CLASS OF ADMISSIBLE PERTURBATIONS OF SPECIAL EXPRESSIONS INVOLVING COMPLETELY MONOTONIC FUNCTIONS

Jerico B. Bacani

Julius Fergy T. Rabago

Department of Mathematics and Computer Science
College of Science
University of the Philippines Baguio
Gov. Pack Rd., Baguio City 2600
Philippines
jicderivative@yahoo.com
jfrabago@gmail.com

Abstract. In this article, a class $\mathbf{M_2}$ of admissible perturbations of the special expression $\mathbf{M}_0 = \sum_{k=0}^{r} c_k t^{\alpha_k} D_t^{\rho_k}$ in the weighted space $\mathcal{L}^2_\omega([1,\infty))$ will be presented. It will be shown that the operator $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$, where ω belongs to the family of completely monotonic functions, is an admissible perturbation of $\mathbf{M}_0$ in the non-weighted space $\mathcal{L}^2([1,\infty))$, and eventually preserves the essential spectrum and nullity of $\mathbf{M}_0$ in that space. Our discussion will be limited only to special expressions with $\alpha_1 < \rho_1$.

Keywords: special expression, admissible perturbations, $\mathcal{L}^2$-space, weighted $\mathcal{L}^2$-space, essential spectrum, nullity, completely monotonic functions.

1. Introduction

In 2014, a new class of admissible perturbations in the weighted space $\mathcal{L}^2_\omega([1,\infty))$ of the *special expression* $\mathbf{M}_0$ was identified and studied by J. B. Bacani in [4]. We recall that this differential operator is of the form

$$\mathbf{M}_0 = \sum_{k=0}^{r} c_k t^{\alpha_k} D_t^{\rho_k} \tag{1}$$

where $c_k \in \mathbb{C}$ and $D_t = \frac{d}{dt}$ with

i) $\rho_k \in \mathbb{N}$ for every k, such that

$$0 = \rho_0 < \rho_1 \cdots < \rho_r = n \tag{2}$$

ii) $\alpha_k \in \mathbb{R}$ for every k, satisfying

$$\alpha_0 = 0 \quad \text{and} \quad \alpha_1 \leqslant \rho_1, \tag{3}$$

and

$$1 \geqslant \frac{\alpha_k - \alpha_{k-1}}{\rho_k - \rho_{k-1}} \geqslant \frac{\alpha_{k+1} - \alpha_k}{\rho_{k+1} - \rho_k}, \tag{4}$$

for $k = 1, \ldots, r-1$ if $r > 1$.

The perturbation, which the author denoted it by $\mathbf{M_1}$, satisfies the following conditions:

AP_1) For every $i > l$,

$$\sup_{t \in \mathbf{I}} \left| \frac{a_i(t)}{t^{i-l} a_l(t)} \right| \quad \text{exists,}$$

where $a_l(t) \in C^l(\mathbf{I}), \mathbf{I} = [1, \infty), l = 0, 1, \ldots, n-1$; and

AP_2) There exist auxiliary functions $b_l(t)$ such that

$$\left| \frac{a_l(t)}{b_l(t)} \right| \quad \text{is bounded,}$$

where $0 < b_l(t) \in C^\infty([1, \infty))$ for all l and $b_l(t) = o(t^{\gamma(l+1)})$ and $b_l(t) = o(t^{\gamma(l)})$ as $t \to \infty$.

The perturbation was shown to be different from the one presented in [3] and the one published in [2].

In the current work, another class of admissible perturbations of the special expression $\mathbf{M}_0$ in the weighted space, which we denote by $\mathbf{M_2}$, will be presented. It will be shown that the operator $\omega^{\frac{1}{2}} \mathbf{M_2} \omega^{-\frac{1}{2}}$, where ω belongs to the family of completely monotonic functions, is an admissible perturbation of $\mathbf{M}_0$ in the non-weighted space $\mathcal{L}^2([1, \infty))$, and eventually preserves the essential spectrum of $\mathbf{M}_0$ in that space. As in [4], our discussion will be limited only to special expressions with $\alpha_1 < \rho_1$. We point out that this new class of perturbation is indeed different from what has been presented in [4] by giving an example. For other related works in the study of special expression $\mathbf{M}_0$, we offer the following articles: [5, 7, 9, 10] and [11].

In this section, we provide further discussion about special expressions, as well as some basic definitions and notations used in this paper.

The special expression $\mathbf{M}_0$, its essential part, and the polygonal path it generates are defined as follows (cf. [10]).

Definition 1.1. *A* special expression $\mathbf{M}_0$ *is a differential expression in* $\mathcal{L}^2([1, \infty))$ *of the form* (1) *where* ρ_k *and* α_k *satisfy* (2), (3), *and* (4). *We denote* $\sigma_1 < \sigma_2 < \cdots < \sigma_{s-1}$ *those indices* $k (k = 1, \ldots, r-1)$ *for which the strong inequality holds in* (3) *and with* $\sigma_0 = 0$ *and* $\sigma_s = r$. *The coefficients* c_k *are arbitrary complex constants and for* $k = \sigma_1, \ldots, r$,

$$c_\sigma \in \mathbf{C} \setminus \{0\}, \qquad \text{and} \qquad \sum_{\substack{\rho_\delta + \rho_x = 2\sigma \\ \sigma_i \leqslant \delta, x \leqslant \sigma_{i+1}}} (-1)^{\rho_\delta + \sigma} c_\delta c_x \geqslant 0$$

where $\sigma = \rho_{\sigma_i}, \ldots, \rho_{\sigma_{i+2}}, i = 1, \ldots, s-1$. *The indices* $\sigma_1, \ldots, \sigma_s$ *are called* kink indices. *The essential part of* $\mathbf{M}_0$ *is given by*

$$\mathbf{M}_{0,0} = \sum_{k=0}^{\sigma_1} c_k t^{\alpha_k} D_t^{\rho_k}.$$

We can give a graphical interpretation of the special expression $\mathbf{M}_0$ in $\mathbb{R}^2$ by plotting and joining by a line the points (ρ_k, α_k) and $(\rho_{k+1}, \alpha_{k+1})$ for $k = 0, 1, \ldots, r$. The resulting graph is called the *polygonal path generated by* $\mathbf{M}_0$. If we let $m_{\sigma_i}(i = 1, \ldots, s)$ be the slope of the line connecting the points $(\rho_{\sigma_{i-1}}, \alpha_{\sigma_{i-1}})$ and $(\rho_{\sigma_{i-1}}, \alpha_{\sigma_{i-1}})$, then

$$1 \geqslant m_{\sigma_1} \geqslant m_{\sigma_2} \geqslant \cdots \geqslant m_{\sigma_s}.$$

Hence, the polygonal path generated by $\mathbf{M}_0$ lies on or below the bisectrix. Furthermore, the polygonal path generated by $\mathbf{M}_0$ corresponds to the graph of the function $\gamma : [0, n] \to \mathbb{R}$ defined by

$$\gamma(k) = \frac{1}{\rho_{\sigma_{i+1}} - \rho_{\sigma_i}} \left\{ (k - \rho_{\sigma_i})\alpha_{\sigma_{i+1}} + (\rho_{\sigma_{i+1}} - k)\alpha_{\sigma_i} \right\} \tag{5}$$

for $k \in [\rho_{\sigma_i}, \rho_{\sigma_{i+1}}], i = 0, 1, 2, \ldots, s-1$.

In [11], Schultze evaluated the essential spectrum and nullity of $\mathbf{M}_0$ and showed that the essential spectrum and nullity of the essential part of $\mathbf{M}_0$ and the essential spectrum and nullity of $\mathbf{M}_0$ are indeed equal. In short, he had the following results for $\alpha_1 < \rho_1$:

Theorem 1.2. *Let* $\mathbf{M}_0$ *be a special expression. Then, for* $\alpha_1 < \rho_1$,

$$\sigma_e(\mathbf{M}_0) = \sigma_e(\mathbf{M}_{0,0}) = \left\{ \sum_{k=0}^{\sigma_1} c_k z^{\rho_k} : Re(z) = 0 \right\}, \tag{6}$$

and for every $x \in \mathbb{C} \setminus \sigma_e(\mathbf{M}_0)$,

$$\operatorname{nul}(\mathbf{M}_0 - x) = \operatorname{nul}(\mathbf{M}_{0,0} - x) + \sum_{i=1}^{s-1} \# \left\{ z | \sum_{k=\sigma_i}^{\sigma_i+1} c_k z^{\rho_k} = x, Re\, z < 0 \right\}, \tag{7}$$

where

$$\operatorname{nul}(\mathbf{M}_{0,0} - x) = \# \left\{ z | \sum_{k=0}^{\sigma_1} c_k z^{\rho_k} = x, Re\, z < 0 \right\}.$$

Also, in the usual $\mathcal{L}^2$-space, a different class of perturbations, called admissible perturbations $\mathbf{M}$ of special expressions was determined by Mumpar-Victoria [9] in her paper, and was shown to preserve essential spectrum and nullity of special expressions.

Definition 1.3. *Let* $\mathbf{M}$ *be a differential expression of the form*

$$\mathbf{M} = \sum_{l=0}^{n-1} a_l(t) D_t^l. \tag{8}$$

We say that $\mathbf{M}$ *is an admissible perturbation of the special expression* $\mathbf{M}_0$ *if there exists a* B *such that the coefficients* $a_l(t)$ *satisfy the following*

$$\sup_{[x,x+1]\subset \mathbf{I}} \int_x^{x+1} \left| \frac{a_l(t)}{b_l(t)} \right|^2 dt < B \tag{9}$$

where $b_l(t) \in C^l(\mathbf{I})$ *for* $l = 0, 1, \ldots, n-1$ *and* $0 < b_l(t)$ *is an auxiliary function in* $C^\infty(\mathbf{I})$ *satisfying*

$$b_l(t) = o(t^{\gamma(l+1)}) \quad \text{and} \quad b_l(t) = o(t^{\gamma(l)}) \tag{10}$$

as $t \longrightarrow \infty$.

For the invariance of nullity, we can only admit a somewhat less general class of perturbations consisting of expressions (8) *satisfying*

$$\sup_{[x,x+1]\subset \mathbf{I}} \int_x^{x+1} \left| \frac{a_j^{(j-l)}(t)}{b_l(t)} \right|^2 dt < \tilde{B} \tag{11}$$

for $l = 0, \ldots, n-1$ *and* $j = l, \ldots, n-1$.

Theorem 1.4. *Let* $\mathbf{M}_0$ *be a special expression and* $\mathbf{M}$ *be an admissible perturbation of* $\mathbf{M}_0$ *of the form* (8) *satisfying* (9) *and* (10). *Then*

$$\sigma_e(\mathbf{M}_0 + \mathbf{M}) = \sigma_e(\mathbf{M}_0).$$

In addition, if $\mathbf{M}$ *satisfies* (11), *then,* $\mathrm{nul}(\mathbf{M}_0 + \mathbf{M} - x) = \mathrm{nul}(\mathbf{M}_0 - x)$ *for every* $x \in \mathbb{C} \setminus \sigma_e(\mathbf{M}_0)$.

2. Completely monotonic functions and the weighted $\mathcal{L}^2$-space

A function f belongs to the class of completely monotonic (c.m.) functions on $\mathbf{I}$ if it possesses derivatives $f^{(n)}(x)$ for $n = 1, 2, \ldots$, and $(-1)^k f^{(k)}(x) \geqslant 0$ for all $k = 0, 1, 2, \ldots$ on $\mathbf{I}$. The well-known Bernstein's Theorem (see [12], p. 161) states that a necessary and sufficient condition for f to be a c.m. function on $(0, \infty)$ is that

$$f(x) = \int_0^\infty e^{-xt} d\chi(t), \tag{12}$$

where $\chi(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. From this, one can easily infer that a non-identically zero c.m. function $f(x)$ cannot vanish for any positive x. A more precise characterization of c.m. functions, also known as their *Bernstein representation*, is given as follows.

Theorem 2.1 (Bernstein)**.** *If $f(x)$ is a c.m. function on $(0,\infty)$, then it is the Laplace transform of a unique Radon measure μ on $[0,\infty)$; that is,*

$$f(x) = \int_0^\infty e^{-xt}\mu(dt), \tag{13}$$

for all $x > 0$. Conversely, if μ is a Radon measure on $[0,\infty)$ such that the above integral is convergent for $x > 0$, then it defines a c.m. function.

The cumulative distribution function for measure μ on $[0,\infty)$ is denoted by $\chi_\mu(t) = \mu[0,t]$. It is often written as $\chi(t)$, as in (12), for simplicity.

Completely monotonic functions appear naturally in various fields, such as in probability theory, numerical analysis, physics and potential theory. A thorough discussion of the main properties of these functions is given in Chapter IV of [12]. Readers may also want to see [8] for a good survey of some properties of completely monotonic functions.

The following elementary functions are examples of c.m. functions:

$$e^{-ax}, \quad \frac{1}{(\beta+\mu x)^\nu}, \quad \text{and} \quad \ln\left(b+\frac{c}{x}\right),$$

where $\min\{a,\beta,\mu,\nu\} \geqslant 0$ with β and μ are not both zero, $b \geqslant 1$, and $c > 0$.

Other examples of elementary c.m. functions are

$$e^{\frac{a}{x}},\ a > 0, \quad \text{and} \quad \frac{\ln(1+x)}{x}.$$

Obviously, given any two c.m. functions f and g, their linear combination and their product are also c.m. (cf. [8]).

The results of Schultze in [11] were generalized in the weighted $\mathcal{L}^2$-space by Agapito (cf. [1]). The weighted space is defined as follows:

Definition 2.2. *Suppose the function $\omega : [1,\infty) \longrightarrow (0,\infty)$ is measurable. The* space $\mathcal{L}^2_\omega(\mathbf{I})$ of weighted square integrable functions *over $\mathbf{I} = [1,\infty)$ is defined by*

$$\mathcal{L}^2_\omega(\mathbf{I}) := \left\{ f : \mathbf{I} \to \mathbb{C} | f \textit{ is measurable and} \int_\mathbf{I} |f(t)|^2\omega(t)dt < \infty \right\},$$

with inner product given by

$$\langle f, g\rangle_\omega := \int_\mathbf{I} f(t)\overline{g(t)}\omega(t)dt$$

for any functions $f, g \in \mathcal{L}^2_\omega(\mathbf{I})$. $\mathcal{L}^2_\omega(\mathbf{I})$ is commonly known as the weighted $\mathcal{L}^2$-space *with ω as the weight function.*

Remark 2.3. It is easy to show that space $\mathcal{L}^2_\omega(\mathbf{I})$ together with the inner product $\langle\cdot,\cdot\rangle_\omega$ is a Hilbert space. Furthermore, one can show that $\mathcal{L}^2(\mathbf{I})$ is equivalent to $\mathcal{L}^2_\omega(\mathbf{I})$ under the isometry $W : \mathcal{L}^2_\omega(\mathbf{I}) \to \mathcal{L}_2(\mathbf{I})$ given by $Wf = \omega^{\frac{1}{2}} f$.

In [4], Bacani considered weight functions $\omega : [1,\infty) \longrightarrow (0,\infty)$ that satisfy the following conditions in the study of a class of admissible perturbations:

(A1) $\omega \in C^{\infty}([1,\infty))$.

(A2) $\dfrac{x^k \omega^{(k)}}{\omega} = O(1)$ for $k = 0, 1, \ldots, n$.

The next Lemma was also used in [4].

Lemma 2.4. *If* $\omega : [1,\infty) \longrightarrow (0,\infty)$ *satisfies* (A1) *or* (A2) *then so does* ω^{α}, *for any* $\alpha \in \mathbb{R}$.

In this current work, we consider weight functions ω belonging to the class of completely monotonic functions but with the restriction that $\omega > 0$ for all $x \in (0,\infty)$.

Clearly, any c.m. function ω satisfies condition (A1) and we claim that it also satisfies (A2). We formally state this result in the following Lemma.

Lemma 2.5. *If* ω *is a c.m. function then it satisfies* (A1) *and* (A2).

Proof. Let ω be a c.m. function. It is clear that (A1) is already satisfied. Hence, we only need to prove (A2). First, we note that the following inequality holds:

$$f_k(x) = x^k e^{-\frac{xk}{k+1}} \leqslant (k+1)ke^{-k}, \quad \forall x \in (0,\infty), k \geqslant 1.$$

Indeed, by taking the derivative of $f_k(x)$ with respect to x, we get

$$f_k'(x) = kx^{k-1}e^{-\frac{xk}{k+1}} - \frac{k}{k+1}x^k e^{-\frac{xk}{k+1}} = kx^{k-1}e^{-\frac{xk}{k+1}}\left(1 - \frac{x}{k+1}\right).$$

Evidently, the above derivative is positive if $x < k+1$ and negative if $x > k+1$. Thus $f_k(x)$ is increasing on $(0, k+1)$ and decreasing on $(k+1,\infty)$. Therefore, the global maximum is achieved at $x = k+1$ with $f_k(k+1) = (k+1)ke^{-k}$.

Now, in view of the above inequality, we find that

$$y^k e^{-y} \leqslant (k+1)^k e^{-k} e^{-\frac{y}{k+1}}, \quad k \geqslant 1, y > 0.$$

From (13), we obtain

$$\begin{aligned}|x^k \omega^{(k)}(x)| &= x^k \int_0^{\infty} e^{-xt} t^k \mu(dt) = \int_0^{\infty} e^{-xt}(xt)^k \mu(dt) \\ &\leqslant \int_0^{\infty} (k+1)^k e^{-k} e^{-\frac{xt}{k+1}} \mu(dt) \\ &= (k+1)^k e^{-k}\left[\omega\left(\frac{x}{k+1}\right) - \mu(\{0\})\right].\end{aligned}$$

Note that $\lim_{x\to\infty} \omega(x) = \mu(\{0\})$. So, as x approaches infinity, we see that $\lim_{x\to\infty} x^k \omega^{(k)}(x) = 0$, for all $n \geqslant 1$. Moreover, dividing both sides of the above inequality by ω, we obtain

$$\frac{|x^k \omega^{(k)}(x)|}{\omega(x)} \leqslant (k+1)^k e^{-k} \frac{\left[\omega\left(\frac{x}{k+1}\right) - \mu(\{0\})\right]}{\omega(x)}.$$

Using the result stated in [6, Proposition D, p. 145], we see that the quotient $\omega(\frac{x}{k+1})/\omega(x)$ is actually bounded. This, in turn, shows that $|x^k \omega^{(k)}(x)|/\omega(x)$ is also bounded as desired. This proves the lemma. $\square$

Some of useful properties of "big $O's$" are the following: $f = O(f)$, $O(f) \cdot O(g) = O(f \cdot g)$, and for $n > 0$, $O(x^{-n}) = O(1)$. Using these identities and by (A2), we have

$$\omega^{(k)}(x)/\omega(x) = x^{-k} \cdot O(1) = O(x^{-k}) \cdot O(1) = O(x^{-k}) = O(1).$$

It can also be seen easily from the previous lemma that $0 \leqslant \lim_{x\to\infty} \omega^{(k)}(x)/\omega(x) \leqslant \lim_{x\to\infty} Cx^{-k} = 0$. Finally, by Squeeze Theorem, we get $\lim_{x\to\infty} \omega^{(k)}(x)/\omega(x) = 0$, or equivalently, $\omega^{(k)}(x)/\omega(x) = o(1) = O(1)$.

3. Main result

A new class of admissible perturbations of special expressions in the weighted space has been identified and it is of the form

$$\mathbf{M_2} = \sum_{l=0}^{n-1} a_l(t) D_t^l \tag{14}$$

satisfying the following conditions:

(AP1) For every $l = 0, 1, \ldots, n-1$, $a_l(t)$ is completely monotonic; and

(AP2) There exist auxiliary functions $b_l(t)$ such that

$$\sup_{l \leqslant i \leqslant n-1} \left| \frac{a_i(t)}{b_l(t)} \right| < \infty,$$

where $0 < b_l(t) \in C^\infty([1,\infty))$ for all l and $b_l(t) = o(t^{\gamma(l+1)})$ and $b_l(t) = o(t^{\gamma(l)})$ as $t \longrightarrow \infty$.

Now we present the main result of our study.

Theorem 3.1. *Let* $\mathbf{M}_0$ *be a special expression of the form* (1) *with* $\alpha_1 < \rho_1$. *Let* $\mathbf{M_2}$, *which is of the form* (14) *that satisfies* (AP1) *and* (AP2), *be an admissible perturbation of* $\mathbf{M}_0$ *in* $\mathcal{L}^2_\omega([1,\infty))$. *If* $\omega > 0$ *is a c.m. function, then* $\omega^{\frac{1}{2}} \mathbf{M_2} \omega^{-\frac{1}{2}}$ *is an admissible perturbation of* $\mathbf{M}_0$ *in* $\mathcal{L}^2([1,\infty))$. *In addition,* $\omega^{\frac{1}{2}} \mathbf{M_2} \omega^{-\frac{1}{2}}$ *preserves the essential spectrum and nullity of* $\mathbf{M}_0$ *in* $\mathcal{L}^2([1,\infty))$.

Proof. We follow [4] for the proof. Consider the admissible perturbation $\mathbf{M_2}$ of the form (14) of $\mathbf{M}_0$ presented above. Let $\mathbf{I} = [1, \infty)$ and for $l = 0, 1, \dots, n - 1$, $a_l(t)$ is completely monotonic. Also, assume that $\omega > 0$ is a c.m. function in $\mathbf{I}$.

Now, note that $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ is an operator that can be transformed, via Leibniz's rule, as follows (cf. [4]):

$$\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}} = \omega^{\frac{1}{2}} \sum_{l=0}^{n-1} a_l(\omega^{-\frac{1}{2}}y)^{(l)} = \sum_{l=0}^{n-1} (\mathbf{R}_l(t))\, y^{(l)},$$

where

$$\mathbf{R}_l(t) = \sum_{i=l}^{n-1} \binom{i}{l} a_i \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-l)}.$$

To show that $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2(\mathbf{I})$, we need to show that Definition 1.3 is satisfied. Equivalently, we need to present that there exists a $B > 0$ such that

$$\sup_{[x,x+1]\subset \mathbf{I}} \int_x^{x+1} \left| \frac{\mathbf{R}_l(t)}{\mathbf{S}_l(t)} \right|^2 dt < B,$$

where $\mathbf{R}_l(t) \in C^1(\mathbf{I})$ for $l = 0, 1, \dots, n-1$ and $0 < \mathbf{S}_l(t)$ is an auxiliary function in $C^\infty(\mathbf{I})$ satisfying

$$\mathbf{S}_l(t) = o(t^{\gamma(l+1)}) \quad \text{and} \quad \mathbf{S}_l(t) = o(t^{\gamma(l)})$$

as $t \longrightarrow \infty$.

Since $a_l(t)$ is c.m. for $l = 0, 1, \dots, n - 1$, $a_l(t) \in C^\infty((0, \infty))$. In particular, $a_l(t) \in C^l(\mathbf{I})$ for $l = 0, 1, \dots, n - 1$. Note that ω is c.m. function. So, by using Lemma 2.5, we can claim that $\omega \in C^\infty(\mathbf{I})$. Also, by using Lemma 2.4, one can show that $\omega^{\frac{1}{2}}$ and $\omega^{-\frac{1}{2}}$ are elements of $C^\infty(\mathbf{I})$. It follows that $\omega^{\frac{1}{2}}, \omega^{-\frac{1}{2}} \in C^l(\mathbf{I})$ for $l = 0, 1, \dots, n - 1$. Therefore, $\mathbf{R}^l(t) \in C^l(\mathbf{I})$ for $l = 0, 1, \dots, n - 1$.

Now let $\mathbf{S}_l(t) = \omega^{\frac{1}{2}} b_l(t) \omega^{-\frac{1}{2}} = b_l(t)$. Since $0 < b_l(t) \in C^\infty(\mathbf{I})$, so is $\mathbf{S}_l(t)$. Clearly, $\mathbf{S}_l(t) = o(t^{\gamma(l+1)})$ and $\mathbf{S}_l(t) = o(t^{\gamma(l)})$ because $b_l(t) = o(t^{\gamma(l+1)})$ and $b_l(t) = o(t^{\gamma(l)})$. Thus, there exists an auxiliary function $\mathbf{S}_l(t)$.

We now proceed on evaluating the integral $\int_x^{x+1} \left| \frac{\mathbf{R}_l(t)}{\mathbf{S}_l(t)} \right|^2 dt$, where $1 \leqslant x < \infty$, as follows:

$$\begin{aligned} \int_x^{x+1} \left| \frac{\mathbf{R}_l(t)}{\mathbf{S}_l(t)} \right|^2 dt &= \int_x^{x+1} \left| \frac{\sum_{i=l}^{n-1} \binom{i}{l} a_i(t) \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-l)}}{b_l(t)} \right|^2 dt \\ &\leqslant J^2 \int_x^{x+1} \left| \frac{\sum_{i=l}^{n-1} a_i(t) \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-l)}}{b_l(t)} \right|^2 dt, \end{aligned}$$

$$
\begin{aligned}
&\left(\text{where } J = \sup_{l \leqslant i \leqslant n-1} \left\{ \binom{i}{l} \right\}\right) \\
\leqslant\ & J^2 \int_x^{x+1} \left(\sum_{i=l}^{n-1} \left| \frac{a_i(t)}{b_l(t)} \frac{(\omega^{-\frac{1}{2}})^{(i-l)}}{\omega^{-\frac{1}{2}}} \right| \right)^2 dt, \\
&\text{(by } \Delta \text{ inequality)} \\
\leqslant\ & J^2 \int_x^{x+1} \left(\sum_{i=l}^{n-1} \left| \frac{a(t)}{b_l(t)} \frac{(\omega^{-\frac{1}{2}})^{(i-l)}}{\omega^{-\frac{1}{2}}} \right| \right)^2 dt, \\
&\left(\text{where } a(t) = \sup_{l \leqslant i \leqslant n-1} \{a_i(t)\}\right) \\
\leqslant\ & J^2 \int_x^{x+1} \left(O(1) \sum_{i=l}^{n-1} 1 \right)^2 \left| \frac{a(t)}{b_l(t)} \right|^2 dt, \\
&\text{(by Lemma 2.4 and 2.5)} \\
\leqslant\ & J^2 K^2 (n-l)^2 \int_x^{x+1} \left| \frac{a(t)}{b_l(t)} \right|^2 dt, \quad \text{for some constant } K, \\
\leqslant\ & J^2 K^2 (n-l)^2 \int_x^{x+1} M^2 dt, \quad \text{for some constant } M \\
=\ & J^2 K^2 (n-l)^2 M^2.
\end{aligned}
$$

Letting $B = J^2 K^2 (n-l)^2 M^2$, we see that

$$
\int_x^{x+1} \left| \frac{\mathbf{R}_l(t)}{\mathbf{S}_l(t)} \right|^2 dt \leqslant B.
$$

Consequently, taking the supremum of both sides over the interval $[x, x+1] \subset [1, \infty)$, we have

$$
\sup_{[x,x+1] \subset \mathbf{I}} \int_x^{x+1} \left| \frac{\mathbf{R}_l(t)}{\mathbf{S}_l(t)} \right|^2 dt < B,
$$

where $B = J^2 K^2 (n-l)^2 M^2$.

For the invariance of nullity we need to show that equation (11) is satisfed. Let $J = \sup_{l \leqslant j \leqslant n-1} \{\binom{i}{j}\}, a(t) = \sup_{l \leqslant j \leqslant n-1} \{a_j(t)\}$, and M be some nonnegative constant. Hence, for $l = 0, \ldots, n-1$ and $j = l, \ldots, n-1$, we have the following

$$
\left| \frac{\mathbf{R}_j^{(j-l)}(t)}{b_l(t)} \right|^2 = \left| \frac{1}{b_l(t)} \sum_{i=j}^{n-1} \binom{i}{j} \left(a_i^{(j-l)} \omega^{\frac{1}{2}} (\omega^{-\frac{1}{2}})^{(i-j)} + a_i (\omega^{\frac{1}{2}})^{(j-l)} (\omega^{-\frac{1}{2}})^{(i-j)} \right. \right.
$$

$$
\begin{aligned}
&+\ a_i\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-l)}\Big)\Big|^2\\
\leqslant\ & J^2\left|\frac{a(t)}{b_l(t)}\sum_{i=j}^{n-1}\Big(\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-j)}+(\omega^{\frac{1}{2}})^{(j-l)}(\omega^{-\frac{1}{2}})^{(i-j)}\right.\\
&+\ \omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-l)}\Big)\Big|^2\\
\leqslant\ & 9J^2\left|\frac{a(t)}{b_l(t)}\sum_{i=j}^{n-1}\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-j)}\right|^2\\
\leqslant\ & 9J^2\left|\frac{a(t)}{b_l(t)}\right|^2\left(\sum_{i=j}^{n-1}\left|\frac{(\omega^{-\frac{1}{2}})^{(i-j)}}{\omega^{-\frac{1}{2}}}\right|\right)^2\\
=\ & 9J^2\left|\frac{a(t)}{b_l(t)}\right|^2\left(O(1)\sum_{i=j}^{n-1}1\right)^2\\
=\ & 9J^2K^2(n-l)^2\left|\frac{a(t)}{b_l(t)}\right|^2\\
\leqslant\ & 9J^2K^2(n-l)^2M^2.
\end{aligned}
$$

Hence,

$$
\left|\frac{\mathbf{R}_j^{(j-l)}(t)}{b_l(t)}\right|^2\leqslant 9J^2K^2(n-l)^2M^2. \tag{15}
$$

Let $\tilde{C}=9J^2K^2(n-l)^2M^2$. Integrating equation (15) over the interval $[x,x+1]$ and using (AP2) we obtain

$$
\int_x^{x+1}\left|\frac{\mathbf{R}_j^{(j-l)}(t)}{b_l(t)}\right|dt\leqslant\int_x^{x+1}\tilde{C}dt=\tilde{C}.
$$

Now, taking the supremum of both sides over the interval $[x,x+1]\subset[1,\infty)$, we have

$$
\sup_{[x,x+1]\subset\mathbf{I}}\int_x^{x+1}\left|\frac{\mathbf{R}_j^{(j-l)}(t)}{b_l(t)}\right|dt\ \leqslant\ \tilde{C}.
$$

Thus, we have shown that

$$
\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}=\sum_{l=0}^{n-1}\left(\mathbf{R}_l(t)\right)y^{(l)},
$$

which is obviously of the form (1.3), is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2([1,\infty))$. Moreover, as what we have shown previously, $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ satisfies

(9) and (10). In addition, it also satisfies equation (11). We now conclude that, using Theorem (1.4), the essential spectrum and the nullity of $\mathbf{M}_0$ are preserved under this kind of admissible perturbation in $\mathcal{L}^2$-space, In short, we have proven the following claim:

$$\sigma_e(\mathbf{M}_0 + \omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}) = \sigma_e(\mathbf{M}_0)$$

and

$$\mathrm{nul}(\mathbf{M}_0 + \mathbf{M_2} - x) = \mathrm{nul}(\mathbf{M}_0 - x)$$

for every $x \in \mathbb{C} \setminus \sigma_e(\mathbf{M}_0)$. □

In the next section we provide an example of the admissible perturbation presented above. We point out that our example does not satisfy condition (AP_1) of [4]. This means that the class of admissible perturbation being studied in the present paper is indeed different from the one presented in [4].

4. Example

Consider the following special expression

$$\mathbf{M}_0 y = y + \frac{1}{2}t^{\frac{1}{2}}y' + t^{\frac{3}{4}}y'' \tag{16}$$

and the weight function $\omega(t) = \left(1 + x^{-\frac{1}{2}}\right)^2$ which is a c.m. function.

We will show that the differential expression $\mathbf{M_2}$ defined by

$$\mathbf{M_2}y = e^{-2t}y + \ln(1 + t^{-1})y',$$

is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2_\omega(\mathbf{I})$, i.e., it should satisfy (AP1) and (AP2). We first show that $a_0(t) = e^{-2t}$ and $a_1(t) = \ln(1 + t^{-1})$ are c.m. functions. Note that

$$a_0^{(k)}(t) = (-1)^k 2^k e^{-2t}, \quad \forall k = 0, 1, 2, \ldots$$

and

$$a_1^{(k)}(t) = (-1)^k \frac{(k-1)!}{t^k(t+1)^k} \sum_{j=0}^{k-1} \binom{n}{j} t^j, \quad \forall k = 1, 2, \ldots.$$

It follows that

$$0 \leqslant (-1)^k a_0^{(k)}(t) = 2^k e^{-2t} < \infty, \quad \forall k \geqslant 0, t \in \mathbf{I},$$

and $0 \leqslant a_1(t) = \ln(1 + t^{-1}) < \infty$, and for $k \geqslant 1$, we have

$$0 \leqslant (-1)^k a_1^{(k)}(t) = \frac{(k-1)!}{t^k(t+1)^k} \sum_{j=0}^{k-1} \binom{n}{j} t^j < \infty, \quad \forall t \in \mathbf{I}.$$

Thus, $a_0(t)$ and $a_1(t)$ are c.m. functions and condition (AP1) is satisfied.

Now, $b_l(t)$ must be of the following forms (for $l = 0, 1$):

$$b_0(y) = o(t^{\gamma(1)}) = o(t^{\frac{1}{2}}) \quad \text{and} \quad b_0(t) = o(t^{\gamma(0)}) = o(1), \tag{17}$$

$$b_1(y) = o(t^{\gamma(2)}) = o(t^{\frac{3}{4}}) \quad \text{and} \quad b_1(t) = o(t^{\gamma(1)}) = o(t^{\frac{1}{2}}), \tag{18}$$

and, in addition,

$$\left| \frac{\sup\{e^{-2t}, \ln(1+t^{-1})\}}{b_l(t)} \right| = \left| \frac{\ln(1+t^{-1})}{b_l(t)} \right| = 1 < \infty, \quad \forall l = 0, 1.$$

Let $b_0(t) = b_1(t) = t^{-1}$. Clearly, (17) and (18) are satisfied. Also, we have, by L'Hôpital's rule, that

$$\lim_{t\to\infty} \frac{\ln(1+t^{-1})}{t^{-1}} = \lim_{t\to\infty} \frac{\frac{1}{-t^{-2}(1+t^{-1})}}{-t^{-2}} = \lim_{t\to\infty} \frac{1}{1+t^{-1}} = 1 < \infty.$$

Hence, condition (AP2) is satisfied. So, we have shown that $\mathbf{M_2}y$ is an admissible perturbation of $\mathbf{M}_0 y$ in $\mathcal{L}^2_\omega(\mathbf{I})$.

Now we proceed on showing that $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}y$ is an admissible perturbation of $\mathbf{M}_0 y$ in $\mathcal{L}^2(\mathbf{I})$. Considering $\mathbf{M_2}y = e^{-2t}y + \ln(1+t^{-1})y'$, we have the following:

$$\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}y = \sum_{l=0}^{n-1} \left(\sum_{i=l}^{1} \binom{i}{l} a_i(t)\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-l)} \right) y^{(l)}.$$

Let $\mathbf{R}_l(t) = \sum_{i=l}^{1} \binom{i}{l} a_i(t)\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-l)}$, where $l = 0, 1$. So,

$$\mathbf{R}_0(t) = \sum_{i=0}^{1} \binom{i}{0} a_i(t)\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i)} = e^{-2t} + \frac{\ln(1+t^{-1})}{2t + 2t^{\frac{3}{2}}};$$

$$\mathbf{R}_1(t) = \sum_{i=1}^{1} \binom{i}{1} a_i(t)\omega^{\frac{1}{2}}(\omega^{-\frac{1}{2}})^{(i-1)} = \ln(1+t^{-1}).$$

It can be verified that $\mathbf{R}_0(t)$ and $\mathbf{R}_1(t)$ are c.m. functions. Hence, $\mathbf{R}_l(t)$ satisfies (AP1) for $l = 0, 1$. We can let $\mathbf{S}_l(t) = b_l(t) = t^{-1}$ for $l = 0, 1$ so that

$$\mathbf{S}_0(y) = o(t^{\frac{1}{2}}) \text{ and } \mathbf{S}_0(t) = o(1), \quad \text{and} \quad \mathbf{S}_1(y) = o(t^{\frac{3}{4}}) \text{ and } \mathbf{S}_1(t) = o(t^{\frac{1}{2}}),$$

and

$$\left| \frac{\sup\left\{ e^{-2t} + \frac{\ln(1+t^{-1})}{2t+2t^{\frac{3}{2}}}, \ln(1+t^{-1}) \right\}}{\mathbf{S}_l(t)} \right| = \left| \frac{\ln(1+t^{-1})}{t^{-1}} \right| = 1 < \infty, \quad \forall l = 0, 1.$$

We conclude that $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ is an admissible perturbation of $\mathbf{M}_0$ in $\mathcal{L}^2(\mathbf{I})$. In addition, $\omega^{\frac{1}{2}}\mathbf{M_2}\omega^{-\frac{1}{2}}$ preserves the essential spectrum and nullity of $\mathbf{M}_0$ in $\mathcal{L}^2([1, \infty))$.

Remark 4.1. We point out that the functions $a_0(t) = e^{-2t}$ and $a_1(t) = \ln(1 + t^{-1})$ do not satisfy (AP_1), since the statement

$$\left|\frac{a_1(t)}{t^{1-0}a_0(t)}\right| = \left|\frac{\ln(1+t^{-1})}{te^{-2t}}\right| \longrightarrow \infty \quad \text{as} \quad t \longrightarrow \infty$$

implies that the supremum

$$\sup_{t\in\mathbf{I}}\left|\frac{a_1(t)}{t^{1-0}a_0(t)}\right| = \sup_{t\in\mathbf{I}}\left|\frac{\ln(1+t^{-1})}{te^{-2t}}\right|$$

does not exist.

Acknowledgment

This work was funded by the UP System Enhanced Creative Work and Research Grant (ECWRG-2015-1-006). The first author would like to thank UP Baguio for the research load credit given in writing the manuscript. Both authors would like to thank the referees for their valuable comments.

References

[1] J.C. Agapito, *On relatively compact perturbations of special expressions in* $\mathcal{L}^2_\omega(\mathrm{I})$, Ph. D. Thesis, Department of Mathematics, University of the Philippines, 1993.

[2] J.C. Agapito, J.B. Bacani and M.P. Roque, *On invariance of nullities of special expressions under admissible perturbations in weighted space*, Matimyas Matematika, 29 (2006), 1–8.

[3] J.B. Bacani, *On admissible perturbations of special expressions in weighted* L_2*-space*, Master's Thesis, Department of Mathematics and Computer Science, University of the Philippines, 2004.

[4] J.B. Bacani, *Another class of admissible perturbations of special expressions*, Int. J. Math. Anal., 8 (1) (2014), 1–8.

[5] E. Balslev and T.W. Gamelin, *The essential spectrum of a class of ordinary differential operators*, Pacific J. Math., 14(3) (1964), 755–776.

[6] C. O'Cinneide, *A property of completely monotonic functions*, J. Austral. Math. Soc. (Series A), 42 (1987), 143–146.

[7] J.A. Collera, *Admissible perturbations of differential expressions with exponentially decaying coefficients preserving the nullities*, Int. J. Math. Anal., 7 (57) (2013), 2803–2810.

[8] K.S. Miller and S.G. Samko, *Completely monotonic functions*, Integral Transform. Spec. Funct., 12 (4) (2001), 389–402.

[9] M.R. Mumpar-Victoria, *Perturbations of a class of ordinary differential expressions preserving the essential spectrum and the nullities*, Science Diliman, 11(1) (1999), 25–33.

[10] M.P. Roque, *Spectral properties of differential expressions with exponentially and logarithmically growing coefficients,* in Functional Analysis and Global Analysis: Proceedings of the Conference Held in Manila, Philippines, October 20-26, 1996, Springer-Verlag Singapore, 1997.

[11] B. Schultze, *Spectral properties of not necessarily self-adjoint linear operators*, Advances in Mathematics, 83 (1990), 75–95.

[12] D.V. Widder, *The Laplace Transforms*, Princeton University Press, 1941.

Accepted: 2.12.2017