, we have

$$\phi U = BU + CU, \tag{3.2}$$

where $BU \in \Gamma(\ker f_*)$ and $CU \in \Gamma(\mu)$. On the additional fact, since $f_*(\Gamma(\ker f_*)^\perp) = TN$ and f is a conformal submersion, for every $X \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$, using equation (3.2) we get $\frac{1}{\lambda^2} g_N(f_*\phi U, f_*CX) = 0$, which denotes that

$$TN = f_*(\phi(\ker f_*)) \oplus f_*(\mu). \tag{3.3}$$

Lemma 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then*

$$g_M(CV, \phi X) = 0, \tag{3.4}$$

and

$$g_M(\nabla^M_U CV, \phi X) = -g_M(CV, \phi\mathcal{A}_U X) + \eta(X) g_M(CU, CV), \tag{3.5}$$

for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, since $BV \in \Gamma(\ker f_*)$ and $\phi X \in (\Gamma(\ker f_*)^\perp)$, using equations (3.2) and (2.3), we have $g_M(CV, \phi X) = 0$. Now, using equations (2.3), (2.11) and (3.4), we get $g_M(\nabla_U CV, \phi X) = -g_M(CV, \nabla_U \phi X), = -g_M(CV, \phi \mathcal{A}_U X) + \eta(X) g_M(CU, CV)$, since $\phi \mathcal{V} \nabla_U X \in \Gamma(\phi(\ker f_*))$. □

Theorem 1. *Let* $(M, \phi, \xi, \eta, g_M)$ *be a Kenmotsu manifold and* (N, g_N) *be a Riemannian manifold. If* $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ *be a conformal anti-invariant submersion, then the followings are equivalent to each other:*

(i) $(\ker f_*)^\perp$ *is integrable,*

(ii) $\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X)$
$- g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) + g_M(\mathcal{H} gradln\lambda, CU) g_M(V, \phi X)$
$- 2 g_M(\mathcal{H} gradln\lambda, \phi X) g_M(CU, V)$, *for* $X \in \Gamma(\ker f_*)$ *and* $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, since $\phi V \in (\Gamma \ker f_* \oplus \mu)$ and $\phi X \in (\Gamma(\ker f_*)^\perp)$. Using equations $(2.1), (2.3), (2.4)$ and (3.2), we get

$$\begin{aligned} g_M([U,V], X) &= g_M(\phi \nabla_U V, \phi X) + \eta(X)\eta(\nabla_U V) \\ &\quad - g_M(\phi \nabla_V U, \phi X) - \eta(X)\eta(\nabla_V U), \\ &= g_M(\nabla_U \phi V, \phi X) - g_M(\nabla_V \phi U, \phi X) - g_M([U,V], \xi)\eta(X), \\ &= g_M(\nabla_U BV, \phi X) + g_M(\nabla_U CV, \phi X) - g_M(\nabla_V BU, \phi X) \\ &\quad - g_M(\nabla_V CU, \phi X) - g_M([U,V], \xi)\eta(X). \end{aligned}$$

Since f is a conformal submersion, using equations (2.11) and (2.12), we get

$$\begin{aligned} g_M([U,V], X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) + \frac{1}{\lambda^2} g_N(f_* \nabla_U CV, f_* \phi X) \\ &\quad - \frac{1}{\lambda^2} g_N(f_* \nabla_V CU, f_* \phi X) - g_M([U,V], \xi)\eta(X). \end{aligned}$$

Using equations $(2.5), (2.15), (3.4)$ and lemma $1(i)$, we get

$$\begin{aligned} g_M([U,V], X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\ &\quad + g_M(\mathcal{H} gradln\lambda, CU) g_M(V, \phi X) \\ &\quad - 2 g_M(\mathcal{H} gradln\lambda, \phi X) g_M(CU, V) \\ &\quad - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X). \end{aligned}$$

which implies $(i) \Leftrightarrow (ii)$. □

Theorem 2. *Let* $(M, \phi, \xi, \eta, g_M)$ *be a Kenmotsu manifold and* (N, g_N) *be a Riemannian manifold. If* $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ *be a conformal anti-invariant submersion, then any two of the following conditions imply the third:*

(i) $(\ker f_*)^\perp$ is integrable,

(ii) f is horizontally homothetic,

(iii) $\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, using Theorem (1), we get

$$\begin{aligned} g_M([U,V],X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(U, \phi X) g_M(\mathcal{H} gradln\lambda, CV) \\ &+ g_M(V, \phi X) g_M(\mathcal{H} gradln\lambda, CU) - 2 g_M(CU, V) g_M(\mathcal{H} gradln\lambda, \phi X) \\ &- \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X). \end{aligned}$$

Now, using conditions (i) and (ii), we get (iii)

$$\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X).$$

Similarly, one can obtain the other assertions. □

Remark 1. Let f be a conformal anti-invariant submersion is conformal Lagrangian submersion, if $\phi(\ker f_*) = (\ker f_*)^{\perp}$. Then equation (3.3), we have $TN = f_*(\phi(\ker f_*)^{\perp})$.

Corollary 1. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal Lagrangian submersion, then the following assertions are equivalent to each other:*
(i) $(\ker f_*)^{\perp}$ *is integrable,*
(ii) $\mathcal{A}_U \phi V = \mathcal{A}_V \phi U$,
(iii) $(\nabla f_*)(V, \phi U) = (\nabla f_*)(U, \phi V)$, *for* $U, V \in (\Gamma(\ker f_*)^{\perp})$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, since $\phi X \in (\Gamma(\ker f_*)^{\perp})$ and $\phi V \in \Gamma(\phi(\ker f_*))$. From Theorem (1), we have

$$\begin{aligned} g_M([U,V],X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\ &+ g_M(\mathcal{H} gradln\lambda, CU) g_M(V, \phi X) - 2 g_M(\mathcal{H} gradln\lambda, \phi X) g_M(CU, V) \\ &- \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X). \end{aligned}$$

Since f conformal Lagrangian submersion, we have

$$g_M([U,V],X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X),$$

which implies $(i) \Leftrightarrow (ii)$. On the further using definition (2) and equation (2.11), we get

$$\begin{aligned} g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) &= g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{A}_V BU, \phi X), \\ &= \frac{1}{\lambda^2} g_N(f_* \mathcal{A}_U BV, f_* \phi X) - \frac{1}{\lambda^2} g_N(f_* \mathcal{A}_V BU, f_* \phi X), \\ &= \frac{1}{\lambda^2} g_N(f_*(\nabla_U BV), f_* \phi X) - \frac{1}{\lambda^2} g_N(f_*(\nabla_V BU), f_* \phi X). \end{aligned}$$

Now, using equation (2.15) we have

$$\begin{aligned}
&\frac{1}{\lambda^2} g_N(f_*(\nabla_U BV), f_*\phi X) - \frac{1}{\lambda^2} g_N(f_*(\nabla_V BU), f_*\phi X) \\
&= \frac{1}{\lambda^2} g_N(-(\nabla f_*)(U, BV) + \nabla_U^f f_* BV, f_*\phi X) \\
&- \frac{1}{\lambda^2} g_N(-(\nabla f_*)(V, BU) + \nabla_V^f f_* BU, f_*\phi X), \\
&= \frac{1}{\lambda^2} [g_N((\nabla f_*)(V, BU) - (\nabla f_*)(U, BV), f_*\phi X)],
\end{aligned}$$

which proves that $(ii) \Leftrightarrow (iii)$. □

Theorem 3. *Let* $(M, \phi, \xi, \eta, g_M)$ *be a Kenmotsu manifold and* (N, g_N) *be a Riemannian manifold. If* $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ *be a conformal anti-invariant submersion, then the followings are equivalent to each other:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,
(ii) $-\frac{1}{\lambda^2} g_N(\nabla_U^f f_* CV, f_*\phi X) = g_M(\mathcal{A}_U BV, \phi X)$
$-g_M(U, \phi X) g_M(\mathcal{H} gradln\lambda, CV) + g_M(U, CV) g_M(\mathcal{H} gradln\lambda, \phi X)$
$-\eta(X) g_M(U, V)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, using equations (2.3), (2.4), (2.11), (2.12) and (3.2), we have

$$\begin{aligned}
g_M(\nabla_U V, X) &= g_M(\nabla_U \phi V, \phi X) + \eta(X)\eta(\nabla_U V), \\
&= g_M(\mathcal{A}_U BV, \phi X) + g_M(\mathcal{H}\nabla_U CV, \phi X) + \eta(X)\eta(\nabla_U V).
\end{aligned}$$

Since f is conformal submersion, using equation (2.15), lemma 1(i), definition (2) and equation (3.4), we get

$$\begin{aligned}
g_M(\nabla_U V, X) = {}& g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\
&- \eta(X) g_M(U, V) + g_M(\mathcal{H} gradln\lambda, \phi X) g_M(U, CV) \\
&+ \frac{1}{\lambda^2} g_N(\nabla_U^f f_* CV, f_*\phi X),
\end{aligned}$$

which implies $(i) \Leftrightarrow (ii)$. □

Theorem 4. *Let* $(M, \phi, \xi, \eta, g_M)$ *be a Kenmotsu manifold and* (N, g_N) *be a Riemannian manifold. If* $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ *be a conformal anti-invariant submersion, then any two of the following conditions imply the third:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,
(ii) f is horizontally homothetic,
(iii) $g_M(\mathcal{A}_U BV, \phi X) - \eta(X) g_M(U, V) = -\frac{1}{\lambda^2} g_N(\nabla_U^f f_* CV, f_*\phi X)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, using Theorem (3), we have

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) - \eta(X) g_M(U, V) + g_M(\mathcal{H} gradln\lambda, \phi X) g_M(U, CV) + \frac{1}{\lambda^2} g_N(\nabla_U^f f_* CV, f_* \phi X).$$

Using conditions (i) and (ii), we get (iii)

$$g_M(\mathcal{A}_U BV, \phi X) - \eta(X) g_M(U, V) = -\frac{1}{\lambda^2} g_N(\nabla_U^f f_* CV, f_* \phi X).$$

Similarly, one can obtain the other assertions. □

Corollary 2. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal Lagrangian submersion, then the followings are equivalent to each other:*

(i) $(\ker f_*)^{\perp}$ defines a totally geodesic foliation on M,
(ii) $g_M(\mathcal{A}_U \phi V, \phi X) = \eta(X) g_M(U, V)$,
(iii) $-\frac{1}{\lambda^2} g_N((\nabla f_*)(U, \phi V), f_* \phi X) = \eta(X) g_M(U, V)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^{\perp}$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, from definition (2), $\phi V \in \Gamma(\phi(\ker f_*))$ and $\phi X \in \Gamma((\ker f_*)^{\perp})$. Using Theorem (3), we have

$$\begin{aligned} g_M(\nabla_U V, X) = {} & g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\ & - \eta(X) g_M(U, V) + g_M(\mathcal{H} gradln\lambda, \phi X) g_M(U, CV) \\ & + \frac{1}{\lambda^2} g_N(\nabla_U^f f_* CV, f_* \phi X). \end{aligned}$$

Since f is conformal Lagrangian submersion, we get

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - \eta(X) g_M(U, V) \\ &= g_M(\mathcal{A}_U \phi V, \phi X) - \eta(X) g_M(U, V), \end{aligned}$$

which implies $(i) \Leftrightarrow (ii)$.

On the further needed, using equation (2.11), we get

$$g_M(\mathcal{A}_U BV, \phi X) = g_M(\nabla_U BV, \phi X).$$

Since f is conformal submersion, we get

$$g_M(\mathcal{A}_U BV, \phi X) = \frac{1}{\lambda^2} g_N(f_* \nabla_U BV, f_* \phi X).$$

Using equation (2.15), we get

$$\begin{aligned} g_M(\mathcal{A}_U BV, \phi X) &= -\frac{1}{\lambda^2} g_N((\nabla f_*)(U, BV), f_* \phi X) \\ &= -\frac{1}{\lambda^2} g_N((\nabla f_*)(U, \phi V), f_* \phi X), \end{aligned}$$

which shows $(ii) \Leftrightarrow (iii)$. □

Theorem 5. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then the followings are equivalent to each other:*

(i) $(\ker f_*)$ *defines a totally geodesic foliation on M,*

(ii) $-\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X, f_*\phi CU)=g_M(\mathcal{T}_X\phi Y,BU)$ $+g_M(\phi Y,\phi X)g_M(\mathcal{H}gradln\lambda,\phi CU)$, *for $X,Y\in\Gamma(\ker f_*)$ and $U\in(\Gamma(\ker f_*)^{\perp})$.*

Proof. For $X,Y\in\Gamma(\ker f_*)$ and $U\in(\Gamma(\ker f_*)^{\perp})$, using equations (2.3), (2.4), (2.10) and (3.2), we get

$$\begin{aligned} g_M(\nabla_X Y,U) &= g_M(\phi\nabla_X Y,\phi U)+\eta(\nabla_X Y)\eta(U),\\ &= g_M(\mathcal{T}_X\phi Y,BU)+g_M(\mathcal{H}\nabla_X\phi Y,CU). \end{aligned}$$

Since ∇ is torsion free and $[X,\phi Y]\in\Gamma(\ker f_*)$, we get

$$g_M(\nabla_X Y,U)=g_M(\mathcal{T}_X\phi Y,BU)+g_M(\nabla_{\phi Y}X,CU),$$

using equations (2.3) and (2.4), we get

$$g_M(\nabla_X Y,U)=g_M(\mathcal{T}_X\phi Y,BU)+g_M(\nabla_{\phi Y}\phi X,\phi CU),$$

here we have used μ is invariant. Since f is conformal submersion, using equation (2.15) and Lemma 1(i), we get

$$\begin{aligned} g_M(\nabla_X Y,U) &= g_M(\mathcal{T}_X\phi Y,BU)-\frac{1}{\lambda}g_M(\mathcal{H}gradln\lambda,\phi Y)g_N(f_*\phi X,f_*\phi CU)\\ &\quad -\frac{1}{\lambda}g_M(\mathcal{H}gradln\lambda,\phi X)g_N(f_*\phi Y,f_*\phi CU)\\ &\quad +\frac{1}{\lambda}g_M(\phi Y,\phi X)g_N(f_*\mathcal{H}gradln\lambda,f_*\phi CU)\\ &\quad +\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X,f_*\phi CU). \end{aligned}$$

Next, using definition (2) and equation (3.4), we have

$$\begin{aligned} g_M(\nabla_X Y,U) &= g_M(\mathcal{T}_X\phi Y,BU)+g_M(\phi Y,\phi X)g_M(\mathcal{H}gradln\lambda,\phi CU)\\ &\quad +\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X,f_*\phi CU), \end{aligned}$$

which shows $(i)\Leftrightarrow(ii)$. □

Theorem 6. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then any two of the followings conditions imply the third:*

(i) $(\ker f_*)$ *defines a totally geodesic foliation on M,*

(ii) *λ is constant on $\Gamma(\mu)$,*

(iii) $\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y}f_*\phi X,f_*\phi CU)=-g_M(\mathcal{T}_X\phi Y,\phi U)$,

for $X,Y\in\Gamma(\ker f_*)$ and $U\in(\Gamma(\ker f_*)^{\perp})$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^{\perp})$, from Theorem (5), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU). \end{aligned}$$

Now, using conditions (i) and (iii), we have$g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) = 0$. From above equation λ is constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

Corollary 3. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal Lagrangian submersion, then the following statements are equivalent to each other:*
(i) $(\ker f_*)$ *defines a totally geodesic foliation on M,*
(ii) $\mathcal{T}_X \phi Y = 0$, *for $X, Y \in \Gamma(\ker f_*)$.*

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in \Gamma(\ker f_*)^{\perp}$, from Theorem (5), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU). \end{aligned}$$

Since f is a conformal Lagrangian submersion, we get $g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, \phi U))$, which proves $(i) \Leftrightarrow (ii)$. □

Theorem 7. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then f is a totally geodesic map if and only if*

$$\begin{aligned} -\nabla^f_V f_* W &= f_*(\phi \mathcal{A}_V \phi W_1 + \phi \mathcal{V} \nabla_V B W_2 + \phi \mathcal{A}_V C W_2 + C \mathcal{H} \nabla_V \phi W_1 \\ &\quad + C \mathcal{A}_V B W_2 + C \mathcal{H} \nabla_V C W_2 - \eta(W_1) V + g_M(V, W_2) \xi), \end{aligned}$$

for any $V \in (\Gamma(\ker f_*)^{\perp}), W \in \Gamma(TM)$, where $W = W_1 + W_2$, $W_1 \in \Gamma(\ker f_*)$ and $W_2 \in (\Gamma(\ker f_*)^{\perp})$.

Proof. Taking equation (2.15) and using equations (2.1), and (2.4), we get

$$(\nabla f_*)(V, W) = \nabla^f_V f_* W + f_*(\phi \nabla_V \phi W - \eta(W) V - \eta(\nabla_V W) \xi),$$

for any $V \in (\Gamma(\ker f_*)^{\perp}), W \in \Gamma(TM)$.

Now using equations (2.15) and (3.2), we get

$$\begin{aligned} (\nabla f_*)(V, W) = \nabla^f_V f_* W &+ f_*(\phi \mathcal{A}_V \phi W_1 + B \mathcal{H} \nabla_V \phi W_1 + C \mathcal{H} \nabla_V \phi W_1 \\ &+ B \mathcal{A}_V B W_2 + C \mathcal{A}_V B W_2 + \phi \mathcal{V} \nabla_V B W_2 + \phi \mathcal{A}_V C W_2 + B \mathcal{H} \nabla_V C W_2 \\ &+ C \mathcal{H} \nabla_V C W_2 + \eta(W_1) B V - \eta(W_1) V + g_M(V, W_2) \xi), \end{aligned}$$

for $W = W_1 + W_2 \in \Gamma(TM)$, where $W_1 \in \Gamma(\ker f_*)$ and $W_2 \in (\Gamma(\ker f_*)^{\perp})$.

Thus taking into account the vertical terms, we get

$$(\nabla f_*)(V,W) = \nabla^f_V f_*W + f_*\{(\phi(\mathcal{A}_V\phi W_1 + \mathcal{V}\nabla_V BW_2 + \mathcal{A}_V CW_2) \\ + C(\mathcal{H}\nabla_V\phi W_1 + \mathcal{A}_V BW_2 + \mathcal{H}\nabla_V CW_2) - \eta(W_1)V) + g_M(V,W_2)\xi\}.$$

Thus

$$\begin{aligned}(\nabla f_*)(V,W) &= 0 \Leftrightarrow \\ -\nabla^f_V f_*W &= f_*(\phi(\mathcal{A}_V\phi W_1 + \mathcal{V}\nabla_V BW_2 + \mathcal{A}_V CW_2) + g_M(V,W_2)\xi \\ &\quad + C(\mathcal{H}\nabla_V\phi W_1 + \mathcal{A}_V BW_2 + \mathcal{H}\nabla_V CW_2) - \eta(W_1)V).\end{aligned}$$

Therefore, we obtain the result. □

Definition 3. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then f is called a $(\phi\ker f_*,\mu)-$totally geodesic map provided $(\nabla f_*)(\phi X,U)=0$, for $X\in\Gamma(\ker f_*)$ and $U\in(\Gamma(\ker f_*)^\perp)$.*

Theorem 8. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then f is called a $(\phi\ker f_*,\mu)-$totally geodesic map if and if f is horizontally homohtetic map.*

Proof. For $X\in\Gamma(\ker f_*)$ and $U\in\Gamma(\mu)$, using lemma $1(i)$, we get

$$(\nabla f_*)(\phi X,U) = \phi X(ln\lambda)f_*(U) + U(ln\lambda)f_*(\phi X) - g_M(\phi X,U)f_*(gradln\lambda),$$

From above equation, if f is a horizontally homothetic, then $(\nabla f_*)(\phi X,U)=0$.

Conversely, if $(\nabla f_*)(\phi X,U)=0$, we find

$$\phi X(ln\lambda)f_*(U) + U(ln\lambda)f_*(\phi X) = 0. \tag{3.6}$$

Taking inner product in above equation with $f_*(\phi X)$ and since f is conformal submersion, we have

$$g_M(\mathcal{H}gradln\lambda,\phi X)g_N(f_*U,f_*\phi X) + g_M(\mathcal{H}gradln\lambda,U)g_N(f_*\phi X,f_*\phi X) = 0.$$

Above equation shows that λ is a constant $\Gamma(\mu)$.

On the other hand taking inner product in equation (3.6) with f_*X, we get

$$g_M(\mathcal{H}gradln\lambda,\phi X)g_N(f_*U,f_*\phi U) + g_M(\mathcal{H}gradln\lambda,U)g_N(f_*\phi X,f_*U) = 0.$$

From above equation shows that λ is a constant on $\Gamma(\phi(\ker f_*))$. Thus λ is a constant on $\Gamma((\ker f_*)^\perp)$. □

Theorem 9. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. Let $f:(M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion. Then f is totally geodesic map if and only if*

(i) $\phi\mathcal{T}_X\phi Y - \eta(Y)X + g_M(X,Y)\xi) = 0$ and $\mathcal{H}\nabla_Y\phi X \in \Gamma(\phi\ker f_*)$,
(ii) f is horizontally homothetic map,
(iii) $\widehat{\nabla}_X BU = -\mathcal{T}_X CU$ and $\mathcal{T}_X BU + \mathcal{H}\nabla_X CU \in \Gamma(\phi\ker f_*)$.

Proof. For $X, Y \in \Gamma(\ker f_*)$, using equations (2.1), (2.4) and (2.15), we get

$$(\nabla f_*)(X,Y) = f_*(\phi(\nabla_X\phi Y) - \eta(Y)X + \eta(X)\eta(Y)\xi - \eta(\nabla_X Y)\xi).$$

Now, using equations (2.10) and (3.2), we get

$$(\nabla f_*)(X,Y) = f_*(\phi\mathcal{T}_X\phi Y + C\mathcal{H}\nabla_X\phi Y - \eta(Y)X + g_M(X,Y)\xi).$$

Thus shows that $\phi\mathcal{T}_X\phi Y - \eta(Y)X + g_M(X,Y)\xi) = 0$ and $\mathcal{H}\nabla_X\phi Y \in \Gamma(\phi\ker f_*)$.

On the other hund lemma $1(i)$, we get

$$(\nabla f_*)(U,V) = U(ln\lambda)f_*(V) + V(ln\lambda)f_*(U) - g_M(U,V)f_*(gradln\lambda),$$

for $U, V \in \Gamma(\mu)$. It is obvious that if f is horizontally homothetic, it follows that $(\nabla f_*)(U,V) = 0$. Conversely, if $(\nabla f_*)(U,V) = 0$, taking $V = \phi U$ in above equation, we have

$$U(ln\lambda)f_*(\phi U) + \phi U(ln\lambda)f_*(U) = 0. \tag{3.7}$$

Taking inner product in equation (3.7) with $f_*\phi U$, we get

$$g_M(\mathcal{H}gradln\lambda, U)g_N(f_*\phi U, f_*\phi U) + g_M(\mathcal{H}gradln\lambda, \phi U)g_N(f_*U, f_*\phi U) = 0.$$

From above equation λ is constant on $\Gamma(\mu)$. On the other hand, for $X, Y \in \Gamma(\ker f_*)$, from lemma $1(i)$, we get

$$(\nabla f_*)(\phi X, \phi Y) = \phi X(ln\lambda)f_*(\phi Y) + \phi Y(ln\lambda)f_*(\phi X) - g_M(\phi X, \phi Y)f_*(gradln\lambda).$$

Again if f is horizontally homothetic, then $(\nabla f_*)(\phi X, \phi Y) = 0$. Conversely, if $(\nabla f_*)(\phi X, \phi Y) = 0$, putting $X = Y$ in above equation, we get

$$2\phi X(ln\lambda)f_*(\phi X) - g_M(\phi X, \phi X)f_*(gradln\lambda) = 0.$$

Taking inner product in above equation with $f_*\phi X$ and since f is conformal submersion, we have $g_M(\phi X, \phi X)g_M(gradln\lambda, \phi X) = 0$.

From above equation, λ is constant on $\Gamma(\phi\ker f_*)$. Thus λ is constant on $\Gamma((\ker f_*)^\perp)$.

Now, for $X \in \Gamma(\ker f_*)$ and $U \in \Gamma((\ker f_*)^\perp)$, using equations (2.1), (2.4) and (2.15), we get $(\nabla f_*)(X,U) = f_*(\phi(\nabla_X\phi U) - \eta(\nabla_X U)\xi)$. Now, again using equations (2.10) and (3.2), we get

$$(\nabla f_*)(X,U) = f_*(C\mathcal{T}_X BU + \phi\widehat{\nabla}_X BU + C\mathcal{H}\nabla_X CU + \phi\mathcal{T}_X CU).$$

Thus $(\nabla f_*)(X,U) = 0 \Leftrightarrow f_*(C\mathcal{T}_X BU + \phi\widehat{\nabla}_X BU + C\mathcal{H}\nabla_X CU + \phi\mathcal{T}_X CU) = 0$. Therefore, we obtain the result. $\square$

From Theorems (3) and (5) in terms of the second fundamental forms of such submersions.

Theorem 10. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. Let $f : (M,\phi,\xi,\eta,g_M) \to (N,g_N)$ be a conformal anti-invariant submersion. Then M is a locally product manifold of the form $M_{(\ker f_*)^\perp}\times M_{(\ker f_*)}$ if*

$$(3.8)\qquad -\frac{1}{\lambda^2}g_N(\nabla^f_U f_*CV, f_*\phi X) = g_M(\mathcal{A}_U BV,\phi X) - g_M(\mathcal{H}gradln\lambda, CV)g_M(U,\phi X) + g_M(\mathcal{H}gradln\lambda,\phi X)g_M(U,CV),$$

and

$$(3.9)\qquad -\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y} f_*\phi X, f_*\phi CU) = g_M(\mathcal{T}_X\phi Y, BU) + g_M(\phi Y,\phi X)g_M(\mathcal{H}gradln\lambda,\phi CU),$$

for $X,Y\in\Gamma(\ker f_*)$ and $U,V,W\in(\Gamma(\ker f_*)^\perp)$, where $M_{(\ker f_*)^\perp}$ and $M_{(\ker f_*)}$ are integral manifolds of the distributions $\Gamma(\ker f_*)$ and $(\Gamma(\ker f_*)^\perp)$. Conversely, if M is a locally product manifold of the form $M_{(\ker f_*)^\perp}\times M_{(\ker f_*)}$ then we have

$$-\frac{1}{\lambda^2}g_N(\nabla^f_U f_*CV, f_*\phi X) = g_M(\mathcal{H}gradln\lambda,\phi X)g_M(U,CV) - g_M(\mathcal{H}gradln\lambda, CV)g_M(U,\phi X),$$

and $-\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y} f_*\phi X, f_*\phi CU) = g_M(\phi Y,\phi X)g_M(\mathcal{H}gradln\lambda,\phi CU)$.

Again, from Corollaries (2) and (3), we have the following theorem.

Theorem 11. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. If $f : (M,\phi,\xi,\eta,g_M)\to(N,g_N)$ be a conformal anti-invariant submersion, then f is a locally product manifold if and only if $g_M(\mathcal{A}_U\phi V, \phi X) = \eta(X)g_M(U,V)$ and $\mathcal{T}_X\phi Y{=}0$, for $X,Y\in\Gamma(\ker f_*)$ and $U,V\in(\Gamma(\ker f_*)^\perp)$.*

Theorem 12. *Let f be a conformal anti-invariant submersion from a Kenmotsu manifolds (M,ϕ,ξ,η,g_M) to a Riemannian manifold (N,g_N). Then M is a locally twisted product manifold of the form $M_{(\ker f_*)}\times M_{(\ker f_*)^\perp}$ if and only if*

$$(3.10)\qquad -\frac{1}{\lambda^2}g_N(\nabla^f_{\phi Y} f_*\phi X, f_*\phi CU) = g_M(\mathcal{T}_X\phi Y, BU) + g_M(\phi Y,\phi X)g_M(\mathcal{H}gradln\lambda,\phi CU),$$

and

$$(3.11)\qquad g_M(U,V)H = -B\mathcal{A}_U BV + CV(ln\lambda)BU - B(\mathcal{H}gradln\lambda)g_M(U,CV) - \phi^* f_*(\nabla^f_U f_*CV),$$

for $X,Y\in\Gamma(\ker f_*)$ and $U,V\in(\Gamma(\ker f_*)^\perp)$, where $M_{(\ker f_*)}$ and $M_{(\ker f_*)^\perp}$ are integral manifolds of the distributions $(\ker f_*)^\perp$ and $(\ker f_*)$ and H is the mean curvature vector field of $M_{(\ker f_*)^\perp}$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^{\perp})$, using equations $(2.3), (2.4)$, (2.10) and (3.2), we get $g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\mathcal{H}\nabla_X \phi Y, CU)$. Since ∇ is torsion free and $[X, \phi Y] \in \Gamma(\ker f_*)$, we get

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\mathcal{H}\nabla_{\phi Y} X, CU).$$

Using equations $(2.3), (2.4)$ and (2.12), we have

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\nabla_{\phi Y} \phi X, \phi CU).$$

Since f is conformal submersion, using equation (2.15) and lemma $1(i)$, we find

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) - \frac{1}{\lambda^2} g_M(\mathcal{H} gradln\lambda, \phi Y) g_N(f_* \phi X, f_* \phi CU) \\ &\quad - \frac{1}{\lambda^2} g_M(\mathcal{H} gradln\lambda, \phi X) g_N(f_* \phi Y, f_* \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_M(\phi X, \phi Y) g_N(f_* \mathcal{H} gradln\lambda, f_* \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU). \end{aligned}$$

Next, using definition (2) and equation (3.2), we obtain

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi X, \phi Y) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU). \end{aligned}$$

Thus shows that $M_{(\ker f_*)}$ is totally geodesic if and only if

$$\begin{aligned} &- \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) \\ &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi X, \phi Y) g_M(\mathcal{H} gradln\lambda, \phi CU). \end{aligned}$$

On the other hand for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^{\perp})$, using equations $(2.3), (2.4), (2.11), (2.12)$ and (3.2), we get

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) + g_M(\mathcal{H}\nabla_U CV, \phi X).$$

Since f is conformal submersion, using equation (2.15) and lemma $1(i)$, we obtain that

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - \frac{1}{\lambda^2} g_M(\mathcal{H} gradln\lambda, U) g_N(f_* CV, f_* \phi X) \\ &\quad - \frac{1}{\lambda^2} g_M(\mathcal{H} gradln\lambda, CV) g_N(f_* U, f_* \phi X) \\ &\quad + \frac{1}{\lambda^2} g_M(U, CV) g_N(f_* \mathcal{H} gradln\lambda, f_* \phi X) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi U} f_* CV, f_* \phi X) + \eta(X)\eta(\nabla_U V). \end{aligned}$$

Moreover, using definition (2) and (3.4), we get

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{A}_U BV, \phi X) - g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X) \\ &+ \eta(X)\eta(\nabla_U V) + g_M(U, CV) g_M(\mathcal{H} gradln\lambda, \phi X) \\ &+ \frac{1}{\lambda^2} g_N(\nabla^f_{\phi U} f_* CV, f_* \phi X) + \eta(X)\eta(\nabla_U V). \end{aligned}$$

Then, we have

$$\begin{aligned} g_M(U, V)H &= -B\mathcal{A}_U BV + CV(ln\lambda)BU - B(\mathcal{H} gradln\lambda) g_M(U, CV) \\ & -\phi f_*(\nabla^f_U f_* CV) - g_M(U, V)\xi + \eta(U)\eta(V)\xi, \end{aligned}$$

which proves. □

Theorem 13. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion with $rank(\ker f_*) > 1$. If M is a locally warped product manifold of the form $M_{(\ker f_*)^\perp} \times_\lambda M_{(\ker f_*)}$, then either f is horizontally homothetic submersion or the fibers are one dimensional.*

Proof. Since f is a conformal submersion, for $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$, using, equations (2.3), (3.4) and lemma 1(i), we obtain

$$-U(ln\lambda) g_M(\phi X, \phi Y) = \phi Y g_M(U, \phi X).$$

For $U \in \Gamma(\mu)$, we get $-U(ln\lambda) g_M(\phi X, \phi Y) = 0$.
From above equation, we find that λ is a constant on $\Gamma(\mu)$.
For $U = \phi X \in \Gamma(\phi(\ker f_*))$, we have

$$\phi X(ln\lambda) g_M(\phi X, \phi Y) = \phi Y(ln\lambda) g_M(\phi X, \phi X). \tag{3.12}$$

Interchanging the roles of Y and X in equation (3.12), we get

$$\phi Y(ln\lambda) g_M(\phi X, \phi X) = \phi X(ln\lambda) g_M(\phi Y, \phi Y). \tag{3.13}$$

From equations (3.12) and (3.13), we get

$$\phi X(ln\lambda) \parallel \phi X \parallel^2 \parallel \phi Y \parallel^2 = \phi X(ln\lambda)(g_M(\phi X, \phi Y))^2. \tag{3.14}$$

From (3.14), either λ is a constant on $\Gamma(\phi(\ker f_*))$ or $\Gamma(\phi(\ker f_*))$ is 1-dimensional. □

4. Conformal anti-invariant submersions admitting horizontal structure vector field

In this section, we study conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifolds such that the characteristic vector field ξ

is horizontal vector field. Using definition (2), we have $(\ker f_*)^\perp = \phi(\ker f_*) \oplus \mu$, where $\mu = \phi(\mu) \oplus <\xi>$. Thus, for any $U \in \Gamma(\ker f_*)^\perp$, we have

$$\phi U = BU + CU, \tag{4.1}$$

where $BU \in \Gamma(\ker f_*)$ and $CU \in \Gamma\phi(\mu)$.

Now, we suppose that X is vertical and U is horizontal vector fields. Using equations (2.3), (4.1), (2.4) and (2.10), we have

$$g_M(CU, \phi X) = 0. \tag{4.2}$$

$$g_M(\nabla_U CV, \phi X) = -g_M(CU, \phi \mathcal{A}_V X). \tag{4.3}$$

Since f is conformal submersion, using equation (4.2), we have

$$\frac{1}{\lambda^2} g_N(f_* CV, f_* \phi X) = 0, \tag{4.4}$$

for $X \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$.

Theorem 14. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then followings are equivalent to each other:*

(i) $(\ker f_*)^\perp$ is integrable,

(ii) $\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_*\phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X)$
$-g_M(\mathcal{H} gradln\lambda, CV) g_M(U, \phi X)$
$+g_M(\mathcal{H} gradln\lambda, CU) g_M(V, \phi X) - 2g_M(\mathcal{H} gradln\lambda, \phi X) g_M(CU, V)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$, since $\phi X \in \Gamma(\ker f_*)^\perp$ and $\phi V \in (\Gamma \ker f_* \oplus \mu)$. Using equations (2.3), (2.4) and (4.1), we have

$$\begin{aligned} g_M([U,V], X) &= g_M(\nabla_U BV, \phi X) - g_M(\nabla_V BU, \phi X) + g_M(\nabla_U CV, \phi X) \\ &\quad - g_M(\nabla_V CU, \phi X). \end{aligned}$$

Since f is a conformal submersion, using equation (2.10), we get

$$\begin{aligned} g_M([U,V], X) &= g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) + \frac{1}{\lambda^2} g_N(f_* \nabla_U CV, f_* \phi X) \\ &\quad - \frac{1}{\lambda^2} g_N(f_* \nabla_V CU, f_* \phi X). \end{aligned}$$

Thus using equation (2.15), (4.2) and lemma 1(i), we have

$$\begin{aligned} g_M([U,V], X) = {} & g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(U, \phi X) g_M(\mathcal{H} gradln\lambda, CV) \\ & + g_M(V, \phi X) g_M(\mathcal{H} gradln\lambda, CU) - 2g_M(CU, V) g_M(\mathcal{H} gradln\lambda, \phi X) \\ & - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X), \end{aligned}$$

which proves $(i) \Leftrightarrow (ii)$. $\square$

Theorem 15. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:*

(i) $(\ker f_*)^\perp$ is integrable,
(ii) f is horizontally homothetic,
(iii) $\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, using Theorem (14), we get

$$\begin{aligned} & g_M([U,V], X) \\ = \; & g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(U, \phi X) g_M(\mathcal{H} gradln\lambda, CV) \\ & + g_M(V, \phi X) g_M(\mathcal{H} gradln\lambda, CU) - 2 g_M(CU, V) g_M(\mathcal{H} gradln\lambda, \phi X) \\ & - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X). \end{aligned}$$

Since $(\ker f_*)^\perp$ is integrable and f is horizontally homothetic, we get

$$\frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X),$$

using conditions (i) and (ii), we get (iii). Similarly, one can obtain the other assertions. $\square$

Remark 2. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion. If $\phi(\ker f_*) \oplus <\xi> = (\ker f_*)^\perp$, then $C = 0$ from equation (4.1).

Corollary 4. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal Lagrangian submersion. Then the following assertions are equivalent to each other:*

(i) $(\ker f_*)^\perp$ is integrable,
(ii) $\mathcal{A}_U \phi V = \mathcal{A}_V \phi U$,
(iii) $(\nabla f_*)(V, \phi U) = (\nabla f_*)(U, \phi V)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, since $\phi X \in \Gamma(\ker f_*)^\perp$ and $\phi V \in \Gamma(\phi \ker f_*)$, from Theorem (14), we have

$$\begin{aligned} g_M([U,V], X) = \; & g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) - g_M(U, \phi X) g_M(\mathcal{H} gradln\lambda, CV) \\ & + g_M(V, \phi X) g_M(\mathcal{H} gradln\lambda, CU) - 2 g_M(CU, V) g_M(\mathcal{H} gradln\lambda, \phi X) \\ & - \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CU - \nabla^f_U f_* CV, f_* \phi X). \end{aligned}$$

Since f is a conformal Lagrangian submersion, we get

$$g_M([U,V], X) = g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X) = 0,$$

which proves $(i) \Leftrightarrow (ii)$.

On the other hand, since f is a conformal submersion, using equations (2.3), (2.11) and (2.15), we get

$$g_M(\mathcal{A}_U BV - \mathcal{A}_V BU, \phi X)$$
$$= \frac{1}{\lambda^2}\{g_N((\nabla f_*)(U, BV), f_*\phi X) - g_N((\nabla f_*)(V, BU), f_*\phi X)\},$$

which proves $(ii) \Leftrightarrow (iii)$. □

Theorem 16. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then the followings are equivalent to each other:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M.
(ii) $-\frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X) = g_M(\mathcal{A}_U BV, \phi X)$
$-g_M(U, \phi X) g_M(\mathcal{H} gradln\lambda, CV)$
$+g_M(U, CV) g_M(\mathcal{H} gradln\lambda, \phi X)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, using equations (2.3), (2.4), (4.1), (2.10) and (2.11), we get

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) + g_M(\mathcal{H}\nabla_U CV, \phi X),$$

Since f is a conformal submersion, using equation (2.15) and lemma 1(i), we have

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) - g_M(U, \phi X) g_M(\mathcal{H} gradln\lambda, CV)$$
$$+ g_M(U, CV) g_M(\mathcal{H} gradln\lambda, \phi X) + \frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X),$$

which shows that $(i) \Leftrightarrow (ii)$. □

Theorem 17. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:*

$(i)$$(\ker f_*)^\perp$ defines a totally geodesic folation on M,
(ii) f is horizontally homothetic,
(iii) $-\frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X) = g_M(A_U BV, \phi X)$, for $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, from Theorem (16), we have

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) - g_M(U, \phi X) g_M(\mathcal{H} gradln\lambda, CV)$$
$$+ g_M(U, CV) g_M(\mathcal{H} gradln\lambda, \phi X) + \frac{1}{\lambda^2} g_N(\nabla^f_U f_* CV, f_*\phi X),$$

Since $(\ker f_*)^\perp$ defines a totally geodesic foliation on M and f is horizontally homothetic, we have $-\frac{1}{\lambda^2}g_N(\nabla^f_U f_*CV, f_*\phi X) = g_M(A_U BV, \phi X)$, which haves any two conditions imply the three. Similarly, one can obtain the other assertions. □

Corollary 5. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If f be a conformal Lagrangian submersion, then the following statements are equivalent to each other:*

(i) $(\ker f_*)^\perp$ defines a totally geodesic foliation on M,
(ii) $\mathcal{A}_U\phi V = 0$,
(iii) $(\nabla f_*)(U, \phi V) = 0$, for $U, V \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X \in \Gamma(\ker f_*)$ and $U, V \in (\Gamma(\ker f_*)^\perp)$, since $\phi V \in \Gamma(\phi \ker f_*)$ and $\phi X \in (\Gamma(\ker f_*)^\perp)$. From theorem (16), we have

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X) - g_M(V, \phi X) g_M(\mathcal{H}gradln\lambda, CV) + g_M(U, CV) g_M(\mathcal{H}gradln\lambda, \phi X) + \frac{1}{\lambda^2} g_N(\nabla^f_U f_*CV, f_*\phi X),$$

Since f is a conformal Lagrangian submersion, we get

$$g_M(\nabla_U V, X) = g_M(\mathcal{A}_U BV, \phi X), \tag{4.5}$$

which proves $(i) \Leftrightarrow (ii)$. On the other hand, using equation (4.5) and (2.11), since f is a conformal Lagrangian submersion and using equation (2.15), we get $g_M(\mathcal{A}_U BV, \phi X) = \frac{1}{\lambda^2} g_N((\nabla f_*)(U, BV) + \eta(V) f_*U, f_*\phi X)$ which proves $(ii) \Leftrightarrow (iii)$. □

Theorem 18. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then the following assertions are equivalent to each other:*

(i) $(\ker f_*)$ defines a totally geodesic foliation on M,
(ii) $-\frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_*\phi X, f_*\phi CU) = g_M(\mathcal{T}_Y \phi X, BU)$
$+g_M(\phi Y, \phi X) g_M(\mathcal{H}gradln\lambda, \phi CU) - \eta(U) g_M(X, Y)$, for $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$, using equations (2.3), (2.4) and (4.1), we have

$$g_M(\nabla_X Y, U) = g_M(T_X \phi Y, BU) + g_M(\nabla_{\phi Y} \phi X, \phi CU) + \eta(U)\eta(\nabla_X Y).$$

Since f is a conformal submersion, using equation (2.15) and lemma 1(i), we have

$$g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H}gradln\lambda, \phi CU) + \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_*\phi X, f_*\phi CU) - \eta(U) g_M(X, Y),$$

If $(\ker f_*)$ is a totally geodesic foliation on M, then we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &+ \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) - \eta(U) g_M(X, Y), \end{aligned}$$

which proves $(i) \Leftrightarrow (ii)$. □

Theorem 19. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion, then any two of the following conditions imply the third:*

(i) $(\ker f_*)$ defines a totally geodesic foliation on M,

(ii) λ is constant on $\Gamma(\mu)$,

(iii) $-\frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) = g_M(\mathcal{T}_X \phi Y, \phi U) - \eta(U) g_M(X, Y)$, for $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in (\Gamma(\ker f_*)^\perp)$, from Theorem (18), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(T_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &+ \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) - \eta(U) g_M(X, Y), \end{aligned}$$

Now, if we have (i) and (iii), then we have

From above equation, λ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the other assertions. □

Corollary 6. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If f be a conformal Lagrangian submersion, then the following assertions are equivalent to each other:*

(i) $(\ker f_*)$ *defines a totally geodesic foliation on M.*

(ii) $\mathcal{T}_X \phi Y = 0$, *for $X, Y \in \Gamma(\ker f_*)$.*

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U \in \Gamma(\ker f_*)^\perp$, from Theorem (18), we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X \phi Y, BU) + g_M(\phi Y, \phi X) g_M(\mathcal{H} gradln\lambda, \phi CU) \\ &+ \frac{1}{\lambda^2} g_N(\nabla^f_{\phi Y} f_* \phi X, f_* \phi CU) - \eta(U) g_M(X, Y). \end{aligned}$$

Since f is conformal Lagrangian submersion, we get $g_M(\nabla_X Y, U) = g_M(\mathcal{T}_X \phi Y, \phi U)$ $-\eta(U) g_M(X, Y)$, which proves $(i) \Leftrightarrow (ii)$. □

Theorem 20. *Let f be a conformal anti-invariant submersion from a Kenmotsu manifolds $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then f is a totally geodesic map if and only if*

$$\begin{aligned} (4.6) -\nabla^f_V f_* W &= f_*(\phi \mathcal{A}_V \phi W_1 + \phi \mathcal{V} \nabla_V BW_2 + \phi \mathcal{A}_V CW_2 + C\mathcal{H} \nabla_V \phi W_1 \\ &+ C\mathcal{A}_V CW_2 + C\mathcal{H} \nabla_V CW_2 + \eta(W_2) CV - \eta(\nabla_V W) \xi), \end{aligned}$$

for any $V \in \Gamma(\ker f_*)^\perp, W \in \Gamma(TM)$, where $W = W_1 + W_2$, $W_1 \in \Gamma(\ker f_*)$ and $W_2 \in (\Gamma(\ker f_*)^\perp)$.

Proof. For any $V \in \Gamma(\ker f_*)^{\perp}, W \in \Gamma(TM)$, using equations (2.1), (2.4), (2.15) and (4.1), we get

$$\begin{aligned}(\nabla f_*)(V,W) &= \nabla^f_V f_*W + f_*(\phi\mathcal{A}_V\phi W_1 + B\mathcal{H}_V\phi W_1 + C\mathcal{H}_V\phi W_1 \\ &\quad + B\mathcal{A}_V BW_2 + C\mathcal{A}_V BW_2 + \phi\mathcal{V}\nabla_V BW_2 + \phi\mathcal{A}_V CW_2 \\ &\quad + B\mathcal{H}\nabla_V CW_2 + C\mathcal{H}\nabla_V CW_2 + \eta(W_2)\phi V - \eta(\nabla_V W)\xi).\end{aligned}$$

If $(\nabla f_*)(V,W) = 0$, then we have

$$\begin{aligned}-\nabla^f_V f_*W &= f_*(\phi A_V\phi W_1 + \phi\mathcal{V}\nabla_V BW_2 + \phi\mathcal{A}_V CW_2 + C\mathcal{H}_V\phi W_1 \\ &\quad + C\mathcal{A}_V BW_2 + C\mathcal{H}\nabla_V CW_2 + \eta(W_2)\phi V - \eta(\nabla_V W)\xi).\end{aligned}$$

Thus $(\nabla f_*)(V,W) = 0 \Leftrightarrow$ we get equation (4.6). □

Theorem 21. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If f be a conformal Lagrangian submersion, then f is called a $(\phi\ker f_*, \mu)-$totally geodesic map if and if f is horizontally homohtetic map.*

Proof. For $U \in \Gamma(\ker f_*)$ and $X \in (\Gamma\mu)$, using lemma 1(i), we have

$$(\nabla f_*)(\phi U, X) = \phi U(ln\lambda)f_*(X) + X(ln\lambda)f_*(\phi U) - g_M(\phi U, X)f_*(\mathcal{H}gradln\lambda).$$

From above equation, if f is a horizontally homothetic then $(\nabla f_*)(\phi U, X) = 0$.

Conversely, if $(\nabla f_*)(\phi U, X) = 0$, we obtain

$$\phi U(ln\lambda)f_*(X) + X(ln\lambda)f_*(\phi U) = 0. \tag{4.7}$$

Taking inner prouct above equation with $f_*(\phi U)$ and since f is a conformal submersion, we write

$$g_M(\mathcal{H}gradln\lambda, \phi U)g_N(f_*X, f_*\phi U) + g_M(\mathcal{H}gradln\lambda, X)g_N(f_*\phi U, f_*\phi U) = 0.$$

Above equation implies that λ is constant on $\Gamma(\mu)$. On the other hand, again taking inner product with f_*X, we get

$$g_M(\mathcal{H}gradln\lambda, \phi U)g_N(f_*X, f_*X) + g_M(\mathcal{H}gradln\lambda, X)g_N(f_*\phi U, f_*X) = 0,$$

From above equation, it follows that λ is constant on $\Gamma(\phi(\ker f_*))$. Thus λ is constant on $(\Gamma(\ker f_*)^{\perp})$. □

Theorem 22. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion. Then f is totally geodesic map if and only if*

(i) $\mathcal{T}_X\phi Y = 0$ and $C\mathcal{H}\nabla_X\phi Y + g_M(X,Y)\xi \in \Gamma(\phi\ker f_*)$,
(ii) f is horizontally homothetic map,
(iii) $\widehat{\nabla}_X BV + \mathcal{T}_X CV = 0$,
and $\mathcal{T}_X BV + \mathcal{H}\nabla_X CV \in \Gamma(\phi\ker f_*)$, for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in \Gamma(\ker f_*)^{\perp}$.

Proof. For $X, Y \in \Gamma(\ker f_*)$, using equations (2.1), (2.4), (4.1) and (2.10), we get $(\nabla f_*)(X,Y) = f_*(\phi\mathcal{T}_X\phi Y + C\mathcal{H}\nabla_X\phi Y + g_M(X,Y)\xi)$. If $(\nabla f_*)(X,Y) = 0$, then we have $f_*(\phi\mathcal{T}_X\phi Y + C\mathcal{H}\nabla_X\phi Y + g_M(X,Y)\xi) = 0$, which prove $\mathcal{T}_X\phi Y = 0$ and $C\mathcal{H}\nabla_X\phi Y + g_M(X,Y)\xi \in \Gamma(\phi\ker f_*)$.

On the other hand, from lemma 1(i) we have $(\nabla f_*)(V,W) = V(ln\lambda)f_*(W) + W(ln\lambda)f_*(V) - g_M(V,W)f_*(\mathcal{H}gradln\lambda)$, for any $V, W \in \Gamma(\mu)$. It is obvious that if f is horizontal homothetic, it follows that $(\nabla f_*)(V,W) = 0$. Conversely, if $(\nabla f_*)(V,W) = 0$, taking $W = \phi V$ in above equation, we have

$$V(ln\lambda)f_*(\phi V) + \phi V(ln\lambda)f_*(V) = 0. \tag{4.8}$$

Taking inner product in equation (4.8) with $f_*\phi V$, we get

$$g_M(\mathcal{H}gradln\lambda, V)g_M(\phi V,\phi V) + g_M(\mathcal{H}gradln\lambda,\phi V)g_M(V,\phi V) = 0.$$

From above equation, λ is constant on $\Gamma(\mu)$. On the other hand, from lemma 1(i) we have $(\nabla f_*)(\phi X,\phi Y) = \phi X(ln\lambda)f_*(\phi Y) + \phi X(ln\lambda)f_*(\phi Y) - g_M(\phi Y,\phi X)f_*(\mathcal{H}gradln\lambda)$,

Again if f is horizontal homothetic, it follows that $(\nabla f_*)(\phi X,\phi Y) = 0$. Conversely, if $(\nabla f_*)(\phi X,\phi Y) = 0$, taking $X = Y$ in above equation, we have $2\phi X(ln\lambda)f_*(\phi X) - g_M(\phi X,\phi X)f_*(\mathcal{H}gradln\lambda) = 0$.

Taking inner product above equation with $f_*\phi X$ and since f is conformal submersion, we get $g_M(\phi X,\phi X)g_M(\mathcal{H}gradln\lambda,\phi X) = 0$. From above equation, λ is constant on $\Gamma(\ker f_*)^\perp$. Thus λ is constant on $\Gamma(\phi\ker f_*)$.

Now, for $U \in \Gamma(\ker f_*)^\perp$ and $X \in \Gamma(\ker f_*)$, from equations (2.15), (2.1), (2.4), (4.1), (2.9) and (2.10), we get

$$\begin{aligned}(\nabla f_*)(X,V) \;=\; & f_*(B\mathcal{T}_XBV + C\mathcal{T}_XBV + \phi\widehat{\nabla}_XBV + B\mathcal{H}\nabla_XCV \\ & + C\mathcal{H}\nabla_XCV + \phi\mathcal{T}_XCV + \eta(V)BX + \eta(V)X).\end{aligned}$$

Thus $(\nabla f_*)(X,V) = 0 \Leftrightarrow f_*(C\mathcal{T}_XBV + \phi\widehat{\nabla}_XBV + C\mathcal{H}\nabla_XCV + \phi\mathcal{T}_XCV)$. □

Theorem 23. *Let (M,ϕ,ξ,η,g_M) be a Kenmotsu manifold and (N,g_N) be a Riemannian manifold. Let $f : (M,\phi,\xi,\eta,g_M) \to (N,g_N)$ be a conformal anti-invariant submersion. Then M is a locally twisted product manifold of the form $M_{(\ker f_*)} \times M_{(\ker f_*)^\perp}$ if and only if*

$$\begin{aligned} & -\frac{1}{\lambda^2}g_N(\nabla_{f_*\phi Y}f_*\phi X, f_*\phi CV) \\ & = g_M(\mathcal{T}_X\phi Y, BV) + g_M(\phi Y,\phi V)g_M(\mathcal{H}gradln\lambda,\phi X) - g_M(X,Y)\eta(V),\end{aligned} \tag{4.9}$$

and

$$\begin{aligned} g_M(V,W)H = & -B\mathcal{A}_VBW + CW(ln\lambda)BV - B\mathcal{H}gradln\lambda g_M(V,CW) \\ & - \phi f^*(\nabla^f_V f_*CW),\end{aligned} \tag{4.10}$$

for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in (\Gamma(\ker f_*)^\perp)$, where $M_{(\ker f_*)}$ and $M_{(\ker f_*)^\perp}$ are integral manifolds of the distributions $(\ker f_*)^\perp$ and $(\ker f_*)$ and H is the mean curvature vector field of $M_{(\ker f_*)^\perp}$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $V \in (\Gamma(\ker f_*)^\perp)$, from equations (2.3), (2.4) and (2.10), we have $g_M(\nabla_X Y, V) = g_M(\phi\nabla_X Y, \phi V) + \eta(V)\eta(\nabla_X Y) = g_M(\mathcal{T}_X \phi Y, BV) + g_M(\mathcal{H}\nabla_X \phi Y, CV) - g_M(X, Y)\eta(V)$.

Since ∇ is torsion free and $[X, \phi Y] \in \Gamma(\ker f_*)$, we get $g_M(\nabla_X Y, V) = g_M(\mathcal{T}_X \phi Y, BV) + g_M(\mathcal{H}\nabla_{\phi Y} X, CV) - g_M(X, Y)\eta(V)$.

Using equation (2.3), (2.4) and (2.11), we get $g_M(\nabla_X Y, V) = g_M(\mathcal{T}_X \phi Y, BV) + g_M(\nabla_{\phi Y} \phi X, \phi CV) - g_M(X, Y)\eta(V)$. Since f is a conformal submersion, using equations (2.15), (4.2) and lemma 1(i), we have

$$\begin{aligned} g_M(\nabla_X Y, V) &= g_M(\mathcal{T}_X \phi Y, BV) + g_M(\phi X, \phi Y) g_M(\mathcal{H} gradln\lambda, \phi CV) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla_{f_*\phi Y} f_*\phi X, f_*\phi CV) - g_M(X, Y)\eta(V), \end{aligned}$$

Thus it follows that $M_{(\ker f_*)}$ is totally geodesic if and only if equation (4.9) is satisfied.

On the other hand, for $X, Y \in \Gamma(\ker f_*)$ and $V, W \in (\Gamma(\ker f_*)^\perp)$, using equations (2.3), (2.4) and (2.11), we have

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{A}_V BW, \phi X) + g_M(\mathcal{H}\nabla_V CW, \phi X) \\ &\quad + \eta(Y) g_M(X, \phi V) \end{aligned}$$

Since f is a conformal submersion, using equation (2.15) and lemma 1(i), we have

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{A}_V BW, \phi X) - \frac{1}{\lambda^2} g_M(\mathcal{H} gradln\lambda, V) g_N(f_* CW, f_*\phi X) \\ &\quad - \frac{1}{\lambda^2} g_M(\mathcal{H} gradln\lambda, CW) g_N(f_* V, f_*\phi X) \\ &\quad + \frac{1}{\lambda^2} g_M(V, CW) g_N(f_*\mathcal{H} gradln\lambda, f_*\phi X) + \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CW, f_*\phi X). \end{aligned}$$

Moreover, using equation (4.2), we get

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{A}_V BW, \phi X) - g_M(\mathcal{H} gradln\lambda, W) g_M(V, \phi X) \\ &\quad + g_M(\mathcal{H} gradln\lambda, \phi X) g_M(V, CW) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla^f_V f_* CW, f_*\phi X) + \eta(W) g_M(V, \phi X) \end{aligned}$$

Thus $M_{(\ker f_*)^\perp}$ is totally umbilical $\Leftrightarrow$ equation (4.10) satisfied. □

Similarly, from Theorem (15), we deduce the following result.

Theorem 24. *Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \phi, \xi, \eta, g_M) \to (N, g_N)$ be a conformal anti-invariant submersion with rank$(\ker f_*) > 1$. If M is a locally warped product manifold of the form $M_{(\ker f_*)^\perp} \times_\lambda M_{(\ker f_*)}$, then either f is horizontally homothetic submersion or the fibers are one dimensional.*

5. Example

Note that given an Euclidean space $(x_1, ...x_{2m}, x_{2m+1})$ with coordinates we can canonically choose an almost contact structure ϕ on R^{2m+1} as follows:

$$\phi(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + ... + a_{2m-1}\frac{\partial}{\partial x_{2m-1}} + a_{2m}\frac{\partial}{\partial x_{2m}} + a_{2m+1}\frac{\partial}{\partial x_{2m+1}})$$
$$= (-a_2\frac{\partial}{\partial x_1} + a_1\frac{\partial}{\partial x_2} + ... - a_{2m}\frac{\partial}{\partial x_{2m-1}} + a_{2m-1}\frac{\partial}{\partial x_{2m}}),$$

where $\xi = \frac{\partial}{\partial x_{2m+1}}$ and $a_1, a_2, ..., a_{2m}, a_{2m+1}$ are $C^\infty-$real valued functions in R. Let $\eta = dx_{2m+1}$ and $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_{2m}}, \frac{\partial}{\partial x_{2m+1}})$ is orthogonal basis of vector fields on R^{2m+1}.

Example 1. Define a map $f : R^5 \to R^2$ by $f(x_1, ..., x_5)$=$(e^{x_2}\cos x_4, e^{x_2}\sin x_4)$. Then we have

$$\ker f_* =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} > \text{ and } (\ker f_*)^\perp =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4} >$$

Thus, f is a conformal anti-invariant submersion with $\lambda = e^{x_2}$.

Example 2. Define a map $f : R^5 \to R^3$ by

$$f(x_1, ..., x_5) = (e^{x_1}\sin x_3, e^{x_1}\cos x_3, e^{x_1}\sin x_5)$$

Then we have

$$\ker f_* =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4} > \text{ and } (\ker f_*)^\perp =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} >$$

Thus, f is a conformal anti-invariant submersion with $\lambda = e^{x_1}$.

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