SEPARATION AXIOMS IN TOPOLOGICAL ORDERED SPACES

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Abstract. In this paper, we introduce and study some new type of separation axioms in topological ordered spaces via ω-open sets.

Keywords: ω-open sets, topological ordered space.

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of semiopen sets introduced by Levine in 1963. A topological ordered space $(X, \tau, \leq)$ is a topological space (X, τ) equipped with a partial order $\leq$ (that is, reflexive, transitive and antisymetric). A subset A of $(X, \tau, \leq)$ is said to be increasing (resp. decreasing) if $A = \{x \in X : a \leq x \text{ for } a \in A\}$ (resp. $A = \{x \in X : x \leq a \text{ for some } a \in A\}$), that is, if $A = \underset{a\in A}{\cup}[a, \rightarrow]$ (resp. $A = \underset{a\in A}{\cup}[\rightarrow, a]$), where $[a, \rightarrow] = \{x \in X : a \leq x\}$ (resp. $[\rightarrow, a] = \{x \in X : x \leq a\}$). In this paper, we introduce and study some new type of separation axioms in topological ordered spaces via ω-open sets.

2. Preliminaries

For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A, respectively.

Definition 2.1. *A subset S of a topological space (X, τ) is said to be semiopen [3] if $S \subset \mathrm{Cl}(\mathrm{Int}(S))$.*

Definition 2.2 ([9])**.** *A subset A of a topological space (X, τ) is said to be an ω-closed set if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is semiopen in (X, τ). The complement of an ω-closed set is called an ω-open set. The family of all ω-open (ω-closed) subsets of (X, τ) is denoted by τ_ω (resp. $\omega C(X)$).*

Definition 2.3 ([10])**.** *The intersection of all ω-closed sets containing $A \subset X$ is called the ω-closure of A and is denoted by $\omega\,\mathrm{Cl}(A)$. The union of all ω-open sets contained in $A \subset X$ is called the ω-interior of A and is denoted by $\omega\,\mathrm{Int}(A)$.*

Definition 2.4 ([7])**.** *A subset $M(x)$ of a topological space (X, τ) is called an ω-neighbourhood of a point $x \in X$ if there exists an ω-open set S such that $x \in S \subset M(x)$.*

Definition 2.5 ([10])**.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω-irresolute if $f^{-1}(U) \in \tau_\omega$ for every $U \in \sigma_\omega$.*

Definition 2.6. *A topological space (X, τ) is said to be:*

1. *ω-T_1 [6] if for every $x, y \in X$, $x \neq y$, there exist $U, V \in \tau_\omega$ such that $x \in U$, $y \notin V$ and $y \in V$, $x \notin U$.*
2. *ω-T_2 [6] if for every $x, y \in X$, $x \neq y$, there exist $U, V \in \tau_\omega$ such that $x \in U$, $y \notin V$ and $y \in V$, $x \notin U$ and $U \cap V = \emptyset$.*
3. *ω-regular [10] if for any closed set F in X and $a \in X \backslash F$, there exist disjoint ω-open sets U and V in X containing a and F, respectively.*

Definition 2.7 ([4])**.** *A topological ordered space $(X, \tau, \leq)$ is said to be upper (resp. lower) T_1-ordered if for each pair of elements $a \nleq b$ (that is, a is not related to b) in X there exists a decreasing (resp. increasing) open set U containing b (resp. a) such that $a \notin U$ (resp. $b \notin U$). $(X, \tau, \leq)$ is said to be T_1-ordered if it is both lower and upper T_1-ordered.*

Definition 2.8 ([4])**.** *A topological ordered space $(X, \tau, \leq)$ is said to be T_2-ordered if for each pair of elements $a \nleq b$ in x there exist disjoint open sets U and V in X containing a and b respectively, U is increasing and V is decreasing.*

Definition 2.9 ([4])**.** *A topological ordered space $(X, \tau, \leq)$ is said to be upper (resp. lower) regularly ordered if for each increasing (resp. decreasing) closed set F in X and $a \notin F$ there exist disjoint open sets U and V containing a and F respectively, U is decreasing (resp. increasing) and V is increasing (resp. decreasing). $(X, \tau, \leq)$ is said to be regularly ordered if it is both lower and upper regularly ordered.*

3. On ω-T_1 and ω-T_2 ordered spaces

Definition 3.1. *Let $(X, \tau, \leq)$ be a topological ordered space and A a subset of X. Define:*
$I^{\omega}(A) = \cap\{F : F$ *is an increasing ω-closed subset of X containing A*$\}$,
$D^{\omega}(A) = \cap\{K : K$ *is a decreasing ω-closed subset of X containing A*$\}$,
$I^{o\omega}(A) = \cup\{G : G$ *is an increasing ω-open subset of X contained in A*$\}$,
$D^{o\omega}(A) = \cup\{H : H$ *is an decreasing ω-open subset of X contained in A*$\}$,
Cleraly, $I^{\omega}(A)$ (resp. $D^{\omega}(A)$) is the smallest increasing (resp. decreasing) ω-closed subset of X containing A and $I^{o\omega}(A)$ (resp.$D^{o\omega}(A)$) is the largest increasing (resp. decreasing) ω-open subset of X contained in A.

Proposition 3.2. *For any subset A of a topological ordered space $(X, \tau, \leq)$, the following hold:*

1. $X \backslash I^{\omega}(A) = D^{o\omega}(X \backslash A)$.
2. $X \backslash D^{\omega}(A) = I^{o\omega}(X \backslash A)$.
3. $X \backslash I^{o\omega}(A) = D^{\omega}(X \backslash A)$.
4. $X \backslash D^{o\omega}(A) = I^{\omega}(X \backslash A)$.

Proof. We shall prove (1) only, (2), (3) and (4) can be proved in a similar manner.

(1) Since $I^{\omega}(A)$ is an ω-closed increasing set containing A, $X \backslash I^{\omega}(A)$ is an ω-open decreasing set such that $X \backslash I^{\omega}(A) \subset X \backslash A$. Let U be an another ω-open decreasing set such that $U \subset X \backslash A$. Then $X \backslash U$ is an ω-closed increasing set such that $X \backslash U \supset A$. It follows that $I^{\omega}(A) \subset X \backslash U$. That is $U \subset X \backslash I^{\omega}(A)$. Thus, $X \backslash I^{\omega}(A)$ is the largest ω-open decreasing set such that $X \backslash I^{\omega}(A) \subset X \backslash A$. That is $X \backslash I^{\omega}(A) = D^{o\omega}(X \backslash A)$. □

Lemma 3.3. *Let $(X, \tau, \leq)$ be a topological ordered space and A a subset of X. Then $x \in I^{\omega}(A)$ (resp. $x \in D^{\omega}(A)$) if and only if for every decreasing (resp. increasing) ω-open subset U of X containing x, $U \cap A \neq \emptyset$.*

Proof. Let U be a decreasing ω-open subset of X containing x such that $U \cap A = \emptyset$. Then $X \backslash U$ is an increasing ω-closed subset of X containing A. Therefore, $I^{\omega}(A) \subset X \backslash U$. Thus $x \notin I^{\omega}(A)$. Conversely, if $x \notin I^{\omega}(A)$. Then $X \backslash I^{\omega}(A)$ is a decreasing ω-open subset of X containing x, but disjoint from A. The case of $D^{\omega}(A)$ can be deal similarly. □

Definition 3.4. *A topological ordered space $(X, \tau, \leq)$ is said to be upper (resp. lower) ω-T_1-ordered if for each pair of elements $a \nleq b$ (that is, a is not related to b) in X there exists a decreasing (resp. increasing) ω-open set U containing b (resp. a) such that $a \notin U$ (resp. $b \notin U$). $(X, \tau, \leq)$ is said to be ω-T_1-ordered if it is both lower and upper ω-T_1-ordered.*

Clearly every T_1-ordered space is ω-T_1-ordered and every ω-T_1-ordered space is ω-T_1 but the converses are not true, in general.

Example 3.5. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $a \leq c, b \leq b, c \leq b, c \leq c$. Then $(X, \tau, \leq)$ is an ω-T_1 space but not T_1-ordered.

Example 3.6. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $a \leq c, b \leq b, c \leq b, c \leq c$. Then $(X, \tau, \leq)$ is an ω-T_1 space but not ω-T_1 ordered.

Theorem 3.7. *For a topological ordered space $(X, \tau, \leq)$, the following statements are equivalent:*

1. $(X, \tau, \leq)$ is lower (resp. upper) ω-T_1-ordered,
2. for each pair $a \nleq b$ of X, there exists an ω-open set U containing a (resp. b) such that $x \nleq b$ (resp. $a \nleq x$) for all $x \in U$,
3. for each $x \in X$, $[\leftarrow, x]$ (resp. $[x, \rightarrow]$) is ω-closed,
4. when the net $\{x_\alpha\}_{\alpha \in A}$ ω-converges to a and $x_\alpha \leq b$ (resp. $b \leq x_\alpha$) for all $\alpha \in A$, then $a \leq b$ (resp. $b \leq a$).

Proof. We shall prove the theorem for lower ω-T_1-ordered spaces only. (1) $\Rightarrow$ (2): Let $a \nleq b$. Then by hypothesis, there exists an increasing ω-open set U containing a such that $b \notin U$. If $x \in U$ and $x \leq b$, then $b \in U$, a contradiction.
(2) $\Rightarrow$ (3): Let $y \in X \backslash [\leftarrow, x]$. Then $y \nleq x$. Then there exists an ω-open set U containing y such that $u \nleq x$ for all $u \in U$. That is, $y \in U \subset X \backslash [\leftarrow, x]$. Hence $[\leftarrow, x]$ is ω-closed.
(3) $\Rightarrow$ (1): Obvious.
(1) $\Rightarrow$ (4): Let $\{x_\alpha\}_{\alpha \in A}$ be a net in X ω-converges to a and $x_\alpha \leq b$ for all $\alpha \in A$. If possible $a \nleq b$, then by hypothesis, $X \backslash [\leftarrow, x]$ is an ω-open set containing a. Then there exists $\lambda \in A$ such that $x_\alpha \in X \backslash [\leftarrow, x]$ for all $\alpha \leq \lambda$. That is $x_\alpha \nleq b$ for all $\alpha \geq \lambda$, which is a contradiction.
(4) $\Rightarrow$ (1): Let X be not lower ω-T_1-ordered. Then there exists a pair $a \nleq b$ in X such that for every ω-open set U containing a, there exists $x_u \in U$ with $x_u \leq b$. Let $\mathcal{U}$ be the collection of all ω-open sets containing a, then $\mathcal{U}$ is directed by inclusion and the net $\{x_u\}_{u \in U}$ ω-converges to a. By hypothesis $a \leq b$, which is a contradiction. □

Corollary 3.8. *If $(X, \tau, \leq)$ is lower (upper) ω-T_1-ordered and $\tau \leq \tau^\star$, then $(X, \tau^\star, \leq)$ is also lower (upper) ω-T_1-ordered.*

Theorem 3.9. *A topological ordered space $(X, \tau, \leq)$ is ω-T_1-ordered if and only if for each $x \in X$, there exists an ω-T_1-ordered ω-open set in X which is both increasing and decreasing containing x.*

Proof. If X is ω-T_1-ordered, then X is the required set for each $x \in X$. Conversely, let $a \nleq b$ in X. Then, by hypothesis, there exist ω-T_1-ordered ω-open sets U_1 and U_2 in X containing a and b, respectively, where U_1 and U_2 are both increasing and decreasing sets. If $b \notin U_1$ and $a \notin U_2$, then there is nothing to prove. But if $b \in U_1$ (resp. $a \in U_2$), then there exist ω-open sets V and W in U_1 (resp. U_2) containing a and b, respectively, V is increasing and W is decreasing in U_1 (resp. U_2) also $b \notin V$ and $a \notin W$. Since U_1 (resp. U_2) is both increasing and decreasing and it also an ω-open subset of X, V is an increasing and W is a decreasing ω-open subsets of X. Hence $(X, \tau, \leq)$ is ω-T_1-ordered. □

Theorem 3.10. *Let f be an order preserving (that is, $x \leq y$ in X if and only if $f(x) \leq^\star f(y)$ in $X^\star$) ω-irresolute function from a topological ordered space $(X, \tau, \leq)$ to a topological ordered space $(X^\star, \tau^\star, \leq^\star)$. If $(X^\star, \tau^\star, \leq^\star)$ is ω-T_1-ordered, then $(X, \tau, \leq)$ is also ω-T_1-ordered.*

Proof. Let $a \nleq b$ in X. Then $f(a) \nleq^\star f(b)$ in X. Then there exists an increasing ω-open set $U^\star$ in $X^\star$ containing $f(a)$ but $f(b) \notin U^\star$. Since f is order preserving and ω-irresolute, $f^{-1}(U^\star) = U$ is an increasing ω-open subset of X. Also, $a \in U$ and $b \notin U$. Hence $(X, \tau, \leq)$ is lower ω-T_1-ordered. Similarly, we can prove $(X, \tau, \leq)$ is upper ω-T_1-ordered. □

Definition 3.11. *A topological ordered space $(X, \tau, \leq)$ is called a C_ω-space if whenever F is an ω-closed subset of X. $i(F)$ and $d(F)$ are also ω-closed subsets of X, where $i(F) = \cup\{[x, \rightarrow] : x \in F\}$ and $d(F) = \cup\{[\leftarrow, x] : x \in F\}$.*

Definition 3.12 ([5]). *Let f be a function from $(X, \tau, \leq)$ onto $(X, \tau^\star, \leq^\star)$. Then $\leq^\star$ is called a quotient order of $\leq$ induced by f if $x^\star \leq^\star y^\star$ for $x^\star, y^\star \in X^\star$ if and only if there exist $x \in f^{-1}(x^\star)$, $y \in f^{-1}(y^\star)$ such that $x \leq y$.*

Theorem 3.13. *Suppose $(X, \tau, \leq)$ is a C_ω-space, ω-T_1-ordered space and f an ω-irresolute $\omega^\star$-closed function of $(X, \tau, \leq)$ onto $(X^\star, \tau^\star, \leq^\star)$, where $\leq^\star$ is the quotient order induced by f. Then $(X^\star, \tau^\star, \leq^\star)$ is also ω-T_1-ordered space.*

Proof. We will first show that $(X^\star, \tau^\star, \leq^\star)$ is a C_ω-space. Let $F^\star$ be an ω-closed subset of $X^\star$. Then $f^{-1}(F^\star)$ is an ω-closed subset of X. Then $i_{X^\star}(F^\star) = f(i_X(f^{-1}(F^\star)))$, $d_{X^\star}(F^\star) = f(d_X(f^{-1}(F^\star)))$. Hence $i_{X^\star}(F^\star)$ and $d_{X^\star}(F^\star)$ are ω-closed sets in $X^\star$ as f is ω-irresolute and ω-closed function. Now, to prove the theorem it is sufficient to show that every ω-T_1, C_ω-space is ω-T_1-ordered. If $(X, \tau, \leq)$ is an ω-T_1 C_ω-space, then $\{x\}$ is ω-closed for all $x \in X$. Since $(X, \tau, \leq)$ is an C_ω-space, $[x, \rightarrow]$ and $[\leftarrow, x]$ are ω-closed subsets of X for all x. Hence $(X^\star, \tau^\star, \leq^\star)$ is also ω-T_1-ordered space. □

Definition 3.14. *A topological ordered space $(X, \tau, \leq)$ is said to be ω-T_2-ordered if for each pair of elements $a \nleq b$ in x there exist disjoint ω-open sets U and V in X containing a and b, respectively, U is increasing and V is decreasing.*

Example 3.15. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $a \leq c, b \leq b, c \leq b, c \leq c$. Then $(X, \tau, \leq)$ is an ω-T_2 space but not T_2-ordered.

Example 3.16. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $a \leq c, b \leq b, c \leq b, c \leq c$. Then $(X, \tau, \leq)$ is an ω-T_2 space but not ω-T_2 ordered.

Theorem 3.17. *A topological ordered space $(X, \tau, \leq)$ is ω-T_2-ordered if and only if for each $x \in X$, there exists an increasing (resp. decreasing) ω-clopen subset of X containing x which is ω-T_2-ordered.*

Proof. If X is ω-T_2-ordered, then X is the required increasing (resp. decreasing) ω-clopen subset of X for all $x \in X$. Conversely, let $x \nleq y$ in X. By hypothesis, there exists an increasing ω-clopen set V in X containing x. If $y \in V$, then nothing to prove. If $y \notin V$. Then $X \backslash V$ is a decreasing ω-clopen subset of X containing y. Hence $(X, \tau, \leq)$ is ω-T_2-ordered. Dually, we can prove the theorem for decreasing ω-clopen subsets of X. □

Theorem 3.18. *A topological ordered space $(X, \tau, \leq)$ is ω-T_2-ordered if and only if for each pair of points $x \nleq y$ in X, there exists an increasing (resp. decreasing) ω-irresolute function f of a space $(X, \tau, \leq)$ into a ω-T_2-ordered space $(X^\star, \tau^\star, \leq^\star)$ such that $f(x) \neq^\star f(y)$ (resp. $f(y) \nleq^\star f(x)$).*

Proof. If $(X, \tau, \leq)$ is ω-T_2-ordered then the identity mapping is the required function. Conversely, let $x \nleq y$ in X. By hypothesis, there exists an increasing ω-irresolute mapping f of a space X into an ω-T_2-ordered space $(X^\star, \tau^\star, \leq^\star)$ with $f(x) \neq^\star f(y)$. Therefore, there exist disjoint ω-open sets $U^\star$ and $V^\star$ in $X^\star$ containing $f(x)$ and $f(y)$, respectively, where $U^\star$ is increasing and $V^\star$ is decreasing. Since f is increasing ω-irresolute, $f^{-1}(U^\star)$ is an increasing ω-open set containing x and $f^{-1}(V^\star)$ is a decreasing ω-open set containing y. Also $f^{-1}(U^\star) \cap f^{-1}(V^\star) = \emptyset$. Hence $(X, \tau, \leq)$ is ω-T_2-ordered. Analogously we can prove the theorem for decreasing functions. □

Theorem 3.19. *A topological ordered space $(X, \tau, \leq)$ is ω-T_2 if and only if for each $x \in X$ the intersection of all increasing (resp. decreasing) ω-closed i-ω-neighbourhoods (resp. d-ω-neighbourhoods) of x is $[x^\star, \rightarrow]$ (resp. $[\leftarrow, x]$).*

Proof. Let $(X, \tau, \leq)$ be an ω-T_2-ordered space and $x \in X$. If $G^\star = \cap\{F : F$ is an increasing ω-closed i-ω-neighbourhood of $x\}$. Clearly, $[x^\star, \rightarrow] \subset G^\star$. Let $y \notin [x^\star, \rightarrow]$. Then $x \nleq y$. Then there exist disjoint ω-open sets U and V containing x and y, respectively such that U is increasing and V is decreasing. Hence $x \in U \subset X \backslash V$. Therefore, $X \backslash V$ is an increasing ω-closed i-ω-neighbourhood of x and $y \notin X \backslash V$. Hence $G^\star = [x^\star, \rightarrow]$. Similarly, we can show that intersection of all decreasing ω-closed i-ω-neighbourhoods of x is $[\leftarrow, x]$. Conversely, let $x \leq y$ in X. Then $y \notin [x^\star, \rightarrow]$. By hypothesis, there exists an increasing ω-closed i-ω-neighbourhood F of x such that $y \notin F$. Then $y \in X \backslash F$, $X \backslash F$ is a decreasing ω-open set. Also. there exists an increasing ω-open set U such that $x \in U \subset F$. Clearly, $U\ (X \backslash F) = \emptyset$. Hence $(X, \tau, \leq)$ is ω-T_2-ordered. By dual argument we can prove that ω-T_2-ordered if intersection of all decreasing ω-closed d-ω-neighbourhoods of x is $[x^\star, \rightarrow]$ for all $x \in X$. □

Theorem 3.20. *Suppose $(X, \tau, \leq)$ is an ω-T_2-ordered space and f an one-to-one $\omega^\star$-open and $\omega^\star$-closed mapping of $(X, \tau, \leq)$ onto $(X^\star, \tau^\star, \leq^\star)$, where $\leq^\star$ is the quotient order induced by f. Then $(X^\star, \tau^\star, \leq^\star)$ is also ω-T_2-ordered.*

Proof. Let $x^\star \in X^\star$ and $\mathcal{F}^\star = \cap\{F^\star : F^\star$ is an increasing ω-closed i-ω-neighborhood of $x^\star\}$. Then $[x^\star, \rightarrow] \subset \mathcal{F}^\star$. Suppose $y^\star \notin [x^\star, \rightarrow]$, that is $x^\star \leq^\star y^\star$ does not hold. Thus $x \nleq y$ for $x, y \in X$ such that $f(x) = x^\star$ and $f(y) = y^\star$. Therefore, there exists an increasing ω-closed i-ω-neighborhood F of x such that $y \notin F$. Since f is one to one $\omega^\star$-open and $\omega^\star$-closed, $f(F)$ is an increasing ω-closed i-ω-neighborhood of $x^\star$ not containing $y^\star$, that is $\mathcal{F}^\star = [x^\star, \rightarrow]$. Hence by Theorem 3.19 $(X^\star, \tau^\star, \leq^\star)$ is ω-T_2-ordered. □

4. Properties of ω-regularly ordered spaces

Definition 4.1. *A topological ordered space $(X, \tau, \leq)$ is said to be upper (resp. lower) ω-regularly ordered if for each increasing (resp. decreasing) closed set F in X and $a \notin F$ there exist disjoint ω-open sets U and V containing a and F respectively, U is decreasing (resp. increasing) and V is increasing (resp. decreasing). $(X, \tau, \leq)$ is said to be ω-regularly ordered if it is both lower and upper ω-regularly ordered.*

Remark 4.2. *Clearly, every regularly ordered space is ω-regularly ordered. However, the converse need not be true as can be true as can be seen from the following example.*

Example 4.3. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $a \leq c, b \leq b, c \leq b, c \leq c$. Then $(X, \tau, \leq)$ is ω-regularly ordered, but not regularly ordered.

Example 4.4. Let $X = \{a, b, c\}$ be equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and with the partial order $\leq$ defined as $a \leq a$, $a \leq b$, $b \leq b, c \leq c$. Then $(X, \tau, \leq)$ is ω-regular but not ω-regularly ordered.

Theorem 4.5. *Every upper (resp. lower) ω-regularly ordered, upper (resp. lower) T_1-ordered space is ω-T_2-ordered.*

Proof. Let $(X, \tau, \leq)$ be an upper ω-regularly ordered, upper T_1-ordered space and $a \nleq b$ in X. Then $[a, \rightarrow]$ is an increasing closed subset of X not containing b. Thus there exist disjoint ω-open sets U, V in X such that $[a, \rightarrow] \subset U$ and $b \in V, U$ is increasing and V is decreasing. Therefore, U and V are disjoint ω-open sets containing a and b respectively, U is increasing and V is decreasing. Hence, $(X, \tau, \leq)$ is ω-T_2-ordered.

Similarly, every lower ω-regularly ordered, lower T_1-ordered space is ω-T_2-ordered. □

Definition 4.6. *A subset A of topological ordered space $(X, \tau, \leq)$ is said to be i-ω-neighborhood (resp. d-ω-neighborhood) of $B \subset X$, if there exists an increasing (resp. decreasing) ω-open set G in X such that $B \subset G \subset A$.*

Theorem 4.7. *For a topological ordered space $(X, \tau, \leq)$, the following are equivalent:*

1. *$(X, \tau, \leq)$ is lower (resp. upper) ω-regularly ordered.*
2. *For each $x \in X$ and each increasing (resp. decreasing) open set U in X containing x, there exists an increasing (resp. decreasing) ω-open set V such that $x \in V \subset I^{\omega}(V) \subset U$ (resp. $x \in V \subset D^{\omega}(V) \subset U$).*
3. *For every decreasing (resp. increasing) closed set F, the intersection of all decreasing (resp. increasing) ω-closed d-ω-neighborhoods (resp. i-ω-neighborhoods) of F is exactly F.*
4. *For every nonempty set A and an increasing (resp. decreasing) open subset B of X such that $A \cap B \neq \emptyset$, there exists an increasing (resp. decreasing) ω-open set V in X such that $A \cap V \neq \emptyset$ and $I^{\omega}(V) \subset B$ (resp. $D^{\omega}(V) \subset B$).*
5. *For every nonempty set A and a decreasing (resp. increasing) closed set B such that $A \cap B = \emptyset$, there exist disjoint ω-open sets U and V with $A \cap U \neq \emptyset$, U is increasing (resp. decreasing) and $B \subset V$, V is decreasing (resp. increasing) in X.*
6. *When a net $\{X_{\alpha}\}_{\alpha \in \Lambda}$ is residually in each increasing (resp. decreasing) ω-open set containing a and a net $\{y_{\alpha}\}_{\alpha \in \Lambda}$ is residually contained in each decreasing (resp. increasing) ω-open set containing a decreasing (resp. increasing) closed set F, $x_{\alpha} \leq y_{\alpha}$ for $\alpha \in \Lambda$, then $a \in F$.*

Proof. We shall prove the theorem for lower ω-regularly ordered spaces only.

(1)⇒(2): Let U be an increasing open subset of X containing x. Then, $X \backslash U$ is a decreasing closed subset of X such that $x \notin X \backslash U$. Thus there exist disjoint ω-open sets V and W containing x and $X \backslash U$ respectively, V is increasing and W is decreasing in X. Since, $V \subset X \backslash W, X \backslash W$ is increasing ω-closed, therefore, $I^{\omega}(V) \subset X \backslash W \subset U$. Hence $x \in V \subset I^{\omega}(V) \subset X \backslash W \subset U$.

(2)⇒(3): Let F be a decreasing closed subset of X and $F^{\star} = \cap\{K :: \text{is a decreasing } \omega\text{-closed, } d\text{-}\omega\text{-neighborhood of } F\}$, then $F \subset F^{\star}$. Let $x \notin F$. Then $x \in X \backslash F, X \backslash F$ is an increasing ω-open set. Thus, there exists an increasing ω-open set U such that $x \notin U \subset I^{\omega}(U) \subset X \backslash F$. Therefore $F \subset X \backslash I^{\omega}(U) \subset X \backslash U$, which implies that $X \backslash U$ is a decreasing ω-closed d-ω-neighborhood of F not containing x. That is $x \notin F^{\star}$. Hence $F = F^{\star}$.

(3)⇒(4): Let A be a nonempty set of B an increasing open subset of X such that $A \cap B \neq \emptyset$. Then, there exists $x \in X$ such that $x \in A \cap B$. By hypothesis, there exists a decreasing ω-closed d-ω-neighborhood K of $X \backslash B$ such that $x \notin K$. Define, $U = X \backslash K$. Then U is an increasing ω-open set such that $U \cap A \neq \emptyset$. Since K is a d-ω-neighborhood of $X \backslash B$, there exists a decreasing ω-open subset W of X such that $X \backslash B \subset W \subset K$. That is, $U = X \backslash K \subset X \backslash W \subset B$. Then $I^{\omega}(U) \subset X \backslash W \subset B$.

(4)⇒(5): Let A be a nonempty set and let B a decreasing closed subset of X such that $A \cap B = \emptyset$. Then, $X \backslash B$ is an increasing open subset of X such that $A \cap (X \backslash B) \neq \emptyset$. Thus by hypothesis, there exists an incresing ω-open set U in X such that $A \cap U \neq \emptyset$ and $I^{\omega}(U) \subset X \backslash B$. Define $X \backslash I^{\omega}(U) = V$. Then, V is a decreasing ω-open subset of X containing B and disjoint from U.

(5)⇒(6): Let a and F be as given in (5) of the theorem with $a \notin F$. Then by hypothesis, there exist disjoint ω-open sets U, V in X such that $a \in U$, U is increasing and $F \subset V$, V is decreasing in X. Thus there exists $\lambda \in \Lambda$ such that $x_{\alpha} \in U$ and $y_{\alpha} \in V$ for all $\alpha \geq \lambda$. Then, $x_{\alpha} \nleq y_{\alpha}$ for all $\alpha \geq \lambda$, otherwise $U \cap V \neq \emptyset$, which is contradiction. Hence $a \in F$.

(6)⇒(1): Suppose $(X, \tau, \leq)$ is not lower ω-regularly ordered. Then, there exists a decreasing closed set F in X and $x \in X \backslash F$ such that every increasing ω-open set containing x intersects every decreasing ω-open set containing F. Let $\mathcal{U}$ denote the family of all increasing ω-open subsets of X containing x and $\mathcal{V}$ be the family of all decreasing ω-open subsets of X containing F. Then both $\mathcal{U}$ and $\mathcal{V}$ are ordered and directed by inclusion. The product $\mathcal{U} \times \mathcal{V}$ may be ordered by agreeing that $(U_1, V_1) \leq (U_2, V_2)$ if and only if $U_2 \subset U_1$ and $V_2 \subset V_1$. For each $(U, V) \in \mathcal{U} \times \mathcal{V}$, an element $x_{(U,V)}$ may be selected in $U \cap V$. The net $\{x_{(U,V)}\}_{(U,V) \in \mathcal{U} \times \mathcal{V}}$ is residually contained in each increasing ω-open subset of X containing x and in each decreasing ω-open subset of X containing F. Hence (6) does not hold. □

Definition 4.8 ([4]). *Let $(X, \tau, \leq)$ be a topological ordered space and $Y \subset X$. then, $(Y, \tau_y, \leq_y)$ with the induced order and induced topology is said to be τ-compatibly ordered if and only if for each τ_y-closed set F, increasing (resp. decreasing) in Y, there exists a τ-closed set $F^{\star}$, increasing (resp. decreasing) in X, such that $F = F^{\star} \cap Y$.*

Definition 4.9 ([1]). *Let F be a mapping of $(X, \tau, \leq)$ onto $(X^{\star}, \tau^{\star}, \leq^{\star})$. Then f is called dual isotomic if $f(x) \leq^{\star} f(y)$ implies $x \leq y$.*

Definition 4.10. *A mapping $f : (X, \tau, \leq) \to (X^{\star}, \tau^{\star}, \leq^{\star})$ is called weakly order continuous if for each increasing (resp. decreasing τ-open, $\tau^{\star}$-closed set $U^{\star}$, $f^{-1}(U^{\star})$ is increasing (resp. decreasing) τ-open, τ-closed, respectively in X.*

Theorem 4.11. *Let $(X, \tau, \leq)$ be an ω-regularly ordered space and f a dual isotonic, $\omega^{\star}$-open and weakly order continuous mapping from $(X, \tau, \leq)$ onto $(X^{\star}, \tau^{\star}, \leq^{\star})$. Then $(X^{\star}, \tau^{\star}, \leq^{\star})$, is also ω-regularly ordered.*

Proof. Let $F^\star$ be an increasing closed subset of $X^\star$ and $a \notin F^\star$. then $f^{-1}(F^\star)$ is an increasing closed subset of X and $f^{-1}(a) \notin f^{-1}(F^\star)$. Thus by hypothesis, there exist disjoint ω-open sets U and V such that $f^{-1}(a) \in U$, U is decreasing and $f^{-1}(F^\star) \subset V$, V is increasing. Since f is dual isotonic and $\omega^\star$-open, $f(U) = U^\star$ and $f(V) = V^\star$ are disjoint ω-open subset of $X^\star$ such that $a \in U^\star$. $U^\star$ is decreasing and $F^\star \subset V$, $V^\star$ is increasing. Hence, $(X^\star, \tau^\star, \leq^\star)$ is upper ω-regularly ordered. Similarly $(X^\star, \tau^\star, \leq^\star)$ is lower ω-regularly ordered. □

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Accepted: 21.01.2018