ON THE JOINT (m, q)-PARTIAL ISOMETRIES AND THE JOINT m- INVERTIBLE TUPLES OF COMMUTING OPERATORS ON A HILBERT SPACE

Ould Ahmed Mahmoud Sid Ahmed
Mathematics Department
College of Science
Jouf University
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Saudi Arabia
sidahmed@ju.edu.sa

Abstract. The study of tuples of commuting operators was the subject of intensive study by many authors. Our aim in this work is to consider a generalization of the notions of m-partial isometries and (m, q)-partial isometries (resp. m- left inverse and m-right inverse) of a single operator done in [23] and [21] (resp. in [14],[19], [22]) to the multivariable operators. We study some of the basic properties of these tuples of commuting operators. A commuting d-tuple of operators $\mathbf{T} = (T_1, \ldots, T_d)$ acting on a Hilbert space $\mathcal{H}$ is called a joint $(m; (q_1, \ldots, , q_d))$-partial isometry, if

$$\mathbf{T}^q \left(\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha} \mathbf{T}^{\alpha} \right) = 0.$$

Keywords: m-isometric tuple, partial isometry, m-left inverse, m-right inverse, joint spectrum, joint approximate spectrum.

1. Introduction and terminologies

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$ we shall write $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\mathcal{N}(T)^{\perp}$ for the null space, the range of T and the orthogonal complement of $\mathcal{N}(T)$ respectively. $I = I_{\mathcal{H}}$ being the identity operator. In what follows $\mathbb{N}, \mathbb{Z}_+$ and $\mathbb{C}$ stands the sets of positive integers, nonnegative integers and complex numbers respectively. Denote by $\overline{\lambda}$ the complex conjugate of a complex number λ in $\mathbb{C}$. We shall henceforth shorten $\lambda I_{\mathcal{H}} - T$ by $\lambda - T$. The spectrum, the point spectrum, the approximate point spectrum of an operator T are denoted by $\sigma(T), \sigma_p(T)$ and $\sigma_{ap}(T)$ respectively. T^* means the adjoint of T.

The study of tuples of commuting operators was the subject of intensive study by many authors as in [4], [9], [10], [11], [12], [13], [25] , [26] and the references therein.

Our aim in this paper is to extend the notions of m-partial isometries ([23]) and (m,q)-partial isometries ([21]) for single variable operators to the tuples of commuting operators defined on a complex Hilbert space.

For $d \in \mathbb{N}$, let $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a tuple of commuting bounded linear operators. Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$ denote tuples of nonnegative integers multi-indices) and set $|\alpha| := \sum_{1\le j\le d} |\alpha_j|$, $\alpha! := \alpha_1! \cdots \alpha_d!$. Further, define $\mathbf{T}^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_d^{\alpha_d}$ where $T_j^{\alpha_j}$ denotes the product of T_j times itself α_j times.

One of the most important subclasses, of the algebra of all bounded linear operators acting on a Hilbert space, the class of partial isometries operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an isometry if $T^*T = I$ and partial isometry if $TT^*T = T$. In recent years this classes has been generalized, in some sense, to the larger sets of operators so-called m-isometries and m-partial isometries. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be m-isometric for some integer $m \ge 1$ if it satisfies the operator equation

$$\sum_{0\le k\le m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} = 0. \tag{1.1}$$

It is immediate that T is m-isometric if and only if

$$\sum_{0\le k\le m} (-1)^k \binom{m}{k} \|T^{m-k}x\|^2 = 0, \tag{1.2}$$

for all $x \in \mathcal{H}$. Major work on m-isometries has been done in a long paper consisting of three parts by Agler and Stankus ([1], [2], [3]) and have since then attracted the attention of several other authors (see for example [6], [7], [8], [15]). More recently a generalization of these operators to m-partial isometries has been studied in the paper of A. Saddi and the present author in [23] and by the present author in [21].

An operator $T \in \mathcal{B}(\mathcal{H})$ is called a m-partial isometry for some integer $m \ge 1$ (see [23]) if

$$T\left(T^{*m}T^m - \binom{m}{1} T^{*m-1}T^{m-1} + \binom{m}{2} T^{*m-2}T^{m-2} - \cdots + (-1)^m I\right) = 0. \tag{1.3}$$

and it is a (m,q)-partial isometry for $m \in \mathbb{N}$ and $q \in \mathbb{Z}_+$ (see ([21]) if

$$T^q\left(T^{*m}T^m - \binom{m}{1} T^{*m-1}T^{m-1} + \binom{m}{2} T^{*m-2}T^{m-2} - \cdots + (-1)^m I\right) = 0. \tag{1.4}$$

Gleason and Richter [16] extend the notion of m-isometric operators to the case of commuting d-tuples of bounded linear operators on a Hilbert space. The defining equation for an m-isometric tuple $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ reads:

$$\sum_{0\le k\le m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha} \mathbf{T}^\alpha = 0 \tag{1.5}$$

or equivalently

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|\mathbf{T}^{\alpha}x\|^2=0 \text{ for all } x\in\mathcal{H}. \tag{1.6}$$

Recently, P. H. W. Hoffmann and M. Mackey in [20] introduced the concept of (m,p)-isometric tuples on normed space. A tuple of commuting linear operators $\mathbf{T} := (T_1,\ldots,T_d)$ with $T_j : X \longrightarrow X$ (normed space) is called an (m,p)-isometry (or an (m,p)-isometric tuple) if, and only if, for given $m\in\mathbb{N}$ and $p\in(0,\infty)$,

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|\mathbf{T}^{\alpha}x\|^p=0 \text{ for all } \ x\in X. \tag{1.7}$$

Definition 1.1. *Let $\mathbf{T}=(T_1,\ldots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a tuple of operators.*

(1) *If $T_iT_j=T_jT_i \quad 1\leq i,\ j\leq d$, we say that $\mathbf{T}$ is a commuting tuple.*

(2) *If $T_iT_j=T_jT_i$, $T_iT_j^*=T_j^*T_i, 1\leq i\neq j\leq d$, we say that $\mathbf{T}$ is a doubly commuting tuple.*

Definition 1.2 ([18])**.** *A commuting tuple $\mathbf{T}=(T_1,\ldots,T_d)\in\mathcal{B}(\mathcal{H})^d$ is called:*

(1) *matricially quasinormal if T_i commutes with $T_j^*T_k$ for all $i,j,k\in\{1,2,\ldots,d\}$.*

(2) *jointly quasinormal if T_i commutes with $T_j^*T_j$ for all $i,j\in\{1,\ldots,d\}$ and*

(3) *spherically quasinormal if T_j commutes with $|\mathbf{T}| := \Big(\sum_{1\leq j\leq d}T_j^*T_j\Big)^{\frac{1}{2}}$ for all $j=1,\ldots,d$.*

If $\mathcal{M}$ is a common invariant subspace of $\mathcal{H}$ for each $T_j\in\mathcal{B}(\mathcal{H})$, then $\mathbf{T}_{|\mathcal{M}}=(T_{1|\mathcal{M}},T_{2|\mathcal{M}},\ldots,T_{d|\mathcal{M}})$ denote an d-tuple of compressions of $\mathcal{M}$.

The contents of this paper are the following. Introduction and terminologies are described in the first part. The second part is devoted to the study of some basic properties of the class of $(m;(q_1,\ldots,q_d))$-partial isometries tuples. Several spectral properties of some $(m;(q_1,\ldots,q_d))$-partial isometries are obtained in section three; concerning the joint point spectrum,the joint approximate spectrum and the spectral radius. In the fourth section we present some results concerning the m-left inverses and the m-right inverses for tuples of operators.

2. The joint $(m;(q_1,\ldots,q_d))$-partial isometries tuples of commuting operators

In this Section, we introduce and study some basic properties of a joint $(m;(q_1,\ldots,q_d))$-partial isometry operators tuples. All of these results are fairly straightforward generalizations of the corresponding single variable results that was proved in [21] and [23].

Definition 2.1. *Given $m \in \mathbb{N}$ and $q = (q_1, \ldots, q_d) \in \mathbb{Z}_+^d$. An commuting d-tuple of operators $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is called a joint $(m; (q_1, \ldots, q_d))$-partial isometry (or joint $(m; (q_1, \ldots, q_d))$-partial isometric d-tuple) if and only if*

$$\mathbf{T}^q \left(\sum_{0 \le k \le m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha} \mathbf{T}^{\alpha} \right) = 0.$$

Remark 2.1. (1) When $d = 1$ and $q \in \mathbb{N}$, the Definition 2.1 coincides with Definition 2.1 in [21].

(2) Every m-isometric d-tuple of commuting operators on $\mathcal{H}$ is a joint $(m; (q_1, \ldots, q_d))$- partial isometry d-tuple.

(3) Every joint $(m; (q_1, \ldots, q_d))$-partial isometry of commuting operators $\mathbf{T} = (T_1, \ldots, T_d)$ such that $\mathbf{T}$ is entry-wise invertible, $\mathbf{T}$ is an m-isometric d-tuple.

Remark 2.2. Let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ be a commuting operator 2-tuple, we have that

(i) $\mathbf{T}$ is a joint $(1; (1, 1))$-partial isometry pair if

$$T_1 T_2 \left(I - T_1^* T_1 - T_2^* T_2 \right) = 0.$$

(ii) $\mathbf{T}$ is a joint $(2; (1, 1))$-partial isometry pair if

$$T_1 T_2 \left(I - 2T_1^* T_1 - 2T_2^* T_2 + T_1^{*2} T_1^2 + T_2^{*2} T_2^2 + 2T_1^* T_2^* T_1 T_2 \right) = 0.$$

Remark 2.3. Let $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting operator d-tuple. Then $\mathbf{T}$ is a joint $(1; (1, \ldots, 1))$-partial isometry if and only if

$$T_1 ... T_d \left(I - T_1^* T_1 - T_2^* T_2 - \cdots - T_d^* T_d \right) = 0.$$

Example 2.1. Consider $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$ and let $\mathbf{T} = (\frac{1}{\sqrt{d}} T, \frac{1}{\sqrt{d}} T, \ldots, \frac{1}{\sqrt{d}} T) \in \mathcal{B}(\mathbb{C}^3)^d$. It is easy to see that $\mathbf{T}$ is a joint $(1; (1, \ldots, 1))$-partial isometry d-tuple.

Remark 2.4. If $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a doubly commuting d-tuple of operators on $\mathcal{H}$. Then $\mathbf{T}$ is a joint $(1; (1, \ldots, 1)$-partial isometry if and only if $\mathbf{T}^* := (T_1^*, T_2^*, \ldots, T_d^*)$ is so.

The following example of a joint $(m; (q_1, \ldots, q_d))$-partial isometry is adopted form [20].

Example 2.2. Let $S \in \mathcal{B}(\mathcal{H})$ be a (m, q_1)-partial isometry operator,$d \in \mathbb{N}$ and $\lambda = (\lambda_1, \dots, \lambda_d) \in (\mathbb{C}^d, \ \|.\|_2)$ with

$$\|\lambda\|_2^2 = \sum_{1 \leq j \leq d} |\lambda_j|^2 = 1.$$

Then the operator tuple $\mathbf{T} = (T_1, \dots, T_d)$ with $T_j = \lambda_j S$ for $j = 1, \dots, d$ is a joint $(m; (q_1, \dots, q_d))$-partial isometry d-tuple.

In fact,it is clair that $T_iT_j = T_jT_i$ for all $1 \leq i; \ j \leq d$. Further, by the multinomial expansion, we get

$$\begin{aligned}
\left(|\lambda_1|^2 + |\lambda_2|^2 + \cdots + |\lambda_d|^2\right)^k &= \sum_{\alpha_1+\alpha_2+\cdots+\alpha_d=k} \binom{k}{\alpha_1, \alpha_2, \dots, \alpha_d} \prod_{1 \leq i \leq d} |\lambda_i|^{2\alpha_i} \\
&= \sum_{|\alpha|=k} \frac{k!}{\alpha!} |\lambda^\alpha|^2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathbf{T}^q \sum_{0 \leq j \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha} \mathbf{T}^\alpha \\
&= \mathbf{T}^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \prod_{1 \leq j \leq d} |\lambda|^{2\alpha_j} S^{*|\alpha|} S^{|\alpha|} \\
&= \prod_{1 \leq j \leq d} \lambda_j^{q_j} S^{|q|} \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} S^{*k} S^k \\
&= \prod_{1 \leq j \leq d} \lambda_j^{q_j} S^{|q|-q_1} \underbrace{S^{q_1} \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} S^{*k} S^k}_{=0} = 0.
\end{aligned}$$

Consequently, $\mathbf{T}$ is a joint $(m; (q_1, \dots, q_d))$-partial isometry d-tuple as required.

The following example shows that the question about joint $(m; (q_1, \dots, q_d))$-partial isometry for d-tuple is non trivial. There exists a d-tuple of commuting operators $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ such that each T_j is a (m, q_j)-partial isometry for $j = 1, \dots, d$, but $\mathbf{T} = (T_1, \dots, T_d)$ is not a joint $(m; (q_1, \dots, q_d))$-partial isometry.

Example 2.3. Let us consider $\mathcal{H} = \mathbb{C}^3$ and define $T_1 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is straightforward that T_1 and T_2 commute. Moreover,T_1 and T_2 are joint $(2; 1)$-partial isometry but (T_1, T_2) is not a joint $(2; (1, 1))$-partial isometry.

Lemma 2.1. *Let $\mathbb{S}_d$ be the group of permutation on d symbols $\{1, 2, \ldots, d\}$ and let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a d-tuple of commuting operators. If $\mathbf{T}$ is a joint $(m; (q_1, \ldots, q_d))$-partial isometry , then for every $\sigma \in \mathbb{S}_d$, $\mathbf{T}_\sigma := (T_{\sigma(1)}, T_{\sigma(2)}, \ldots, T_{\sigma(d)})$ is a joint $(m; (q_{\sigma(1)}, \ldots, q_{\sigma(d)}))$-partial isometry.*

Proof. The proof follows from the condition that $\prod_{1\leq j\leq d} T_j = \prod_{1\leq j\leq d} T_{\sigma(j)}$ and the identity

$$\prod_{1\leq j\leq d} T_j^{q_j} \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \prod_{1\leq j\leq d} T_j^{*\alpha_j} \prod_{1\leq j\leq d} T_j^{\alpha_j} = 0.$$

□

Theorem 2.1. *Let $m \in \mathbb{N}$ and $q = (q_1, \ldots, q_d) \in \mathbb{Z}_+^d$. Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for T_j for all $j = 1, 2, \ldots, d$. Then the following properties are equivalent.*

(1) *$\mathbf{T}$ is a joint $(m; (q_1, \ldots, q_d))$-partial isometry.*

(2)
$$\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 = 0, \quad \textit{for all } x \in \mathcal{H}.$$

Proof. First, assume that $\mathbf{T}$ is a joint $(m; (q_1, \ldots, q_d))$-partial isometry. We have that for all $x \in \mathcal{H}$

$$\begin{aligned}
&\mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^\alpha \mathbf{T}^{*q} x = 0 \\
&\Longrightarrow \langle \mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^\alpha \mathbf{T}^{*q} x,\ x\rangle = 0 \\
&\Longrightarrow \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 = 0.
\end{aligned}$$

Thus, (2) holds.

To prove the converse, assume that the equality in (2) is holds. It follows that,

$$\begin{aligned}
&\langle \mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^\alpha \mathbf{T}^{*q} x,\ x\rangle = 0,\ \forall\, x \in \mathcal{H} \\
&\Longrightarrow \mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^\alpha \mathbf{T}^{*q} x = 0, \forall\, x \in \mathcal{H}.
\end{aligned}$$

Hence,

$$\mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^\alpha = 0 \ \text{ on } \ \overline{\mathcal{R}(\mathbf{T}^{*q})} = \mathcal{N}(\mathbf{T}^q)^\perp.$$

As $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j $(1 \leq j \leq d)$, we have that

$$\mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha} = 0 \text{ on } \mathcal{N}(\mathbf{T}^q)$$

and hence,

$$\mathbf{T}^q \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha} = 0.$$

□

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.1. *Let $m \in \mathbb{N}$ and $q = (q_1, \ldots, q_d) \in \mathbb{Z}_+^d$. Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple of operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j, $1 \leq j \leq d$. Then the following properties are equivalent*

1. *$\mathbf{T}$ is a joint $(m; q_1, \ldots, q_d))$-partial isometry.*
2. *$\mathbf{T}|_{\mathcal{N}(\mathbf{T}^q)^\perp} := \Big(\mathbf{T}_1|_{\mathcal{N}(\mathbf{T}^q)^\perp}, T_2|_{\mathcal{N}(\mathbf{T}^q)^\perp}, \ldots, T_d|_{\mathcal{N}(\mathbf{T}^q)^\perp}\Big)$ is an m-isometric tuple.*

Remark 2.5. It is easy to see that every joint $(m; (1, \ldots, 1))$-partial isometry d-tuple of commuting operators is a joint $(m; (q_1, \ldots, q_d))$-partial isometry.

In the following theorem we show that by imposing certain conditions on a joint $(m; (q_1, \ldots, q_d))$-partial isometry of operators it becomes a joint $(m; (1, \ldots, 1))$-partial isometry.

Theorem 2.2. *If $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple of operators such is a joint $(m; (q_1, \ldots, q_d))$-partial isometry and $\mathcal{N}(T_j) = \mathcal{N}(T_j^2)$ for each j, $1 \leq j \leq d$, then $\mathbf{T}$ is a joint $(m; (1, \ldots, 1))$-partial isometry.*

Proof. By the assumption we have for $j = 1, \ldots, d$ that $\mathcal{N}(T_j) = \mathcal{N}(T_j^n)$ for all positive integer n. It follows that

$$\mathbf{T}^q \left(\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha} \right) = 0$$

implies

$$\prod_{1\leq j\leq d} T_j \left(\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha} \right) = 0.$$

□

The following proposition generalized ([23], Proposition 3.1)

Proposition 2.1. *If* $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ *is a jointly quasinormal and a joint* $(m;(1,\dots,1))$*-partial isometry, then* $\mathbf{T}$ *is a joint* $(1;(1,\dots,1))$*-partial isometry.*

Proof. Since $\mathbf{T} = (T_1, \dots, T_d)$ is a jointly quasinormal and a joint $(m;(1,\dots,1))$-partial isometry, it follows that

$$\prod_{1\le j\le d} T_j\Big(I - \sum_{1\le j\le d} T_j^*T_j\Big)^m = 0.$$

A straightforward computation using this last equation yields that

$$\prod_{1\le j\le d} T_j\Big(I - \sum_{1\le j\le d} T_j^*T_j\Big) = 0.$$

The proof is complete. □

Definition 2.2. *Let* $\mathbf{T} = (T_1, \dots, T_d)$ *and* $\mathbf{S} = (S_1, \dots, S_d)$ *are two commuting d-tuples of operators on a common Hilbert space* $\mathcal{H}$*. We said that* $\mathbf{S}$ *is unitary equivalent to* $\mathbf{T}$ *if there exists an unitary operator* $V \in \mathcal{B}(\mathcal{H})$ *such that*

$$\mathbf{S} = (S_1, \dots, S_d) = (V^*T_1V, V^*T_2V, \dots, V^*T_dV).$$

Proposition 2.2. *Let* $\mathbf{T} = (T_1, \dots, T_d)$ *and* $\mathbf{S} = (S_1, \dots, S_d) \in \mathcal{B}(\mathcal{H})^d$ *are two commuting d-tuple of operators such that* $\mathbf{S}$ *is unitary equivalent to* $\mathbf{T}$*, then* $\mathbf{T}$ *is a joint* $(m,(q_1,\dots,q_d)$*-partial isometry if and only if* $\mathbf{S}$ *is a joint* $(m,(q_1,\dots,q_d)$*-partial isometry.*

Proof. Suppose that $\mathbf{S}$ and $\mathbf{T}$ are unitary equivalent,that is there exists a unitary operator $V \in \mathcal{B}(\mathcal{H})$ such that $S_j = V^*T_jV, (1 \le j \le d)$. Since $T_iT_j = T_jT_i$; it follows that

$$(V^*T_jV)(V^*T_iV) = (V^*T_iV)(V^*T_jV) \text{ for all } 1 \le i,j \le d.$$

Using the observations above, we get the following identity

$$\begin{aligned}
&\mathbf{S}^q \sum_{0\le k\le m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathbf{S}^{*\alpha}\mathbf{S}^\alpha \\
&= V^*\mathbf{T}^qV \sum_{0\le k\le m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!}V^*\mathbf{T}^{*\alpha}\mathbf{T}^\alpha V \\
&= V^*\Bigg(\mathbf{T}^q \sum_{0\le k\le m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!}V^*\mathbf{T}^{*\alpha}\mathbf{T}^\alpha\Bigg)V.
\end{aligned}$$

□

In the proof of the following theorem, we need the following formula

Remark 2.6. For $n, d, k_1, k_2, \ldots, k_d \in \mathbb{N}$ with $k_1 + ... + k_d = n$, $n \geq 1$ and $d \geq 2$, we have

$$\binom{n}{k_1 \cdots k_d} = \sum_{1 \leq j \leq d} \binom{n-1}{k_1 \cdots k_j - 1 \cdots k_d}.$$

It will known that if $\mathbf{T} = (T_1, \ldots, T_d)$ is an m-isometric tuple then $\mathbf{T}$ is an n-isometric tuple for $n \geq m$. This result is not true for joint $(m; (q_1, \ldots, q_d))$-partial isometric tuple. Since it was shown in [21] that a (m, q)-partial isometry operator need not be a $(m+1, q)$-partial isometry and vice versa.

Theorem 2.3. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a $(m; (q_1, \ldots, q_d))$-partial isometry d-tuple of commuting operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j for $1 \leq j \leq d$. Then $\mathbf{T}$ is a $(m+n; (q_1, \ldots, q_d))$-partial isometry d-tuple for $n \in \mathbb{N}$.*

Proof. To prove that $\mathbf{T}$ is a joint $(m+n; (q_1, \ldots, q_d))$-partial isometry, it suffices to prove that $\mathbf{T}$ is a joint $(m+1; (q_1, \ldots, q_d))$-partial isometry.

Indeed, for $x \in \mathcal{H}$ we have

$$\begin{aligned}
&\sum_{0 \leq k \leq m+1} (-1)^k \binom{m+1}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&= \|\mathbf{T}^{*q} x\|^2 + \sum_{1 \leq k \leq m} (-1)^k \Big[\binom{m}{k} + \binom{m}{k-1}\Big] \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 - (-1)^m \\
&\sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&= \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|-k} \frac{k!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&- \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&- (-1)^m \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&= - \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\alpha|=k+1} \frac{k!(\alpha_1 + ... + \alpha_d)}{\alpha_1!.\alpha_2....\alpha_d!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&- (-1)^m \sum_{|\alpha|=m+1} \frac{m!(\alpha_1 + ... + \alpha_d)}{\alpha_1!.\alpha_2....\alpha_d!} \|\mathbf{T}^\alpha \mathbf{T}^{*q} x\|^2 \\
&= - \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\alpha|=k+1} (-1)^k \binom{m}{k}
\end{aligned}$$

$$\frac{k!\alpha_j}{\alpha_1!.\alpha_2!....\alpha_d!}\|T^{\alpha_1}...T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}...T_d^{\alpha_d}T_j\mathbf{T}^{*q}x\|^2$$
$$-(-1)^m\sum_{1\le j\le d}\sum_{|\alpha|=m+1}\frac{m!\alpha_j}{\alpha_1!.\alpha_2!....\alpha_d!}\|T^{\alpha_1}...T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}...T_d^{\alpha_d}T_j\mathbf{T}^{*q}x\|^2$$
$$=-\sum_{1\le j\le d}\sum_{0\le k\le m-1}(-1)^k\binom{m}{k}\sum_{|\beta|=k}(-1)^k\binom{m}{k}\frac{k!}{\beta!}\|\mathbf{T}^{\beta}T_j\mathbf{T}^{*q}x\|^2$$
$$-(-1)^m\sum_{1\le j\le d}\sum_{|\alpha|=m}\frac{m!}{\beta!}\|\mathbf{T}^{\beta}T_j\mathbf{T}^{*q}x\|^2$$
$$=-\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}\|\mathbf{T}^{\beta}T_j\mathbf{T}^{*q}x\|^2=0.$$

This completes the proof. □

Proposition 2.3. *Let $\mathbf{T}=(T_1,\dots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple of operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j for $j=1,2,\dots,d$. Assume that $\mathbf{T}$ satisfies*

$$\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T_j\mathbf{T}^{\alpha}\mathbf{T}^{*q}x\|^2=0 \ \textit{for all}\ x\in\mathcal{H}. \tag{2.1}$$

Then $\mathbf{T}$ is a joint $(m;(q_1,\dots,q_d))$-partial isometry if and only if $\mathbf{T}$ is a joint $(m+1;(q_1,\dots,q_d))$-partial isometry.

Proof. If $\mathbf{T}$ is a joint $(m;(q_1,\dots,q_d))$-partial isometry, then $\mathbf{T}$ is a joint $(m+1;(q_1,\dots,q_d))$-partial isometry by Theorem 2.3.

Conversely, assume that $\mathbf{T}$ is a joint $(m+1;(q_1,\dots,q_d))$-partial isometry and satisfies the equation (2.1). It follows that

$$\mathbf{T}^q\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}T_j\mathbf{T}^{*q}=0$$

and hence

$$\mathbf{T}^q\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}T_j^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}T_j=0\quad\text{on}\quad\overline{\mathcal{R}(\mathbf{T}^{*q})}=\mathcal{N}(\mathbf{T}^q)^{\perp}.$$

On the other hand, since $T_j\big(\mathcal{N}(\mathbf{T}^q)\big)\subset\mathcal{N}(\mathbf{T}^q)$ and $T_j^*\big(\mathcal{N}(\mathbf{T}^q)\big)\subset\mathcal{N}(\mathbf{T}^q)$, we have $T_j^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}T_j\,\mathcal{N}\big(\mathbf{T}^q\big)\subseteq\mathcal{N}\big(\mathbf{T}^q\big)$. Thus,

$$\mathbf{T}^q\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\beta}\mathbf{T}^{\beta}T_j=0\quad\text{on}\quad\mathcal{N}\big(\mathbf{T}^q\big).$$

From the above equalities we conclude that

$$\mathbf{T}^q \sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}T_j = 0 \text{ on } \mathcal{H}=\mathcal{N}(\mathbf{T}^q)\oplus\mathcal{N}(\mathbf{T}^q)^{\perp}.$$

Since $\mathbf{T}$ is a joint $(m+1,q)$-isometry, we get

$$\begin{aligned}0 &= \mathbf{T}^q\sum_{0\le k\le m+1}(-1)^k\binom{m+1}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\\ &= \mathbf{T}^q\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\\ &- \mathbf{T}^q\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbf{T}^q\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\\ &= \mathbf{T}^q\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\\ &= \mathbf{T}^q\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}T_j = 0.\end{aligned}$$

Consequently, $\mathbf{T}$ is a joint $(m,(q_1,\dots,q_d))$-isometry. □

Proposition 2.4. *Let $\mathbf{T}=(T_1,\dots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a joint $(m;(q_1,\dots,q_d))$-partial isometry d-tuple. The following statements hold:*

(i) $\mathbf{T}$ is a joint $(m+1;(q_1,\dots,q_d))$-partial isometry d-tuple if and if $\mathbf{T}$ satisfies

$$\mathbf{T}^q\Big(\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\Big)=0. \tag{2.2}$$

(ii) If $\mathcal{N}(T_j)=\mathcal{N}(T_j^2)$ for each $j; 1\le j\le d$, then $\mathbf{T}$ is a joint $(m+1,(q_1,\cdots,q_d))$-partial isometry d-tuple if and only if $\mathbf{T}$ satisfies (2.1)

Proof. (i) Since $\mathbf{T}$ is a joint $(m,(q_1,\cdots,q_d))$-partial isometry d-tuple, we have

$$\begin{aligned}&\mathbf{T}^q\sum_{0\le k\le m+1}(-1)^k\binom{m+1}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\\ &= -\mathbf{T}^q\sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\beta}\mathbf{T}^{\beta}T_j.\end{aligned}$$

Thus, $\mathbf{T}$ is a joint $(m+1,(q_1,\ldots,q_d))$-partial isometry d-tuple if and only if

$$\mathbf{T}^q \sum_{1\leq j\leq d}\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\beta}\mathbf{T}^{\beta}T_j=0, \tag{2.3}$$

which is equivalent to the equation (2.2).

(ii) Suppose that $\mathbf{T}$ is a joint $(m+1,(q_1,\cdots,q_d))$-partial isometry d-tuple. Then

$$\mathbf{T}^q \sum_{1\leq j\leq d}\sum_{0\leq k\leq m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}T_j^*\mathbf{T}^{*\beta}\mathbf{T}^{\beta}T_j=0,$$

and the equation (2.1) is satisfied.

Conversely, suppose that (2.1) is fulfilled. Under the conditions $\mathcal{N}(T_j)=\mathcal{N}(T_j^2)$ for each j; $1\leq j\leq d$, we see easily that

$$\mathcal{N}(\mathbf{T}^q)=\mathcal{N}(\prod_{1\leq j\leq d}T_j)=\bigcap_{1\leq j\leq d}\mathcal{N}(T_j),$$

and therefore we have (2.3). Now, from (i), $\mathbf{T}$ is a joint $(m+1,(q_1,\cdots,q_d)$-partial isometry d-tuple.

□

3. Spectral properties of joint $(m;(q_1,\ldots,q_d))$-partial isometries tuples

Spectral properties of commuting d-tuples of operators received important attention during last decades. For more details, the interested reader is referred to [9], [10], [11], [12], [13], [24] and the references therein.

First, we recapitulate very briefly the following definitions.

Definition 3.1. *Let $\mathbf{T}=(T_1,\ldots,T_d)$ be a d-tuple of operators on a complex Hilbert space $\mathcal{H}$.*

1. *A point $\lambda=(\lambda_1,\ldots,\lambda_d)\in\mathbb{C}^d$ is called a joint point eigenvalue of $\mathbf{T}$ if there exists a non zero vector $x\in\mathcal{H}$ such that*

   $$(T_j-\lambda_j)x=0 \ \textit{for}\ j=1,\ldots,d.$$

   Or equivalently if there exists a non-zero vector $x\in\mathcal{H}$ such that $x\in\bigcap_{1\leq j\leq d}\mathcal{N}(T_j-\lambda_j)$,i.e.;

   $$\sigma_p(\mathbf{T})=\{\lambda\in\mathbb{C}^d:\ \bigcap_{1\leq j\leq d}\mathcal{N}(T_j-\lambda_j)\neq\{0\}\}.$$

2. *The joint point spectrum, denoted by $\sigma_p(\mathbf{T})$ of $\mathbf{T}$ is the set of all joint eigenvalues of $\mathbf{T}$.*

Definition 3.2. *For a commuting d-tuple* $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$*. A number* $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d$ *is in the joint approximate point spectrum* $\sigma_{ap}(\mathbf{T})$ *if and only if there exists a sequence of unit vector* $(x_n)_n$ *such that*

$$(T_j - \lambda_j)x_n \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty \ \text{ for every } \ \ j = 1, ..., d.$$

Lemma 3.1. *(*[16]*) Let* $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ *be a commuting tuples of bounded operators. Then*

$$\sigma_{ap}(\mathbf{T}) = \Big\{ \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d : \ \exists \, (x_n)_n \subset \mathcal{H},$$

$$\|x_n\| = 1, \ \ / \lim_{n \longrightarrow \infty} \sum_{1 \le j \le d} \|(T_j - \lambda_j)x_n\| = 0 \Big\}.$$

Definition 3.3 ([26])**.** *The Taylor spectrum of commuting d-tuple* $= (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ *is the set of all complex d-tuple* $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d$ *with the property that the translated d-tuple* $(T_1 - \lambda_1, \dots, T_d - \lambda_d)$ *is note invertible. The symbol* $\sigma(\mathbf{T})$ *will stand for the Taylor spectrum of* $\mathbf{T}$*.*

Remark 3.1 ([26])**.** Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be an d-tuple of commuting operators on $\mathcal{H}$. $(\lambda_1, \dots, \lambda_d) \notin \sigma(\mathbf{T})$ if there exist operators $U_1, \dots, U_d, V_1, \dots, V_d \in \mathcal{B}(\mathcal{H}))$ such that

$$\sum_{1 \le k \le d} U_k(T_k - \lambda_k I) = I \text{ and } \sum_{1 \le k \le d} (T_k - \lambda_k I)V_k = I.$$

The spectral radius of $\mathbf{T}$ is

$$r(\mathbf{T}) = \max\{\|\lambda\|_2, \ \ \lambda \in \sigma(\mathbf{T})\}$$

where $\|\lambda\|_2 = \Big(\sum_{1 \le j \le d} |\lambda_j|^2 \Big)^{\frac{1}{2}}$.

In the following results we examine some spectral properties of a joint $(m; (q_1, \dots, q_d))$-partial isometries.That extend the case of single variable m-partial isometries studied in [23].

We put

$$\mathbb{B}(\mathbb{C}^d) := \{ \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d \ / \ \|\lambda\|_2 = \Big(\sum_{1 \le j \le d} |\lambda_j|^2 \Big)^{\frac{1}{2}} < 1 \ \}$$

and

$$\partial\mathbb{B}(\mathbb{C}^d) := \{ \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d \ / \ \|\lambda\|_2 = \Big(\sum_{1 \le j \le d} |\lambda_j|^2 \Big)^{\frac{1}{2}} = 1 \ \}.$$

In ([16], Lemma 3.2), the authors proved that if $\mathbf{T}$ is a m-isometric tuple, then the joint approximate point spectrum of $\mathbf{T}$ is in the boundary of the unit ball $\mathbb{B}(\mathbb{C}^d)$. This is not true for a joint $(m; (q_1, \dots, q_d))$-partial isometry tuple as shown in the following example.

Example 3.1. Let $\mathbf{T} = (T,0,...,0) \in \mathcal{B}(\mathbb{C}^2)^d$, where T is the matrix operator $T = \begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix}$ with $|a|^2 = \frac{1+\sqrt{5}}{2}$. It is easily to show that $\mathbf{T}$ is a joint $(2;(1,0,0,\dots,0))$-partial isometry and further $\sigma(\mathbf{T}) = \{0,a\} \times \{0\} \times ... \times \{0\}$.

However, if in addition assume that T_j reduces $\mathcal{N}(\mathbf{T}^q)$ for $1 \le j \le d$, we obtain the following result.

Theorem 3.1. *Let $\mathbf{T} = (T_1,\dots,T_d) \in \mathcal{B}(\mathcal{H})^d$ be a joint $(m;(q_1,\dots,q_d))$-partial isometry of d-tuple of operators such that $\mathcal{N}(\mathbf{T^q})$ is a reducing subspace for each T_j $(1 \le j \le d)$. Then $\sigma_{ap}(\mathbf{T}) \subset \partial\mathbb{B}(\mathbb{C}^d) \cup \big[0\big]$ where*

$$\big[0\big] := \{(\lambda_1,\dots,\lambda_d) \in \mathbb{C}^d : \prod_{1\le k\le d} \lambda_k = 0\}.$$

Proof. Let $\lambda = (\lambda_1,\dots,\lambda_d) \in \sigma_{ap}(\mathbf{T})$, then there exists a sequence $(x_n)_{n\ge 1} \subset \mathcal{H}$, with $||x_n|| = 1$ such that $(T_j - \lambda_j I)x_n \longrightarrow 0$ for all $j = 1,2,\dots,d$. Since for $\alpha_j > 1$,

$$T_j^{\alpha_j} - \lambda_j^{\alpha_j} = (T_j - \lambda_j) \sum_{1\le k\le \alpha_j} \lambda_j^{k-1} T_j^{\alpha_j - k}.$$

By induction, for $\alpha \in \mathbb{Z}_+^d$, we have

$$(T^\alpha - \lambda^\alpha I) = \sum_{1\le k\le d} \Big(\prod_{i\le k} \lambda_i^{\alpha_i}\Big)\Big(T_j^{\alpha_j} - \lambda_j^{\alpha_i}\Big) \prod_{i>k} T_i^{\alpha_i}.$$

Since, $\mathcal{R}(\mathbf{T}^q) \subset \mathcal{N}(\mathbf{T}^q)^\perp$ we have from Corollary 2.1 that, for all $n \ge 1$

$$\begin{aligned}
0 &= \lambda^q \langle \sum_{0\le k\le m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\mathbf{T}^{q} x_n \,, x_n \rangle \\
&= \lambda^q \langle \sum_{0\le k\le m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}(\mathbf{T}^{q} - \lambda^q) x_n,\ x_n \rangle \\
&+ \lambda^{2q} \sum_{0\le k\le m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} ||\mathbf{T}^{\alpha} x_n||^2 \\
&= \lambda^q \langle \sum_{0\le k\le m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}(\mathbf{T}^{q} - \lambda^q x_n | x_n \rangle \\
&+ \lambda^{2q} \Big\{ \sum_{0\le k\le m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \Big(||(\mathbf{T}^{\alpha} - \lambda^\alpha) x_n||^2 \\
&+ 2Re\langle (\mathbf{T}^{\alpha} - \lambda^\alpha)x_n | \lambda^\alpha x_n \rangle + |\lambda^\alpha|^2 \Big) \Big\}
\end{aligned}$$

as $\big(\mathbf{T}^\alpha - \lambda^\alpha I\big)x_n \to 0$ as $n \longrightarrow \infty$ for all $\alpha \in \mathbb{Z}_+^d$ we obtain that

$$0 = \lambda^q \sum_{0\le k\le m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} |\lambda^\alpha|^2 = \lambda^q (1 - \|\lambda\|_2^2)^m,$$

where $\|\lambda\|_2 = \left(\sum_{1\leq k\leq d}|\lambda_k|^2\right)^{\frac{1}{2}}$. We deduce that $\lambda^q = 0$ or $\|\lambda\|_2 = 1$. This implies that

$$\lambda \in \{(\lambda_1, \lambda_2, ..., \lambda_d) \in \mathbb{C}^d : \prod_{1\leq k\leq d} \lambda_k = 0\} \text{ or } \lambda \in \partial\mathbb{B}(\mathbb{C}^d).$$

□

Corollary 3.1. *If $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is a joint $(m; (q_1, \ldots, q_d))$-partial isometry d-tuple of operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j $(1 \leq j \leq d)$, then $r(\mathbf{T}) = 1$.*

Proof. It is known (see for example [24]) that the convex envelopes of all spectra coincide. Thus from Theorem 3.1 we have that the approximate point spectrum of $\mathbf{T} = (T_1, \ldots, T_d)$ is contained in the boundary of the unit ball, it follows that $r(\mathbf{T}) = 1$. □

We have also, the following properties.

Proposition 3.1. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a joint $(m; (q_1, \ldots, q_d))$-partial isometry d-tuple such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for T_j $(1 \leq j \leq d)$. The following properties hold.*

1. *If $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(\mathbf{T})\backslash[0]$, then $1 \in \sigma_{ap}(\sum_{1\leq j\leq d} \lambda_j T_j^*)$.*
2. *If $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_p(\mathbf{T})\backslash[0]$, then $1 \in \sigma_p(\sum_{1\leq j\leq d} \lambda_j T_j^*)$.*
3. *Eigenvectors of $\mathbf{T}$ corresponding to two joint eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$ such that $\sum_{1\leq j\leq d} \lambda_j \overline{\mu}_j \neq 1$ are orthogonal.*

Proof. 1. Let $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(\mathbf{T})\backslash\{(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d : \prod_{1\leq k\leq d} \lambda_k = 0\}$, choose a sequence $(x_n)_n \subset \mathcal{H}$, such that $\|x_n\| = 1$ and $(T_j - \lambda_j)x_n \longrightarrow 0$ for all $j = 1, 2, \ldots, d$. Following similar arguments it is easy to see that for all $\alpha_j \geq 0$. $(T_j^{\alpha_j} - \lambda_j^{\alpha_j})x_n \longrightarrow 0$ as $n \longrightarrow \infty$ and

$$(T^\alpha - \lambda^\alpha)x_n \longrightarrow 0.$$

On the other hand

$$\begin{aligned}
\mathbf{T}^{*\alpha}\mathbf{T}^\alpha(\mathbf{T}^q - \lambda^q)x_n &= \mathbf{T}^{*\alpha}\mathbf{T}^\alpha\mathbf{T}^q x_n - \lambda^q\mathbf{T}^{*\alpha}\mathbf{T}^\alpha x_n \\
&= \mathbf{T}^{*\alpha}\mathbf{T}^\alpha\mathbf{T}^q x_n - \lambda^q\mathbf{T}^{*\alpha}\big(\mathbf{T}^\alpha - \lambda^\alpha\big)x_n + \lambda^q\mathbf{T}^{*\alpha}\lambda^\alpha x_n \longrightarrow 0.
\end{aligned}$$

Since $\mathbf{T}$ is a joint $(m; (q_1, \ldots, q_d))$-partial isometry, we observe that

$$\lambda^q \sum_{1\leq k\leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\lambda\mathbf{T}^*)^\alpha x_n \longrightarrow 0.$$

and hence,

$$\lambda^q\Big(\mathbf{I}-\sum_{1\le j\le d}\lambda_jT_j^*\Big)^m x_n\longrightarrow 0.$$

Using the fact that $\lambda=(\lambda_1,\ldots,\lambda_d)\in\sigma_{ap}(\mathbf{T})\backslash[0]$, we get

$$\Big(I-\sum_{1\le j\le d}\lambda_jT_j^*\Big)^m x_n\longrightarrow 0.$$

We deduce that $(I_{\mathcal{H}}-\sum_{1\le j\le d}\lambda_jT_j^*)$ is not bounded below and hence $1\in\sigma_{ap}(\sum_{1\le j\le d}\lambda_jT_j^*)$,

and the proof of this implication is over.

2. Let $\lambda=(\lambda_1,\ldots,\lambda_d)\in\sigma_p(\mathbf{T})\backslash[0]$, there exists a non zero vector $x\in\mathcal{H}$ such that

$$T_jx=\lambda_jx \text{ for } j=1,2,..,d.$$

By using a similar argument as in 1 we show $(I_{\mathcal{H}}-\sum_{1\le j\le d}\lambda_jT_j^*)x=0$. From which it follows that $1\in\sigma_p(\sum_{1\le j\le d}\lambda_jT_j^*)$.

3. Let $\lambda=(\lambda_1,\ldots,\lambda_d)$ and $\mu=(\mu_1,\ldots,\mu_d)$ be eigenvalues of $\mathbf{T}$ such that $\sum_{1\le j\le d}\lambda_j\overline{\mu}_j\ne 1$. Assume that $T_jx=\lambda_jx$ and $T_jy=\mu_jy$ for $j=1,\ldots,d$. Since $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j for $j=1,\ldots,d$, we have $\mathcal{R}(\mathbf{T}^q)\subseteq\mathcal{N}(\mathbf{T}^q)^\perp$. Moreover since $\mathbf{T}$ is a $(m;(q_1,\ldots,q_d)$-partial isometry tuple,it follows from Corollary 2.1 that

$$\begin{aligned}
0 &= \langle\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\mathbf{T}^{q}x,\ y\rangle\\
&= \lambda^q\langle\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}x,\ y\rangle\\
&= \lambda^q\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}(\lambda.\overline{\mu})^{\alpha}\langle x,\ y\rangle\\
&= \Big(1-\sum_{1\le j\le d}\lambda_j\overline{\mu}_j\Big)^m\langle x,\ y\rangle.
\end{aligned}$$

where $\lambda.\overline{\mu}=(\lambda_1\overline{\mu}_1,\lambda_2\overline{\mu}_2,,...,\lambda_d\overline{\mu}_d)$.

Since $1-\sum_{1\le j\le d}\lambda_j\overline{\mu}_j\ne 0$, we obtain that $\langle x\mid y\rangle=0$. □

Lemma 3.2. *Let $\mathbf{T}=(T_1,\ldots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be an joint $(m;(q_1,\ldots,q_d))$-partial isometry such that $\mathcal{N}(\mathbf{T^q})$ is a reducing subspace for T_j, $j=1,\ldots,d$. Let $\lambda=(\lambda_1,\ldots,\lambda_d)$ and $\mu=(\mu_1,\ldots,\mu_d)\in\sigma_{ap}(\mathbf{T})$ such that $\sum_{1\le j\le d}\lambda_j\overline{\mu_j}\ne 1$. If $(x_n)_n$ and $(y_n)_n$ are two sequences of unit vectors in $\mathcal{H}$ such that*

$$\|(T_j-\lambda_j)x_n\|\longrightarrow 0 \text{ and } \|(T_j-\mu_j)y_n\|\longrightarrow 0\ \ (as\ n\longrightarrow\infty)\ \text{ for all } j=1,2,...,d,$$

then we have

$$\langle x_n|\ y_n\rangle\longrightarrow 0\ \ (as\ n\longrightarrow\infty). \tag{3.1}$$

Proof. Let $\{x_n\}_n$ and $\{y_n\}_n$ be two sequences of unit vectors in $\mathcal{H}$ such that

$\|(T_j-\lambda_j)x_n\| \longrightarrow 0$ and $\|(T_j-\mu_j)y_n\| \longrightarrow 0$ (as $n \longrightarrow \infty$) for all $j = 1, 2, ..., d$.

Then for all $\alpha_j, q_j \in \mathbb{N}$, we have $\lim_{n\to\infty}(T_j^{\alpha_j+q_j} - \lambda_j^{\alpha_j+q_j})x_n = 0$ and $\lim_{n\to\infty}(T_j^{\alpha_j} - \mu_j^{\alpha_j})y_n = 0$. On the other hand, we also have

$$\lim_{n\to\infty}(\mathbf{T}^{\alpha+q} - \lambda^{\alpha+q})x_n = 0 \quad \text{and} \quad \lim_{n\to\infty}(\mathbf{T}^{\alpha} - \mu^{\alpha})y_n = 0.$$

Since $\mathbf{T}$ is a joint $(m; (q_1, \dots, q_d))$-partial isometric tuple such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each T_j for $j = 1, \dots, d$, it follows that

$$\begin{aligned}
0 &= \lim_{n\to\infty}\langle\Big(\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\mathbf{T}^{\alpha}\mathbf{T}^{q}x_n \mid y_n\rangle\\
&= \lim_{n\to\infty}\langle\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\big(\mathbf{T}^{\alpha+q} - \lambda^{\alpha+q} + \lambda^{\alpha+q}\big)x_n \mid y_n\rangle\\
&= \lim_{n\to\infty}\langle\Big(\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^{*\alpha}\lambda^{\alpha+q}\Big)x_n \mid y_n\rangle\\
&= \lim_{n\to\infty}\langle x_n \mid \sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\overline{\lambda}^{\alpha}\mathbf{T}^{\alpha}y_n\rangle\\
&= \lim_{n\to\infty}\langle x_n \mid \sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\overline{\lambda}^{\alpha}\big(\mathbf{T}^{\alpha} - \mu^{\alpha} + \mu^{\alpha}\big)y_n\rangle\\
&= \lim_{n\to\infty}\langle x_n \mid \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\overline{\lambda}^{\alpha}\mu^{\alpha}y_n\rangle\\
&= \lim_{n\to\infty}\langle x_n \mid \sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\big(\overline{\lambda}\mu\big)^{\alpha}y_n\rangle\\
&= \lim_{n\to\infty}\langle x_n \mid \Big(1 - \sum_{1\le j\le d}\overline{\lambda_j}\mu_j\Big)^m y_n\rangle\\
&= \Big(1 - \sum_{1\le j\le d}\lambda_j\overline{\mu_j}\Big)^m \lim_{n\to\infty}\langle x_n \mid y_n\rangle.
\end{aligned}$$

By the assumption $\sum_{1\le j\le d}\lambda_j\overline{\mu_j} \neq 1$, we obtain that $\lim_{n\to\infty}\langle x_n \mid y_n\rangle = 0$. □

4. The joint m-left inverse and joint m-right inverse of commuting tuples of operators

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a left invertible if there is an operator $S \in \mathcal{B}(\mathcal{H})$ such that $ST = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$, denotes the identity operator. The operator S is called a left inverse of T. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a right invertible if there is an operator $R \in \mathcal{B}(\mathcal{H})$ such that $TR = I_{\mathcal{H}}$. The operator R is called a right inverse of T.

The left and right m-invertibility of operator have been introduced by the present author in [22] and by B. P. Duggal and V. Müller in [14].

Given a positive integer m. A bounded linear operator T is called a m-left invertible (resp. m-right invertible) if there exists a bounded linear operator S such that

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}S^kT^k = 0\Big(\text{resp.} \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}T^kS^k = 0\Big).$$

The m-invertibility have been extensively studied in the recent paper [19] by C. Gu. In [17] the author extends the notions of m-left and m-right invertibility respectively to the notions of m-left generalized inverse and m-right generalized inverse on Banach spaces.

The following definition generalize the definition of m-left invertibility and m-right invertibility of a single operator to tuples of commuting operators.

Definition 4.1. *Let $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting d-tuple of operators on $\mathcal{H}$,we say that $\mathbf{T}$ is a joint m-left invertible (resp. joint m-right invertible) operator for some integer $m \geq 1$, if there exists a commuting d-tuple of operators $\mathbf{S} = (S_1, \dots, S_d) \in \mathcal{B}(\mathcal{H})^d$ such that*

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha = 0$$

$$\Big(\textit{resp.} \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{T}^\alpha\mathbf{S}^\alpha = 0\Big).$$

$\mathbf{S}$ *is called a joint m-left (resp. m-right) inverse of* $\mathbf{T}$.

We say that $\mathbf{T} = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is m-invertible d-tuple of commuting operators if it has both a m-left inverse and a m-right inverse.

An interesting example of a m-left invertible commuting tuple of operators is that of an m-isometric tuple of operators.

Remark 4.1. It is clear that $\mathbf{S}$ is a m-left inverse of $\mathbf{T}$ if and only if $\mathbf{T}^*$ is a m-left inverse of $\mathbf{S}^*$.

Example 4.1. Let $T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Then the pair $\mathbf{T} = (T_1, T_1)$ is m-invertible tuple with m-inverse $\mathbf{S} = (S_1, S_2)$ in $\mathcal{B}(\mathbb{C}^2)^2$.

Remark 4.2. 1. If $\mathbf{S} = (S_1, S_2)$ and $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ be commuting pairs of operators then $\mathbf{S}$ is a joint left inverse of $\mathbf{T}$ if $S_1T_1 + S_2T_2 = I$ and it is a joint 2-left inverse of $\mathbf{T}$ if

$$S_1^2T_1^2 + S_2^2T_2^2 + 2S_1S_2T_1T_2 - 2\big(S_1T_1 + S_2T_2\big) + I = 0. \tag{4.1}$$

2. $\mathbf{S} = (S_1, \ldots, S_d)$ is a joint left inverse (or joint 1-left inverse) of $\mathbf{T} = (T_1, T_2, ..., T_d)$ if

$$S_1T_1 + S_2T_2 + ... + S_dT_d = I_{\mathcal{H}} \tag{4.2}$$

and it is a joint 2-left inverse of $\mathbf{T}$ if

$$I_{\mathcal{H}} - 2\sum_{1\le j\le d} S_jT_j + \sum_{1\le j\le d} S_j^2T_j^2 + 2\sum_{1\le j<k\le d} S_jS_kT_jT_k = 0. \tag{4.3}$$

For $\mathbf{T} = (T_1, \ldots, T_d)$ and $\mathbf{S} = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d$, set

$$\beta_m(\mathbf{S}, \mathbf{T}) = \sum_{0\le k\le m} (-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha.$$

Lemma 4.1. *Let $\mathbf{T} = (T_1, \ldots, T_d)$ and $\mathbf{S} = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d$ be commuting d-tuples of operators, then the following identity holds for $m \in \mathbb{N}$:*

$$\beta_{m+1}(\mathbf{S}, \mathbf{T}) = -\beta_m(\mathbf{S}, \mathbf{T}) + \sum_{1\le j\le d} S_j\beta_m(\mathbf{S}, \mathbf{T})T_j.$$

Proof.

$$\beta_{m+1}(\mathbf{S}, \mathbf{T}) = (-1)^{m+1}I_{\mathcal{H}} + \sum_{1\le k\le m}(-1)^k\Big[\binom{m}{k} + \binom{m}{k-1}\Big]\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$

$$+\sum_{|\alpha|=m+1}\frac{(m+1)!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha = -\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$

$$+\sum_{0\le k\le m-1}(-1)^k\binom{m}{k}\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$

$$\sum_{|\alpha|=m+1}\frac{(m+1)!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$

$$= -\beta_m(\mathbf{S}, \mathbf{T}) + \sum_{0\le k\le m-1}(-1)^k\binom{m}{k}\sum_{|\alpha|=k+1}\frac{k!(\alpha_1 + \cdots + \alpha_d)}{\alpha_1!.\cdots.\alpha_d!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$

$$+\sum_{|\alpha|=m+1}\frac{m!(\alpha_1 + \cdots + \alpha_d)}{\alpha_1!.\alpha_2....\alpha_d!}\mathbf{S}^\alpha\mathbf{T}^\alpha = -\beta_m(\mathbf{S}, \mathbf{T})$$

$$+\sum_{1\le j\le d}\sum_{0\le k\le m-1}(-1)^k\binom{m}{k}\sum_{|\alpha|=k+1}\frac{k!\alpha_j}{\alpha_1!.\alpha_2!\cdots\alpha_d!}$$

$$\Big(S_jS^{\alpha_1}\cdots S_j^{\alpha_j-1}S_{j+1}^{\alpha_{j+1}}\cdots S_d^{\alpha_d}T^{\alpha_1}\cdots T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}\cdots T_d^{\alpha_d}T_j\Big)$$

$$+\sum_{1\le j\le d}\sum_{|\alpha|=m+1}\frac{m!\alpha_j}{\alpha_1!.\alpha_2!....\alpha_d!}S_jS^{\alpha_1}...S_j^{\alpha_j-1}S_{j+1}^{\alpha_{j+1}}...S_d^{\alpha_d}T^{\alpha_1}...T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}...T_d^{\alpha_d}T_j$$

$$
\begin{aligned}
&= -\beta_m(\mathbf{S},\mathbf{T}) + \sum_{1\le j\le d}\sum_{0\le k\le m-1}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k!}{\beta!}S_j\mathbf{S}^\beta\mathbf{T}^\beta T_j\\
&+\sum_{1\le j\le d}\sum_{|\alpha|=m}\frac{m!}{\beta!}S_j\mathbf{S}^\beta\mathbf{T}^\beta T_j\\
&= -\beta_m(\mathbf{S},\mathbf{T}) + \sum_{1\le j\le d}\sum_{0\le k\le m}(-1)^k\binom{m}{k}\sum_{|\beta|=k}\frac{k1}{\beta!}S_j\mathbf{S}^\beta\mathbf{T}^\beta T_j\\
&= -\beta_m(\mathbf{S},\mathbf{T}) + \sum_{1\le j\le d}S_j\beta_m(\mathbf{S},\mathbf{T})T_j.
\end{aligned}
$$

Which completed the proof. □

Remark 4.3. From Lemma 4.1, it is clear that if $\mathbf{S}$ is a joint m-left inverse (resp. m-right inverse) of $\mathbf{T}$, then $\mathbf{S}$ is an joint m-left inverse (resp. m-right inverse) of $\mathbf{T}$ for all integer $n\ge m$.

For $k,n\in\mathbb{N}$ denote the (descending Pochhammer) symbol by $n^{(k)}$, i.e.

$$
n^{(k)} = \begin{cases} 0, \ \text{if} \ n=0\\ 0 \ \text{if} \ n>0 \ \text{and} \ k>n\\ \binom{n}{k}k! \ \text{if} \ n>0 \ \text{and} \ k\le n.\end{cases}
$$

Proposition 4.1. *Let* $\mathbf{S}=(S_1,\dots,S_d)\in\mathcal{B}(\mathcal{H})^d$ *and* $\mathbf{T}=(T_1,\dots,T_d)\in\mathcal{B}(\mathcal{H})^d$ *are commuting operators. Then the following properties hold:*

1.

$$
\sum_{|\alpha|=n}\frac{n!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha = \sum_{0\le k\le n}\frac{n^{(k)}}{k!}\beta_k(\mathbf{S},\mathbf{T}), \ \ \textit{for all} \ n=0,1,....
$$

2. If $\mathbf{S}$ *is a joint* m*-left inverse of* $\mathbf{T}$*, then*

$$
\sum_{|\alpha|=n}\frac{n!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha = \sum_{0\le k\le m-1}\frac{n^{(k)}}{k!}\beta_k(\mathbf{S},\mathbf{T}), \ \ \textit{for all} \ n=0,1,....
$$

Proof. 1. We prove the statement by indication on n. For $n=0,1$ the statement is true. Suppose that the statement is true for n.

Form the identity

$$
\beta_{n+1}(\mathbf{S},\mathbf{T}) = \sum_{0\le k\le n+1}(-1)^{n+1-k}\binom{n+1}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha
$$

it follows that

$$
\sum_{|\alpha|=n+1}\frac{(n+1)!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha = \beta_{n+1}(\mathbf{S},\mathbf{T}) - \sum_{0\le k\le n}(-1)^{n+1-k}\binom{n+1}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha.
$$

By the assumption and similar calculation as in [5] we obtained

$$\begin{aligned}
&\sum_{|\alpha|=n+1}\frac{(n+1)!}{\alpha!}\mathbf{S}^{\alpha}\mathbf{T}^{\alpha}\\
&=\beta_{n+1}(\mathbf{S},\mathbf{T})-\sum_{0\le k\le n}(-1)^{n+1-k}\binom{n+1}{k}\sum_{0\le j\le k}\frac{k^{(j)}}{j!}\beta_j(\mathbf{S},\mathbf{T})\\
&=\beta_{n+1}(\mathbf{S},\mathbf{T})-\sum_{0\le j\le n}\beta_j(\mathbf{S},\mathbf{T})\Bigg(\sum_{j\le k\le n}(-1)^{n+1-k}\binom{n+1}{k}\frac{k^{(j)}}{j!}\Bigg)\\
&=\beta_{n+1}(\mathbf{S},\mathbf{T})-\sum_{0\le k\le n}\frac{(n+1)^{(j)}}{j!}\beta_j(\mathbf{S},\mathbf{T})\underbrace{\Bigg(\sum_{0\le r\le n-j}(-1)^{n+1-j-r}\binom{n+1-j}{r}\Bigg)}_{=-1}\\
&=\sum_{0\le j\le n+1}\frac{(n+1)^{(j)}}{j!}\beta_j(\mathbf{S},\mathbf{T}).
\end{aligned}$$

2. The result follows immediately from the fact that if $\mathbf{S}$ is a m-left inverse of $\mathbf{T}$ then $\beta_k(\mathbf{S},\mathbf{T})=0$ for all $k\ge m$ (see Lemma 4.1). □

The following proposition is a generalization of [[22], Lemmas 3.1 and 3.2].

Proposition 4.2. *Let $\mathbf{T}=(T_1,\dots,T_d)\in\mathcal{B}(\mathcal{H})^d$ be a commuting tuple of operators. The following statements hold.*

(1) *If $\mathbf{T}$ possesses a joint 2-left inverse $\mathbf{S}=(S_1,\dots,S_d)$, then*

$$\sum_{|\alpha|=n}\frac{n!}{\alpha!}\mathbf{S}^{\alpha}\mathbf{T}^{\alpha}=(1-n)I_{\mathcal{H}}+n\Bigg(\sum_{1\le j\le d}S_jT_j\Bigg),\ \ \forall\, n\in\mathbb{N}. \tag{4.4}$$

(2) *If $\mathbf{T}$ possesses a joint 2-right inverse $\mathbf{R}=(R_1,\dots,R_d)$, then*

$$\sum_{|\alpha|=n}\frac{n!}{\alpha!}\mathbf{T}^{\alpha}\mathbf{R}^{\alpha}=(1-n)I_{\mathcal{H}}+n\Bigg(\sum_{1\le j\le d}T_jR_j\Bigg),\ \ \forall\, n\in\mathbb{N}. \tag{4.5}$$

Proof. We shall prove equality (4.4) by induction on n. For $n=0$ or $n=1$ it is clear. Assume that (4.4) is true for n and prove it for $n+1$. Indeed, a simple calculation shows that

$$\sum_{|\alpha|=n+1}\frac{(n+1)!}{\alpha!}\mathbf{S}^{\alpha}\mathbf{T}^{\alpha}=\sum_{\alpha_1+\cdots+\alpha_d=n+1}\frac{(n+1)n!}{\alpha_1!\cdots\alpha_d!}\mathbf{S}^{\alpha}\mathbf{T}^{\alpha}$$

$$= \sum_{\alpha_1+\cdots+\alpha_d=n+1} \frac{(\alpha_1+\cdots+\alpha_d)n!}{\alpha_1!\cdots\alpha_d!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$
$$= \sum_{1\le k\le d}\Bigg(\sum_{\alpha_1+\cdots+\alpha_k-1+\cdots+\alpha_d=n} \frac{n!}{\alpha_1!\cdots(\alpha_k-1)!\cdots\alpha_d!}S_kS_1^{\alpha_1}\cdots S_k^{\alpha_k-1}\cdots S_d^{\alpha_d}$$
$$\cdot\, T_1^{\alpha_1}\cdots T_k^{\alpha_k-1}\cdots T_d^{\alpha_d}T_k\Bigg)$$
$$= \sum_{1\le k\le d} S_k\Bigg(\sum_{|\beta|=n}\frac{n!}{\beta!}\mathbf{S}^\beta\mathbf{T}^\beta\Bigg)T_k.$$

Since $\mathbf{T}$ possesses a joint 2-left inverse tuple $\mathbf{S}$, it follow from the induction hypothesis and (4.3) that

$$\sum_{|\beta|=n+1}\frac{(n+1)!}{\alpha!}\mathbf{S}^\alpha\mathbf{T}^\alpha$$
$$= \sum_{1\le k\le d} S_k\Bigg(\sum_{|\beta|=n}\frac{n!}{\beta!}\mathbf{S}^\beta\mathbf{T}^\beta\Bigg)T_k$$
$$= \sum_{1\le k\le d} S_k\Bigg((1-n)I_{\mathcal{H}} + n\sum_{1\le j\le d}S_jT_j\Bigg)T_k$$
$$= (1-n)\sum_{1\le k\le d}S_kT_k + n\sum_{1\le j,\, k\le d}S_kS_jT_kT_j$$
$$= (1-n)\sum_{1\le k\le d}S_kT_k + n\Bigg(\sum_{1\le k\le d}S_k^2T_k^2 + 2\sum_{1\le j<k\le d}S_jS_kT_jT_k\Bigg)$$
$$= (1-n)\sum_{1\le k\le d}S_kT_k + n\Bigg(-I_{\mathcal{H}} + 2\sum_{1\le j\le d}S_jT_j\Bigg)$$
$$= -nI_{\mathcal{H}} + (n+1)\Bigg(\sum_{1\le k\le d}S_kT_k\Bigg),$$

so that (4.4) holds for $n+1$. Exchanging $\mathbf{S} = \mathbf{R}$ and $\mathbf{T}$ and similar to the above proof, we can prove that (4.5) holds.

□

Theorem 4.1. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ *be a commuting d-tuple such that* $\mathbf{T}$ *possesses a joint m-left inverse* $\mathbf{S} = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d$*, then the following statements hold:*

(1) $[0] \not\subset \sigma_{ap}(\mathbf{T})$,

(2) If $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(\mathbf{T})$*, then* $1 \in \sigma_{ap}(\sum_{1\le j\le d}\lambda_jS_j)$*,*

(3) *If* $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_p(\mathbf{T})$*, then* $1 \in \sigma_p(\sum_{1\le j\le d}\lambda_jS_j)$*.*

Proof. (1) Suppose contrary to our claim that $[0] \subset \sigma_{ap}(\mathbf{T})$ and let $\lambda = (\lambda_1, ..., \lambda_d) \in [0]$. Then there exists a sequence $(x_n)_n \in \mathcal{H}$ such that

$$\|x_n\| = 1 \quad \text{and} \quad (T_j - \lambda_j)x_n \longrightarrow 0 \text{ as } n \longrightarrow +\infty \text{ for } j = 1, 2, ..., d.$$

For $\alpha_j \geq 1$ we deduce that

$$(T_j^{\alpha_j} - \lambda_j^{\alpha_j})x_n \longrightarrow 0 \text{ as } n \longrightarrow +\infty \text{ for } j = 1, 2, ..., d$$

which mean that $(\mathbf{T}^\alpha - \lambda^\alpha)x_n \longrightarrow 0$ and hence,$(\mathbf{S}^\alpha\mathbf{T}^\alpha - \lambda^\alpha\mathbf{S}^\alpha)x_n \longrightarrow 0$.

Now, we get

$$\begin{aligned}
&\Big(\mathbf{S}^\alpha\mathbf{T}^\alpha - \lambda^\alpha\mathbf{S}^\alpha\Big)x_n \longrightarrow 0 \\
&\Longrightarrow \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\Big(\mathbf{S}^\alpha\mathbf{T}^\alpha - \lambda^\alpha\mathbf{S}^\alpha\Big)x_n \longrightarrow 0 \\
&\Longrightarrow (-1)^m x_n + \sum_{1\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\Big(\prod_{1\leq j\leq d}\lambda_j^{\alpha_j}\Big)\mathbf{S}^\alpha x_n \longrightarrow 0 \\
&\Longrightarrow x_n \longrightarrow 0 \ \text{ as } \ n \longrightarrow 0 \ \ (\text{since } \ \lambda \in [0]),
\end{aligned}$$

which is impossible.

(2) Let $\lambda = (\lambda_1, ..., \lambda_d) \in \sigma_{ap}(\mathbf{T})$, then there exists a sequence $(x_n)_n \in \mathcal{H}$ such that

$$\|x_n\| = 1 \quad \text{and} \quad (T_j - \lambda_j)x_n \longrightarrow 0 \text{ as } n \longrightarrow +\infty \text{ for } j = 1, \ldots, d.$$

$$\begin{aligned}
&\Big(\mathbf{S}^\alpha\mathbf{T}^\alpha - \lambda^\alpha\mathbf{S}^\alpha\Big)x_n \longrightarrow 0 \\
&\Longrightarrow \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\Big(\mathbf{S}^\alpha\mathbf{T}^\alpha - \lambda^\alpha\mathbf{S}^\alpha\Big)x_n \longrightarrow 0 \\
&\Longrightarrow \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\Big(\prod_{1\leq j\leq d}\lambda_j^{\alpha_j}\Big)\mathbf{S}^\alpha x_n \longrightarrow 0 \\
&\Longrightarrow \Big(I_{\mathcal{H}} - \sum_{1\leq j\leq d}\lambda_j S_j\Big)^m x_n \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.
\end{aligned}$$

We deduce that $(I - \sum_{1\leq j\leq d}\lambda_j S_j)$ is not bounded from below. Consequently, $1 \in \sigma_{ap}(\sum_{1\leq j\leq d}\lambda_j S_j)$ as required.

(3) The argument is similar to one given in (2). This achieves the proof of the Theorem. □

The proof of the following theorem is similar to the proof of Theorem 4.1, so we omit it.

Theorem 4.2. *Let* $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ *be a commuting tuple of operators such that* $\mathbf{T}$ *possesses a joint* m*-right inverse* $\mathbf{R} = (R_1, \ldots, R_d) \in \mathcal{B}(\mathcal{H})^d$*, then the following statements hold:*

(1) $[0] \not\subset \sigma_{ap}(\mathbf{R})$.

(2) *If* $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(\mathbf{R})$*, then* $1 \in \sigma_{ap}(\sum_{1 \le j \le d} \lambda_j T_j)$.

(3) *If* $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_p(\mathbf{R})$*, then* $1 \in \sigma_p(\sum_{1 \le j \le d} \lambda_j T_j)$.

Remark 4.4. When $d = 1$, the Theorem 4.2 is proved in ([22], Lemma 3.4).

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