THE TOPOLOGICAL INDICES OF THE CAYLEY GRAPHS OF DIHEDRAL GROUP D_{2n} AND THE GENERALIZED QUATERNION GROUP Q_{2^n}

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Abstract. A topological index of a simple connected graph Γ is a numeric quantity related to the structure of the graph Γ. The set of all automorphisms of Γ under the composition of mapping forms a group which is denoted by $\mathrm{Aut}(\Gamma)$. Let G be a group, and let $S \subset G$ be a set of group elements such that the identity element $1 \notin S$. The Cayley graph associated with (G, S) is defined as the directed graph with vertex set G and edge set E such that $e = xy$ is an edge of E if $(x^{-1}y) \in S$ for every vertices x, y in G. In this paper we define the Cayley graph of the Dihedral group D_{2n} and the Cayley graph of the generalized quaternion group Q_{2^n} on the specified subsets of these groups, and compute the Wiener, Szeged and PI indices of these graphs.

Keywords: Cayley graph, Dihedral group, generalized Quaternion group, topological index.

1. Introduction

Let G be a group with identity element 1 and let S be a nonempty subset of G such that, $1 \notin S$. The Cayley graph of G relative to S is denoted by $\Gamma = \Gamma(G, S)$ and it is a directed graph with vertex set G and edge set E consisting of the

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ordered pairs (x,y) such that $(x^{-1}y) \in S$. It is obvious that the Cayley graph $\Gamma(G, S)$ of a group G is undirected iff $S = S^{-1}$, i.e, $(x, y) \in E$ iff $(y, x) \in E$ in this case $\Gamma(G, S)$ is called the Cayley graph of G relative to S. The Cayley graph $\Gamma(G, S)$ is connected iff the set S is a generating set of the group G. The Cayley graph has been studied in [10, 11]. Let Γ (V, E) be a graph with vertex set V and edge set E, An automorphism θ of Γ is a bijective function on V which preserves the edges of Γ, i.e, e=uv is an edge of Γ iff $e^\theta = u^\theta v^\theta$ is an edge of Γ too. The set of all automorphisms of Γ forms a group $Aut(\Gamma)$ under combination of functions. $Aut(\Gamma)$ acts transitively on V if for two arbitrary vertices u and v there is an automorphisms $\theta \in Aut(\Gamma)$ such that $u^\theta = v$, then in this case Γ is said to be a vertex-transitive graph. In graph theory it has been shown that the Cayley graph (directed or undirected) of a group is always vertex-transitive.

The topological index of Γ is a numerical quantity which is constant under any arbitrary automorphism of Γ. The Wiener index is the oldest topological index of the simple connected graph, it first was introduced by a chemist named H.Wiener for molecular graphs while he was studying properties of chemical compounds [2, 3, 5, 9]. The Wiener index of a connected graph Γ is denoted by $W(\Gamma)$ and it is defined by:

$$W(\Gamma) = \sum_{\{u,v\} \subseteq V} d(u, v) = \frac{1}{2} \sum_{u \in V} d(u).$$

Where d(u,v) is the distance between two vertices u and v, and $d(u)$ is the distances between u with other vertices of Γ.

The Szeged index [4, 8] is another invariant of a graph Γ which is related to the distribution of the vertices of the graph and denoted by $Sz(\Gamma)$ and it is defined as follows:

Let $e = uv$ be an edge of the graph Γ, the sets $N_u(e|\Gamma)$ and $N_v(e|\Gamma)$ are defined as: $N_u(e|\Gamma) = \{x \in V | d(x, u) < d(x, v)\}$, $N_v(e|\Gamma) = \{x \in V | d(x, v) < d(x, u)\}$. Then $N_u(e|\Gamma)$ is the set of all vertices of Γ which are closer to u than v, $N_v(e|\Gamma)$ is defined similarly. The cardinalities of $N_u(e|\Gamma)$ and $N_v(e\Gamma)$ are denoted by $n_v(e|\Gamma)$ and $n_v(e|\Gamma)$ respectively.

Now, the Szeged index of Γ which is denoted by $Sz(\Gamma)$ is defined as:

$$Sz(\Gamma) = \sum_{e=uv \in E} n_u(e|\Gamma).n_v(e|\Gamma).$$

The Szeged index is closely related to the Wiener index such that in the case that the graph Γ is a tree, they are the same. Thus the Szeged index concerned with the distribution of the vertices of the graph, there is a topological index named Padmakar-Ivan index [6, 7] which is related to the distribution of the edges of the graph. The Padmakar-Ivan index of graph Γ is denoted by $PI(\Gamma)$ and is defined as follows:

$$PI(\Gamma) = \sum_{e=uv \in E} (n_{eu}(e|\Gamma) + n_{ev}(e|\Gamma)).$$

Where $n_{eu}(e|\Gamma)$ (resp. $n_{ev}(e|\Gamma)$) is the number of edges of the subgraph of Γ which has the vertex set $N_u(e|\Gamma)$(resp. $N_v(e|\Gamma)$).

The Dihedral group D_{2n} is the symmetry group of an n-sided regular polygonal which it has the following presentation $D_{2n} =< a, b \,|a^2 = b^n = I, (ab)^2 = I >$. Considering subset $S_1 = \{a, ab, b^{n-1}, b\}$ of D_{2n}, we can define the Cayley graph Cay(D_{2n}, S_1) in the cases that n is odd or even.

The generalized Quaternion Q_{2^n} is the non-abelian group of order 2^n and it has the following presentation $Q_{2^n} =< a, b|a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, b^{-1}ab = a^{-1} >$. For some integer $n \geq 3$. The ordinary Quaternion group corresponds to the case n=3. Now, we regard the subset $S_2 = \{a, a^{2^{n-1}-1}, b, a^{2^{n-2}}\}$ of Q_{2^n} and define the Cayley graph $Cay(Q_{2^n}, S_2)$

It is obvious that $S_1 = S_1^{-1}$ and $S_2 = S_2^{-1}$ also S_1 and S_2 are generating sets of groups (D_{2n}, (Q_{2^n} respectively, so the Cayley graphs Cay(D_{2n}, S_1) and Cay(Q_{2^n}, S_2) both are undirected connected graphs. Here we try to compute the Wiener and Szeged indices of the Cayley graph Cay(D_{2n}, S_1).

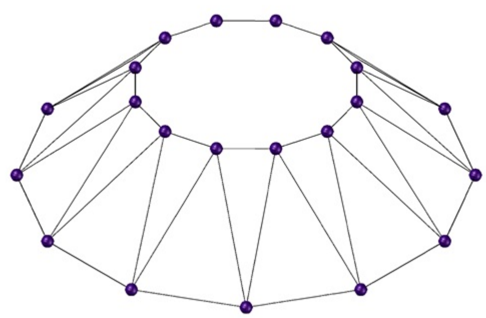

Figure 1: The $Cay(D_{2n}, S_1)$

2. Computation of Wiener, Szeged and PI indices of Cay(D_{2n}, S_1)

In this paper the following lemma [1] are frequently used.

Lemma 1. *Let $G = (V, E)$ be a simple connected graph with vertex set V and edge set E. suppose Aut(G) has orbits $\Delta_i = \Delta_i(u_i)$, $1 \leq i \leq k$, on V, where u_i is orbit representative. Then $W(G) = \frac{1}{2}\sum_{i=1}^{k} |\Delta_i|d(u_i)$. Where $d(u_i) = \sum_{x\in V} d(u_i, x)$. As a result of this lemma, when Aut(G) acts transitively on V, i.e, G is a vertex transitive graph, then $W(G) = \frac{1}{2}|V|d(u)$, for some $u \in V$.*

Lemma 2. *Let G=(V,E) be a simple connected graph with vertex set V and edge set E. If Aut(G) on E has orbits $E_1, E_2, ..., E_r$ with representatives $e_1, e_2, ..., e_r$,* *where* $e_i = u_i v_i \in E$ *then*:

$$Sz(G) = \sum_{i=1}^{r} |E_i|(n_{u_i}(e_i|G).n_{v_i}(e_i|G)), PI(G) = \sum_{i=1}^{r} |E_i|(n_{e_i u_i}(e_i|G)+n_{e_i v_i}(e_i|G)).$$

Proposition 1. *The Wiener index of the graph* $\Gamma = Cay(D_{2n}, S_1)$ *is* $W(\Gamma) = \frac{n^2(n+1)}{2}$

Proof. Because Γ is a vertex transitive graph by the lemma 1. the Wiener index of Γ is:

$$W(G) = \frac{1}{2}|D_{2n}|d(x). \tag{1}$$

Where x is an arbitrary vertex of Γ. We have $D_{2n} = \{1, b, b^2, ..., b^{n-1}, a, ab, ab^2, ..., ab^{n-1}\}$. Suppose that n is even, without less of generality we may choose $x = 1$ and calculate d(1) as follows:

$$d(1) = \sum_{u \in V} d(1, u) = \sum_{i=1}^{n-1} d(1, b^i) + \sum_{i=1}^{n} d(1, ab^i). \tag{2}$$

Because the vertices $1, b, b^2, ..., b^{n-1}$ form a cycle so, we have $d(1, b^i) = i, 1 \leq i \leq \frac{n}{2}$, $d(1, b^i) = n - i$, $\frac{n}{2} + 1 \leq i \leq n - 1$. Therefore:

$$\begin{aligned} \sum_{i=1}^{n-1} d(1, b^i) &= \sum_{i=1}^{\frac{n}{2}} d(1, b^i) + \sum_{i=\frac{n}{2}+1}^{n-1} d(1, b^i) = \sum_{i=1}^{\frac{n}{2}} i + \sum_{i=\frac{n}{2}+1}^{n-1} n - i \\ &= \frac{1}{2}(\frac{n}{2})(\frac{n}{2} + 1) + \frac{1}{2}(\frac{n}{2})(\frac{n}{2} - 1) = \frac{n^2}{4}. \end{aligned} \tag{3}$$

Also the vertices $a, ab, ab^2, ..., ab^{n-1}$ form a cycle we have: $d(1, ab^i) = i$, $1 \leq i \leq \frac{n}{2}$, $d(1, ab^i) = n - i + 1$, $\frac{n}{2} + 1 \leq i \leq n$. Therefore:

$$\begin{aligned} \sum_{i=1}^{n} d(1, ab^i) &= \sum_{i=1}^{\frac{n}{2}} d(1, ab^i) + \sum_{i=\frac{n}{2}+1}^{n} d(1, ab^i) = \sum_{i=1}^{\frac{n}{2}} i \\ &+ \sum_{i=\frac{n}{2}+1}^{n} n - i + 1 = \frac{1}{2}(\frac{n}{2})(\frac{n}{2} + 1) + \frac{1}{2}(\frac{n}{2})(\frac{n}{2} + 1) = (\frac{n}{2})(\frac{n}{2} + 1). \end{aligned} \tag{4}$$

Considering (3), (4) and replacing in (2), $d(1) = \frac{n}{2}(n+1)$. Now, suppose that n is odd, $d(1, b^i) = i$, $1 \leq i \leq \frac{n-1}{2}$, $d(1, b^i) = n - i$, $\frac{n+1}{2} + 1 \leq i \leq n-1$. Therefore:

$$\begin{aligned} \sum_{i=1}^{n-1} d(1, b^i) &= \sum_{i=1}^{\frac{n-1}{2}} d(1, b^i) + \sum_{i=\frac{n+1}{2}}^{n-1} d(1, b^i) = \sum_{i=1}^{\frac{n-1}{2}} i + \sum_{i=\frac{n+1}{2}}^{n-1} n - i \\ &= \frac{1}{2}(\frac{n-1}{2})(\frac{n+1}{2}) + \frac{1}{2}(\frac{n-1}{2})(\frac{n+1}{2}) = \frac{(n-1)(n+1)}{4} \end{aligned} \tag{5}$$

Also, $d(1, ab^i) = i$, $1 \leq i \leq \frac{n-1}{2}$, $d(1, ab^i) = n-i+1$, $\frac{n+1}{2}+1 \leq i \leq n$. Therefore:

$$\begin{aligned}
\sum_{i=1}^{n} d(1, ab^i) &= \sum_{i=1}^{\frac{n-1}{2}} d(1, ab^i) + \sum_{i=\frac{n+1}{2}}^{n} d(1, ab^i) = \sum_{i=1}^{\frac{n-1}{2}} i + \sum_{i=\frac{n+1}{2}}^{n} n - i + 1 \\
&= \frac{1}{2}(\frac{n-1}{2})(\frac{n+1}{2}) + \frac{1}{2}(\frac{n+1}{2})(\frac{n+3}{2}) \\
&= \frac{n+1}{4}(\frac{n-1}{2})(\frac{n+3}{2}) = \frac{n+1}{4}(n+1).
\end{aligned} \tag{6}$$

Considering (5),(6) and replacing in (2) we have:

$$d(1) = \frac{n}{2}(n+1). \tag{7}$$

We see that in two cases n is even or odd the same result is obtained, so replacing (7) in (1) the proof is completed. □

Proposition 2. *The Szeged index of the graph* $\Gamma = Cay(D_{2n}, S_1)$ *is:*

$$Sz(\Gamma) = \begin{cases} n(\frac{5}{2}n^2 - 4n + 2), & n \ \ is \ \ even \\ n(\frac{5}{2}n^2 - 3n + \frac{5}{2}), & n \ \ is \ \ odd. \end{cases}$$

Proof. Let E be the edge set of the graph Γ. If we consider the action of $Aut(\Gamma)$ on the E, it is obvious that it has two orbits, the first is the set of edges which are the sides of the upper and lower polygons and the second is the edges that are located between these polygons, which are denoted by E_1 and E_2 respectively. $Aut(\Gamma)$ acts transitively on each orbit and the set E breaks into E_1 and E_2 whose union is E. By the lemma 2, we have:

$$Sz(G) = \sum_{i=1}^{2} |E_i|(n_{u_i}(e_i|\Gamma).n_{v_i}(e_i|\Gamma))$$

Where $e_i = u_i v_i$ is an arbitrary edge in E_i. We choose edge $e_1 = u_1 v_1 \in E_1$ as a representative of E_1, and suppose that n is even, $(\frac{n}{2} - 1)$ vertices on upper polygon and $(\frac{n}{2})$ vertices on lower polygon form the set of vertices which are closer to u_1 than v_1. $(N_{u_1}(e_1|\Gamma))$. So is the set $(N_{v_1}(e_1|\Gamma))$. Also if the edge $e_2 = u_1 v_2 \in E_2$ is a representative 0f E_2 then $(\frac{n}{2})$ vertices on upper polygon and $(\frac{n}{2})$ vertices on lower polygon form the set of vertices which are closer to u_1 than v_2, $(N_{u_1}(e_2|\Gamma))$. So is the set $(N_{v_2}(e_2|\Gamma))$. Therefore we have:

$$|N_{u_1}(e_1|\Gamma| = |N_{v_1}(e_1|\Gamma)| = n - 1, \ |N_{u_1}(e_2|\Gamma| = |N_{v_2}(e_2|\Gamma)| = \frac{n}{2}.$$

It is clear that $\rho(u) = 4$ for every vertex, therefore $|E| = 4n$, and in two cases n is even or odd, $|E_1| = |E_2| = 2n$, so, we have:

$$\begin{aligned}Sz(G) &= \sum_{i=1}^{2} |E_i|(n_{u_i}(e_i|\Gamma).n_{v_i}(e_i|\Gamma))\\ &= |E_1|((n_{u_1}(e_1|\Gamma).n_{v_1}(e_1|\Gamma)) + |E_2|(n_{u_1}(e_2|\Gamma).n_{v_2}(e_2|\Gamma))\\ &= 2n(n-1)(n-1) + 2n(\frac{n}{2})(\frac{n}{2}) = 2n(n-1)^2 + \frac{n^2}{2} = n(\frac{5}{2}n^2 - 4n + 2).\end{aligned}$$

Now, when n is odd and $e_1 = u_1v_1$ and $e_2 = u_1v_2$ are representative of E_1 and E_2, respectively considering, $(\frac{n}{2}-1)$ vertices on upper polygon and $(\frac{n}{2}-1)$ vertices on lower polygon form the set $N_{u_1}(e_1|\Gamma)$. So is the set $N_{v_1}(e_1|\Gamma)$. Also $(\frac{n+1}{2})$ vertices on upper polygon and $(\frac{n+1}{2})$ vertices on lower polygon form the set $N_{u_1}(e_2|\Gamma)$. So is the set $N_{v_2}(e_2|\Gamma)$. Therefore we have:

$$|N_{u_1}(e_1|\Gamma| = |N_{v_1}(e_1|\Gamma)| = n-1,\ |N_{u_1}(e_2|\Gamma| = |N_{v_2}(e_2|\Gamma)| = \frac{n+1}{2}.$$

The Szeged index in this case is obtained as follows:

$$\begin{aligned}Sz(\Gamma) &= \sum_{i=1}^{2} |E_i|(n_{u_i}(e_i|\Gamma).n_{v_i}(e_i|\Gamma))\\ &= |E_1|((n_{u_1}(e_1|\Gamma).n_{v_1}(e_1|\Gamma)) + |E_2|(n_{u_1}(e_2|\Gamma).n_{v_2}(e_2|\Gamma))\\ &= 2n(n-1)(n-1) + 2n(\frac{n+1}{2})(\frac{n+1}{2})\\ &= 2n(n-1)^2 + 2n(\frac{n+1}{2})^2 = n(\frac{5}{2}n^2 - 3n + \frac{5}{2}).\end{aligned}$$

□

Proposition 3. *The PI index of the graph* $\Gamma = Cay(D_{2n}, S_1)$ *is:*

$$PI(\Gamma) = \begin{cases} 2n(5n-12), & n \ \ is \ \ even \\ 2n(5n-11), & n \ \ is \ \ odd. \end{cases}$$

Proof. Similar to the proof of previous proposition, the action of $Aut(\Gamma)$ on the E has two orbits E_1, E_2 and by the lemma 2, the PI index of Γ can be obtained by the formula as follows:

$$PI(\Gamma) = \sum_{i=1}^{2} |E_i|(n_{e_iu_i}(e_i|\Gamma) + n_{e_iv_i}(e_i|\Gamma)). \tag{8}$$

Where $e_i = u_iv_i$ is an arbitrary edge in E_i. When n is even by considering, $n_{u_1}(e_1|\Gamma)$, $n_{v_1}(e_1|\Gamma)$, $n_{u_1}(e_2|\Gamma)$, $n_{v_2}(e_2|\Gamma)$ in the proof of proposition 2.2 we have:

$$(n_{e_1u_1}(e_1|\Gamma) + n_{e_1v_1}(e_1|\Gamma)) = (2n-5), (n_{e_2u_1}(e_2|\Gamma) + n_{e_2v_2}(e_2|\Gamma)) = \frac{n}{2} - 1.$$

So, by (8) $PI(\Gamma) = 2n(2(2n-5)) + 2n(2(\frac{n}{2}-1)) = 2n(5n-12)$. Similarly in the case that n is odd we have:

$$(n_{e_1u_1}(e_1|G) + n_{e_1v_1}(e_1|\Gamma)) = (2n-5),\ (n_{e_2u_1}(e_2|G) + n_{e_2v_2}(e_2|\Gamma)) = \frac{n-1}{2}.$$

Therefore $PI(\Gamma) = 2n(2(2n-5)) + 2n(2(\frac{n-1}{2})) = 2n(5n-11)$. □

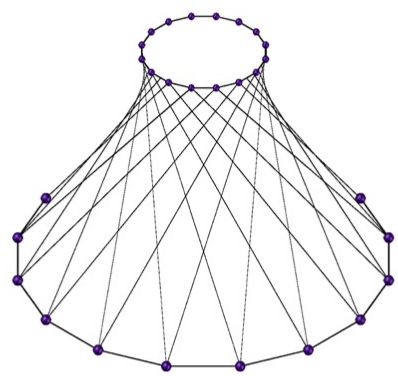

Figure 2: The Cay (Q_{2^n}, S_2)

3. Computation of Wiener, Szeged and PI indices of Cay(Q_{2^n}, S_2)

Proposition 4. *The Wiener index of the graph* $H = Cay(Q_{2^n}, S_2)$ *is* $W(H) = 2^{n-1}(2^n(2^{n-4}+1)-2)$.

Proof. As what was said in introduction, the Cayley graph is a vertex-transitive graph so by the lemma1 we have:

$$W(H) = \frac{1}{2}|Q_{2^n}|d(x). \tag{9}$$

Where x is an arbitrary vertex of H. We have $Q_{2^n} = \{1, a, a^2, ..., a^{2^{n-1}-1}, b, ab, a^2b, ..., a^{2^{n-1}-1}b\}$. Now let $x = 1$ and calculate d(1), by Fig. 2 we have:

$$d(1) = \sum_{u \in V} d(1,u) = \sum_{i=1}^{2^{n-1}-1} d(1,a^i) + \sum_{i=1}^{2^{n-1}} d(1,a^ib). \tag{10}$$

Because the vertices $1, a, a^2, ..., a^{2^{n-1}-1}$ form a cycle we have $d(1,a^i)$=$d(1,a^{2^{n-1}-i})$, $1 \le i \le 2^{n-1}$,

$$\sum_{i=1}^{2^{n-1}-1} d(1,a^i) = 2(\sum_{i=1}^{2^{n-2}-1} d(1,a^i)) + d(1,a^{2^{n-2}}), \tag{11}$$

$$d(1,a^i)=i, 1\leq i\leq 2^{n-3}+1,$$
$$d(1,a^i)=2^{n-2}-i+2, 2^{n-3}+2\leq i\leq 2^{n-2},$$
$$d(1,a^{2^{n-2}})=2.$$

So:

$$\sum_{i=1}^{2^{n-2}-1} d(1,a^i)=\sum_{i=1}^{2^{n-3}+1} d(1,a^i)+\sum_{i=2^{n-3}+2}^{2^{n-2}-1} d(1,a^i)$$

(12)
$$\sum_{i=1}^{2^{n-3}+1} i+\sum_{i=2^{n-3}+2}^{2^{n-2}-1} 2^{n-2}-i+2$$
$$=\frac{(2^{n-3}+1)(2^{n-3}+2)}{2}+\frac{(2^{n-3})(2^{n-3}+1)}{2}-3$$
$$=(2^{n-3}+1)(2^{n-3}+1)-3=(2^{n-3}+1)^2-3.$$

Considering (11),(12) we have:

(13)
$$\sum_{i=1}^{2^{n-1}-1} d(1,a^i)=2((2^{n-3}+1)^2-3)+2=2(2^{n-3}+1)^2-4.$$

Also, the vertices $b, ab, a^2b, ..., a^{2^{n-1}-1}b$ form a cycle so we have $d(1,a^ib)=d(1,a^{2^{n-2}+i}b), 1\leq i\leq 2^{n-2}$,

(14)
$$\sum_{i=1}^{2^{n-1}} d(1,a^ib)=2(\sum_{i=1}^{2^{n-2}-2} d(1,a^ib)).$$

$$d(1,a^ib)=i+1, 1\leq i\leq 2^{n-3},$$
$$d(1,a^ib)=2^{n-2}-i+1, 2^{n-3}+1\leq i\leq 2^{n-2},$$
$$\sum_{i=1}^{2^{n-2}} d(1,a^ib)=\sum_{i=1}^{2^{n-3}} d(1,a^ib)+\sum_{i=2^{n-3}+1}^{2^{n-2}} d(1,a^ib),$$
$$=\sum_{i=1}^{2^{n-3}}(i+1)+\sum_{i=2^{n-3}+1}^{2^{n-2}}(2^{n-2}-i+1)=\sum_{i=1}^{2^{n-3}}(i+1)+\sum_{i=2^{n-3}+1}^{2^{n-2}} i$$
$$=\frac{(2^{n-3}+1)(2^{n-3}+2)}{2}-1+\frac{(2^{n-3})(2^{n-3}+1)}{2}$$

(15)
$$=(2^{n-3}+1)(2^{n-4}+1)-1+(2^{n-4})(2^{n-3}+1)$$
$$=(2^{n-3}+1)(2^{n-4}+1+2^{n-4})-1$$
$$=(2^{n-3}+1)(2^{n-3}+1)-1=(2^{n-3}+1)^2-1=(2^{n-2})(2^{n-4}+1).$$

Now replacing (15) in (14) we have:

$$\sum_{i=1}^{2^{n-1}} d(1, a^i b) = (2^{n-1})(2^{n-4} + 1). \tag{16}$$

Therefore considering (16) and (13) and replacing in (10) we see:

$$\begin{aligned} d(1) &= 2(2^{n-3} + 1)^2) - 4 + (2^{n-1})(2^{n-4} + 1) \\ &= (2^{n-1})(2^{n-4} + 1) - 2 + (2^{n-1})(2^{n-4} + 1) \\ &= (2^{n-1})(2^{n-3} + 1) - 2 = (2^n)(2^{n-4} + 1) - 2. \end{aligned} \tag{17}$$

So by replacing (17) in (9) the proof is done. □

Proposition 5. *The Szeged index of the graph $H = Cay(Q_{2^n}, S_2)$ is:*

$$Sz(H) = 2^{3n-1}.$$

Proof. We denote the vertex set and the edge set of H by V, E respectively. Note that there is no vertex in V which it's distances from the nodes of the edge $e \in E$ be the same, for any arbitrary edge $e = uv$ of E. So we have $|N_u(e|H)| + |N_v(e|H)| = |V|$. Also, because the shape of the graph is symmetric, Fig. 2, so the number of vertices which are closer to u than v are the same to the vertices which are closer to v than u, for any edge $e = uv \in E$, therefore:

$$|N_u(e|H)| = |N_v(e|H)| = \frac{1}{2}|V| = \frac{1}{2}|Q_{2^n}| = 2^{n-1}.$$

By the Fig. 2, it is clear that:

$$|E| = 2^{n-1} + 2^{n-1} + 4.(2^{n-2}) = 2^{n+1}. \tag{18}$$

So $Sz(H) = \sum_{e=uv\in E}(n_u(e|H).n_v(e|H)) = (2^{n-1})(2^{n-1})|E| = 2^{3n-1}$. □

Proposition 6. *The PI index of the graph $H = Cay(Q_{2^n}, S_2)$ is:*

$$PI(\Gamma) = \begin{cases} 2^n(7.2^{n-2} - 9), & n > 3 \\ 96, & n = 3. \end{cases}$$

Proof. We have $PI(H) = \sum_{e=uv\in E}(n_{eu}(e|H) + n_{ev}(e|H))$. In the case n=3, we see for any edge $e = uv$ of E, $n_{eu}(e|H) = n_e v(e|H) = 3$, so by (18) and the formula of PI index it is done. According to the Fig. 2, it is clear that the action of Aut(H) on the edge set E of H has two orbits E_1,E_2. First, the set of the edges which are the sides of the upper and lower polygons, second the rest of the edges. we let the edges $e_1 = u_1v_1$ and $e_2 = u_1v_2$ as representative of E_1 and E_2, respectively. By Fig. 2 we have:

$$\begin{aligned} n_{e_1u_1}(e_1|H) &= n_{e_1v_1}(e_1|H) \\ &= 2^{n-3} - 1 + 2^{n-3} - 1 + 2^{n-1} + 2^{n-3} - 2 + (2^{n-3} - 1).4 + 2 = 2^n - 6. \end{aligned}$$

And $n_{e_2u_1}(e_2|H) = n_{e_2v_2}(e_2|H) = 2^{n-2} + 2^{n-2} - 2 + 2^{n-2} - 1 = 3(2^{n-2}) - 3$. So, By lemma.2,

$$PI(H) = \sum_{i=1}^{2} |E_i|(n_{e_iu_i}(e_i|H + n_{e_iv_i}(e_i|H))$$
$$= |E_1|(n_{e_1u_1}(e_1|H) + n_{e_1v_1}(e_1|H)) + |E_2|(n_{e_2u_1}(e_2|H) + n_{e_2v_2}(e_2|H))$$
$$2^n(2^n - 6) + 2^n(3.2^{n-2} - 3) = 2^n(7.2^{n-2} - 9).$$

□

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