DERIVABLE MAPPINGS AND COMMUTATIVITY OF ASSOCIATIVE RINGS

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Abstract. Let R be a ring with center $Z(R)$. A mapping $F : R \to R$ (not necessarily additive) is called a multiplicative (generalized)-derivation of R if it is uniquely determined by a mapping $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for each $x, y \in R$. In the present paper, we investigate the commutativity of a semiprime (prime) ring via studying a number polynomial constraints involving multiplicative (generalized)-derivations. Moreover, some annihilator conditions are also examined.

Keywords: Prime ring, Semiprime ring, Derivation, Generalized derivation, Multiplicative (generalized)-derivation.

1. Introduction

All through this paper R be an associative ring with center $Z(R)$. A ring R is said to be a prime ring if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and semiprime if $aRa = (0)$ implies that $a = 0$. Obviously, every prime ring is semiprime. For any nonempty subset S of R the right annihilator $r_R(S)$ of S in R is the set of all $r \in R$ such that $Sr = (0)$. Accordingly, the left annihilator $l_R(S)$ is the set of all $r \in R$ such that $rS = (0)$. The intersection of right and left annihilators of S in R *i.e.*

$$Ann_R(S) = \{r \in R : sr = 0 \text{ and } rs = 0 \text{ for all } s \in S\}$$

is called an annihilator of S in R . Recall that, for any $x, y \in R$ the commutator and anti-commutator are denoted by the symbols $[x, y] = xy - yx$ and $x \circ y = xy + yx$ respectively. We shall frequently use the basic commutator identities:

$$[xy, z] = x[y, z] + [x, z]y, [x, yz] = y[x, z] + [x, y]z,$$

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for all $x, y, z \in R$. For any nonempty subset Q of R, a mapping $f : R \to R$ is said to be centralizing on Q if $[f(x), x] \in Z(R)$ and commuting if $[f(x), x] = 0$ for all $x \in Q$. A derivation (or left multiplier) of R is a map such that $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$ (or $d(xy) = d(x)y$) for all $x, y \in R$. The notion of derivation was extended to generalized derivation by Brešar [10]. A generalized derivation of R is an additive map uniquely determined by a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

Inspired by Martindale's [22] remarkable paper on the additivity of multiplicative bijective mappings, Daif [12] introduced multiplicative derivation, which is a map $d : R \to R$ satisfying Leibnitz rule and not necessarily additive on R. The complete description of such mappings was explained by Goldmann and Šemrl [18]. Daif and Tammam-El-Sayiad [14] extended this notion to multiplicative generalized derivation by dropping the additivity assumption of generalized derivation F. Recently, Dhara and Ali [16] made a slight generalization in this definition of multiplicative generalized derivation by relaxing the conditions on d and call it multiplicative (generalized)-derivation, which is a map $F : R \to R$ (not necessarily additive) along with a map $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ where $x, y \in R$. Observe that every multiplicative derivation is a multiplicative (generalized)-derivation, so multiplicative (generalized)-derivation covers both the concepts of multiplicative derivation (if $F = d$) and multiplicative left multiplier (if $d = 0$). In this way, multiplicative (generalized)-derivation is a more satisfactory generalization of multiplicative derivation.

2. Some preliminary results

Throughout this paper, we shall use the following well known lemmas to prove our results:

Lemma 1 (Lemma 2, [13]). *If R is a prime ring containing nonzero central ideal, then R is commutative.*

Lemma 2 (Corollary 2, [20]). *If R is a semiprime ring and I is an ideal of R, then $I \cap Ann_R(I) = (0)$.*

Lemma 3 (Lemma 2.3, [17]). *If R is a prime ring, I a nonzero ideal and d is derivation of R. If for some $0 \neq a \in R$, $[ad(x), x] = 0$ for all $x \in I$, then $d = 0$ or R is commutative.*

Lemma 4 (Corollary, [20]). *Let R be a semiprime ring and let I be a nonzero right ideal of R. If I is commutative as a ring, then $I \subseteq Z(R)$.*

Lemma 5 (Theorem, [21]). *Let g be a polynomial in n noncommuting variables $u_1, u_2..., u_n$ with relatively prime integer coefficients. Then the following are equivalent:*

(*i*) *Every ring satisfying the polynomial identity* $g = 0$ *has nil commutator ideal.*

(*ii*) *Every semiprime ring satisfying* $g = 0$ *is commutative.*

(*iii*) *For every prime* p *the ring of* 2×2 *matrices over* Z_p *fails to satisfy* $g = 0$.

Throughout this paper, R will denote a semiprime ring with nonzero ideal I, unless otherwise stated.

3. Main results

3.1 On central value conditions

During the last seven decades, there has been a large amount of results concerning the conditions that force a ring to be commutative. In this direction, Posner [23] proved a classical result: *Every prime ring admitting a nonzero centralizing derivation is commutative.* This theorem has been generalized in many ways. Towards the commutativity of prime rings with derivations Ashraf et al. [5] proved: *Let* R *be a prime ring and* I *be a nonzero ideal of* R*. Suppose that* d *is a nonzero derivation of* R *such that* $d(xy) \pm xy \in Z(R)$ *where* $x, y \in I$*, then* R *is commutative.* In [3], Ashraf et al. extend these results for generalized derivations and obtained the following theorem: *Let* R *be a prime ring and* I *a nonzero ideal of* R*. Suppose* F *is a generalized derivation associated with a derivation* d *on* R*. If one of the following:*

(*i*) $F(xy) \pm xy \in Z(R)$,

(*ii*) $F(xy) \pm yx \in Z(R)$,

(*iii*) $F(x)F(y) \pm xy \in Z(R)$ *holds on* I*, then* R *is commutative.*

After that, Atteya [6] studied these situations on semiprime rings and obtained the following results: *Let* R *be a semiprime ring and* I *be a nonzero ideal of* R*. If* R *admits a generalized derivation* F *associated with a derivation* d *such that any one of the following:*

(*i*) $F(xy) \pm xy \in Z(R)$,

(*ii*) $F(xy) \pm yx \in Z(R)$,

(*iii*) $F(x)F(y) \pm xy \in Z(R)$ *holds on* I*, then* R *contains a nonzero central ideal.*

It is a fact of interest to generalize these results to multiplicative (generalized)-derivations. In this line of investigation Dhara and Ali [16] studied the following identities: (i) $F(xy) \pm xy = 0$, (ii) $F(xy) \pm yx = 0$, (iii) $F(xy) \pm xy \in Z(R)$, (iv) $F(x)F(y) \pm yx \in Z(R)$, where x, y varies over some suitable subset of semiprime ring R. In this section, we study central valued conditions involving multiplicative (generalized)-derivations and consequently give a generalized version of some known results.

Theorem 1. *Let $F : R \to R$ be a multiplicative (generalized)-derivation of R together with a mapping $d : R \to R$. If ϕ is a mapping of R such that $F(xy) + xy \pm [\phi(x), y] \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Furthermore, if ϕ is an automorphism of R, then $I \subseteq Z(R)$.

Proof. For each $x, y \in I$, we consider

$$F(xy) + xy + [\phi(x), y] \in Z(R). \tag{3.1}$$

Replace y by yz in (3.1), where $z \in I$ and we get $(F(xy) + xy + [\phi(x), y])z + xyd(z) + y[\phi(x), z] \in Z(R)$. On commuting with z and using our hypothesis, we find

$$[xyd(z), z] + [y[\phi(x), z], z] = 0. \tag{3.2}$$

Again replace y by zy in (3.2), we have

$$[xzyd(z), z] + z[y[\phi(x), z], z] = 0. \tag{3.3}$$

Left multiply (3.2) by z and subtract from (3.3) in order to obtain

$$[[x, z]yd(z), z] = 0. \tag{3.4}$$

Since I is an ideal of R so we substitute $xd(z)$ in place of x in (3.4) and get

$$[x[d(z), z]yd(z), z] + [[x, z]d(z)yd(z), z] = 0. \tag{3.5}$$

Now, substitute $d(z)y$ instead of y in (3.4) and subtract from (3.5) to obtain

$$[x[d(z), z]yd(z), z] = 0. \tag{3.6}$$

Putting $x = d(z)x$ in (3.6) and we obtain $d(z)[x[d(z), z]yd(z), z]+[d(z), z]x[d(z), z]yd(z) = 0$. Relation (3.6) reduces it to $[d(z), z]x[d(z), z]yd(z) = 0$. That is, $(I[d(z), z])^3 = (0)$. But R has no nonzero nilpotent ideal, hence $I[d(z), z] = (0)$. Clearly, $[d(z), z] \in I$ as well as $[d(z), z] \in Ann_R(I)$. That means $[d(z), z] \in I \cap Ann_R(I)$. Therefore, Lemma 2 implies that $[d(z), z] = 0$ for each $z \in I$. This process also shows that every nonzero ideal of a semiprime ring is a semiprime ring itself.

Next, we assume that ϕ is an automorphism of R. Replacing y by yz in (3.2), we get

$$[xyzd(z), z] + [yz[\phi(x), z], z] = 0. \tag{3.7}$$

Multiplying (3.2) from right by z, we get

$$[xyd(z)z, z] + [y[\phi(x), z]z, z] = 0. \tag{3.8}$$

Subtracting (3.7) from (3.8) and we find $[xy[d(z), z], z] + [y[[\phi(x), z], z], z] = 0$. Since $[d(z), z] = 0$, we left with the expression

$$[y[[\phi(x), z], z], z] = 0. \tag{3.9}$$

Putting $y = ty$ in (3.9), where $t \in I$, we have $t[y[[\phi(x), z], z], z] + [t, z]y[[\phi(x), z], z] = 0$ for each $x, y, z, t \in I$. Use of Eq. (3.9) gives

$$[t, z]y[[\phi(x), z], z] = 0. \tag{3.10}$$

Replace t by $t[\phi(x), z]$ in (3.10) and we obtain

$$t[[\phi(x), z], z]y[[\phi(x), z], z] + [t, z][\phi(x), z]y[[\phi(x), z], z] = 0. \tag{3.11}$$

Replace y by $[\phi(x), z]y$ in (3.10) and combine with (3.11) in order to find $t[[\phi(x), z], z]y[[\phi(x), z], z] = 0$. In particular, we have $y[[\phi(x), z], z]Ry[[\phi(x), z], z] = (0)$. Hence, we obtain $y[[\phi(x), z], z] = 0$. That is, $I[[\phi(x), z], z] = (0)$. Thus, semiprimeness of I assures that, for each $x, z \in I$

$$[[\phi(x), z], z] = 0. \tag{3.12}$$

Linearizing (12) w.r.t.z, we get

$$[[\phi(x), t], z] + [[\phi(x), z], t] = 0. \tag{3.13}$$

Substituting zt in place of z in (3.13), where $t \in I$. We obtain

$$[[\phi(x), t], z]t + z[[\phi(x), t], t] + [[\phi(x), z], t]t + [z[\phi(x), t], t] = 0. \tag{3.14}$$

Using (3.12) and (3.13) in (3.14), it follows that

$$[z, t][\phi(x), t] = 0. \tag{3.15}$$

Replace x by $x\phi^{-1}(z)$ in (3.15), we obtain $[z, t]\phi(x)[z, t] = 0$ for any $x, z, t \in I$. Since ϕ is an automorphism of R so $\phi(I)$ is an ideal of R. Thus, we may infer that I is commutative as a ring. Hence, by Lemma 4 we infer that $I \subseteq Z(R)$.

On substituting $-\phi$ in place of ϕ in (3.1) and following the same argument with necessary variations, we get the same conclusions for the situation $F(xy) + xy - [\phi(x), y] \in Z(R)$. □

Theorem 2. *Let $F : R \to R$ be a multiplicative (generalized)-derivation of R together with a mapping $d : R \to R$. If ϕ is a mapping of R such that $F(xy) - xy \pm [\phi(x), y] \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Furthermore, if ϕ is an automorphism of R, then $I \subseteq Z(R)$.

Proof. On replacing F by $-F$ and d with $-d$ in Theorem 1, we can get the desired results. □

Theorem 3. *Let $F : R \to R$ be a multiplicative (generalized)-derivation of R together with a mapping $d : R \to R$. If ϕ is a mapping of R such that $F(x)F(y) + xy \pm [\phi(x), y] \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Furthermore, if ϕ is an automorphism of R, then $I \subseteq Z(R)$.

Proof. For any $x, y \in I$, we consider

$$F(x)F(y) + xy + [\phi(x), y] \in Z(R). \tag{3.16}$$

On replacing y by yz in (3.16), where $z \in I$, we find $(F(x)F(y)+xy+[\phi(x), y])z+F(x)yd(z)+y[\phi(x), z] \in Z(R)$. On commuting with z, our hypothesis forces that

$$[F(x)yd(z), z] + [y[\phi(x), z], z] = 0. \tag{3.17}$$

Put $y = zy$ in (3.17) and we get

$$[F(x)zyd(z), z] + z[y[\phi(x), z], z] = 0. \tag{3.18}$$

Left multiply (3.17) by z and subtract from (3.18), we have

$$[[F(x), z]yd(z), z] = 0. \tag{3.19}$$

Replace x by xz in (3.19) and we obtain

$$[[F(x), z]zyd(z), z] + [[xd(z), z]yd(z), z] = 0. \tag{3.20}$$

Replace y by zy in (3.19) and subtract from (3.20) to obtain $[[xd(z), z]yd(z), z] = 0$. That is, $[x[d(z), z]yd(z), z] + [[x, z]d(z)yd(z), z] = 0$. This expression is same as (3.5), so the similar arguments imply that $[d(z), z] = 0$ for each z in I. Now, we replace y by yz in (3.17) and get

$$[F(x)yzd(z), z] + [yz[\phi(x), z], z] = 0. \tag{3.21}$$

Right multiply (3.17) by z, we get

$$[F(x)yd(z)z, z] + [y[\phi(x), z]z, z] = 0. \tag{3.22}$$

Combining relations (3.21) and (3.22), we have $[F(x)y[d(z), z], z]+[y[[\phi(x), z], z], z] = 0$. Utilizing the fact $[d(z), z] = 0$, for all $z \in I$, we get $[y[[\phi(x), z], z], z] = 0$. This expression is same as equation (3.9), again the proof follows from Theorem 1.

On substituting $-\phi$ in place of ϕ in (3.16) and following the same technique with necessary variations, we get the same conclusions for the situation $F(x)F(y) + xy - [\phi(x), y] \in Z(R)$. □

Theorem 4. *Let $F : R \to R$ be a multiplicative (generalized)-derivation of R together with a mapping $d : R \to R$. If ϕ is a mapping of R such that $F(x)F(y) - xy \pm [\phi(x), y] \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Furthermore, if ϕ is an automorphism of R, then $I \subseteq Z(R)$.

Proof. On replacing F by $-F$ and d by $-d$ in Theorem 3, we can get the desired results. □

Now, we extend some theorems of Tiwari et al. [25].

Theorem 5. *Let $F, G : R \to R$ be multiplicative (generalized)-derivations of R together with mappings d, g respectively. If ϕ is a mapping of R such that $G(xy) + F(x)F(y) \pm [\phi(x), y] \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ and $[g(x), x] = 0$ for all $x \in I$.*

Furthermore, if R is prime and ϕ is an automorphism of R, then R is commutative.

Proof. For each $x, y \in I$, we consider

$$G(xy) + F(x)F(y) + [\phi(x), y] \in Z(R). \tag{3.23}$$

Putting $y = yz$ in (3.23), where $z \in I$, we get $(G(xy) + F(x)F(y) + [\phi(x), y])z + xyg(z) + F(x)yd(z) + y[\phi(x), z] \in Z(R)$. On commuting with z, our hypothesis yields

$$[xyg(z), z] + [F(x)yd(z), z] + [y[\phi(x), z], z] = 0. \tag{3.24}$$

Replace y by zy in (3.24) and we get

$$[xzyg(z), z] + [F(x)zyd(z), z] + z[y[\phi(x), z], z] = 0. \tag{3.25}$$

Left multiply (3.24) by z and subtract from (3.25), we have

$$[[x, z]yg(z), z] + [[F(x), z]yd(z), z] = 0. \tag{3.26}$$

On replacing x by xz in (3.26), we get

$$[[x, z]zyg(z), z] + [[F(x), z]zyd(z), z] + [[xd(z), z]yd(z), z] = 0. \tag{3.27}$$

Replace y by zy in (3.26) and subtract from (3.27) to find

$$[[xd(z), z]yd(z), z] = 0. \tag{3.28}$$

That is, $[x[d(z), z]yd(z), z] + [[x, z]d(z)yd(z), z] = 0$. This equation is same as (3.5), so similar arguments imply that $[d(z), z] = 0$ for each $z \in I$. Now, we substitute yz instead of y in (3.26) in order to obtain

$$[[x, z]yzg(z), z] + [[F(x), z]yzd(z), z] = 0. \tag{3.29}$$

Right multiply (3.26) by z, we get

$$[[x, z]yg(z)z, z] + [[F(x), z]yd(z)z, z] = 0. \tag{3.30}$$

Subtract (3.29) from (3.30), we obtain $[[x,z]y[g(z),z],z]+[[F(x),z]y[d(z),z],z]=0$. Utilizing the fact $[d(z),z]=0$, for all $z\in I$, we find

$$[[x,z]y[g(z),z],z]=0. \tag{3.31}$$

Put $x=xg(z)$ in (3.31), we get

$$[x[g(z),z]y[g(z),z],z]+[[x,z]g(z)y[g(z),z],z]=0. \tag{3.32}$$

Put $y=g(z)y$ in (3.31) and subtract from (3.32) in order to get

$$[x[g(z),z]y[g(z),z],z]=0. \tag{3.33}$$

Substituting $g(z)x$ for x in (3.33) and we get $g(z)[x[g(z),z]y[g(z),z],z]+[g(z),z]x[g(z),z]y[g(z),z]=0$. Eq. (3.33) reduces it to $[g(z),z]x[g(z),z]y[g(z),z]=0$. It implies that $(I[g(z),z])^3=(0)$. Since R has no nonzero ideal, we have $I[g(z),z]=(0)$. Semiprimeness of I yields that for each $z\in I$, $[g(z),z]=0$.

Next, let us assume that R is a prime ring and ϕ is an automorphism of R. Replace y by yz in (3.24) and we find

$$[xyzg(z),z]+[F(x)yzd(z),z]+[yz[\phi(x),z],z]=0. \tag{3.34}$$

Right multiply (3.24) by z in order to get

$$[xyg(z)z,z]+[F(x)yd(z)z,z]+[y[\phi(x),z]z,z]=0. \tag{3.35}$$

Subtracting (3.34) from (3.35) and we find

$$[xy[g(z),z],z]+[F(x)y[d(z),z],z]+[y[[\phi(x),z],z],z]=0. \tag{3.36}$$

Since d and g are commuting on I, Eq. (3.36) reduces to $[y[[\phi(x),z],z],z]=0$. This is same as equation (3.9), again from Theorem 1, we get $I\subseteq Z(R)$. By Lemma 1, R is commutative.

On substituting $-\phi$ in place of ϕ in (3.23) and following the same argument with necessary variations, we get the same conclusions for the identity $G(xy)+F(x)F(y)-[\phi(x),y]\in Z(R)$. □

Corollary 1 (Theorem 1, [25]). *Let $F,G:R\to R$ be multiplicative (generalized)-derivations of R together with mappings d,g respectively. If ϕ is a mapping of R such that $G(xy)+F(x)F(y)\pm[\phi(x),y]=0$ for all $x,y\in I$, then $[d(x),x]=0$ and $[g(x),x]=0$ for all $x\in I$.*

Furthermore, if R is prime and ϕ is an automorphism of R, then R is commutative.

Theorem 6. *Let $F,G:R\to R$ be multiplicative (generalized)-derivations of R together with mappings d,g respectively. If ϕ is a mapping of R such that $G(xy)-F(x)F(y)\pm[\phi(x),y]\in Z(R)$ for all $x,y\in I$, then $[d(x),x]=0$ and $[g(x),x]=0$ for all $x\in I$.*

Furthermore, if R is prime and ϕ is an automorphism of R, then R is commutative.

Proof. On replacing G by $-G$ and g by $-g$ in Theorem 5, we can get the desired results. □

Corollary 2 (Theorem 2, [25]). *Let $F, G : R \to R$ be multiplicative (generalized)-derivations of R together with mappings d, g respectively. If ϕ is a mapping of R such that $G(xy) - F(x)F(y) \pm [\phi(x), y] = 0$ for all $x, y \in I$, then $[d(x), x] = 0$ and $[g(x), x] = 0$ for all $x \in I$.*

Furthermore, if R is prime and ϕ is an automorphism of R, then R is commutative.

Corollary 3. *Let $F, G : R \to R$ be multiplicative (generalized)-derivations of R together with mappings d, g respectively. If any of the following condition*

(*i*) $G(xy) \pm F(x)F(y) \pm [x, y] \in Z(R)$

(*ii*) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$

(*iii*) $G(xy) \pm F(x)F(y) \in Z(R)$

(*iv*) $G(xy) \pm F(x)F(y) \pm xy \in Z(R)$

holds on R. Then R is commutative.

Proof. (i) Firstly, we consider $G(xy) + F(x)F(y) \pm [x, y] \in Z(R)$ for each $x, y \in R$. In particular, for $\phi = i_d$ (identity map), Theorem 5 gives us that $[y[[x, z], z], z] = 0$ where $x, y, z \in R$. From Theorem 5 commutativity of R easily follows. We also can prove the same conclusion with an alternative way. Since for each $x, y, z \in I$, we have $[y[[z, x], z], z] = 0$, which is a polynomial identity in noncommuting three variables on R. If possible assume that, for some prime integer p the ring $M_2(GF(p))$ satisfies the polynomial identity $[y[[z, x], z], z] = 0$. But, if we choose $x = e_{11}$, $y = e_{12}$, and $z = e_{12} + e_{21}$, where e_{ij} denotes the 2×2 matrix with 1 in $(ij)^{th}$-entry and 0 elsewhere. With these choices we see that $[y[[z, x], z], z] = 2(e_{11} - e_{22})$, which is a contradiction. Hence by Lemma 5, R must be commutative.

Similarly, we can prove the commutativity of R for the constraint $G(xy) - F(x)F(y) \pm [x, y] \in Z(R)$.

The proof of (ii), (iii) and (iv) is straight forward from the fact that if G is a multiplicative (generalized)-derivation of R associated with a mapping g, then so is $G \pm i_d$, where i_d is the identity map of R. □

Immediately after Theorem 5 and Theorem 6 with Corollary 4.2 of [9], we give the following result:

Corollary 4. *Let $F, G : R \to R$ be multiplicative generalized derivations of R together with derivations d, g respectively. If for any map ϕ on R, $G(xy) \pm F(x)F(y) \pm [\phi(x), y] \in Z(R)$ where $x, y \in R$, then there exist $\lambda_1, \lambda_2 \in C$ and additive mappings $\zeta_1, \zeta_2 : R \to C$ respectively such that $d(x) = \lambda_1 x + \zeta_1(x)$ and $g(x) = \lambda_2 x + \zeta_2(x)$ for all $x \in R$.*

Next, we give a generalization of Theorem 2.7 of [3] as a consequence of above results in the setting of generalized derivations:

Remark 1. Let I be a nonzero ideal of a prime ring R. If F and G are generalized derivations of R together with derivations d and g, then the following conditions are equivalent:

(i) $G(xy) + F(x)F(y) \pm [x, y] \in Z(R)$ or $G(xy) - F(x)F(y) \pm [x, y] \in Z(R)$ for all $x, y \in I$.

(ii) $G(xy) + F(x)F(y) \pm yx \in Z(R)$ or $G(xy) - F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in I$.

(iii) $G(xy) + F(x)F(y) \in Z(R)$ or $G(xy) - F(x)F(y) \in Z(R)$ for all $x, y \in I$.

(iv) $G(xy) + F(x)F(y) \pm xy \in Z(R)$ or $G(xy) - F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$.

(v) R is commutative.

Proof. Clearly, $(v) \Rightarrow (i), (v) \Rightarrow (ii), (v) \Rightarrow (iii)$ and $(v) \Rightarrow (iv)$.

$(i) \Rightarrow (v)$ Let $x \in I$ be a fixed element. Let $A_x = \{y \in I : G(xy) + F(x)F(y) \pm [x, y] \in Z(R)\}$ and $B_x = \{y \in I : G(xy) - F(x)F(y) \pm [x, y] \in Z(R)\}$. Since F and G are additive mappings so both A_x and B_x are additive subgroups of I such that $I = A_x \cup B_x$. Therefore, Brauer's trick forces that either $I = A_x$ or $I = B_x$. Now, for some fixed $y \in I$, let $A_y = \{x \in I : G(xy) + F(x)F(y) \pm [x, y] \in Z(R)\}$ and $B_y = \{x \in I : G(xy) - F(x)F(y) \pm [x, y] \in Z(R)\}$. By the same arguments as above, we find that either $I = A_y$ or $I = B_y$. Hence, the commutativity of R follows from Theorem 5 and Theorem 6 with $\phi = i_d$ the identity map.

$(ii) \Rightarrow (v)$ By substituting $\phi = i_d$ and $G = G \mp i_d$ together with g in Theorem 5 and Theorem 6, we may infer that R is commutative if any one of

(a) $G(xy) + F(x)F(y) \pm yx \in Z(R)$

(b) $G(xy) - F(x)F(y) \pm yx \in Z(R)$

holds on I. For a fixed element $x \in I$ we set $A_x = \{y \in I : G(xy) + F(x)F(y) \pm yx \in Z(R)\}$ and $B_x = \{y \in I : G(xy) - F(x)F(y) \pm yx \in Z(R)\}$. Further, by repeating the same arguments we can get the required results.

$(iii) \Rightarrow (v)$ By substituting $\phi = 0$ in Theorem 5 and Theorem 6, we infer that R is commutative if any one of

(a) $G(xy) + F(x)F(y) \in Z(R)$

(b) $G(xy) - F(x)F(y) \in Z(R)$

holds on I. For a fixed element $x \in I$ we set $A_x = \{y \in I : G(xy) + F(x)F(y) \in Z(R)\}$ and $B_x = \{y \in I : G(xy) - F(x)F(y) \in Z(R)\}$. Again, by repeating the same arguments we can get the desired results.

$(iv) \Rightarrow (v)$ As we just shown that if either $G(xy) + F(x)F(y) \in Z(R)$ or $G(xy) - F(x)F(y) \in Z(R)$ holds on I, then R is commutative. By replacing G by $G \pm i_d$ in these equations, we can easily get the desired conclusion. □

3.2 On annihilator conditions

Let S be any subset of R. A derivation d is said to be acting as a homomorphism or as an anti-homomorphism on a set S if $d(xy) - d(x)d(y) = 0$ for all $x, y \in S$ or $d(xy) - d(y)d(x) = 0$ for all $x, y \in S$ respectively. Study of the derivations acting as homomorphisms or as anti-homomorphisms on associative rings was initiated by Bell and Kappe in [7]. After that a number of results has been obtained with various types of derivations acting as homomorphisms or as anti-homomorphisms on some appropriate subsets of associative rings (see [1], [2], [15], [17], [19], [24] and references therein). In [19], Gusic proved the following: *Let R be an associative prime ring, let d be any function on R (not necessarily a derivation nor an additive function), let F be any function on R (not necessarily additive) satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, and let I be a non-zero ideal in R.*

Assume that $F(xy) - F(x)F(y) = 0$ for all $x, y \in I$. Then $d = 0$, and $F = 0$ or $F(x) = x$ for any $x \in R$.

Assume that $F(xy) - F(y)F(x) = 0$ for all $x, y \in I$. Then $d = 0$, and $F = 0$ or $F(x) = x$ for any $x \in R$ (in this case R should be commutative) . Ali et al. [2] studied the same functional identities on square closed Lie ideal of 2-torsion free prime ring. In [27], Dhara et al. extend this notion by studying the algebraic identities $F(x)G(y) \pm H(xy) \in Z(R)$ and $F(x)G(y) \pm H(yx) \in Z(R)$ on square-closed Lie ideals of prime ring of char $\neq 2$, where F, G, H are generalized derivations of R. Further, Rehman and Raza in [26] gave a study of generalized derivations acting as homomorphism or anti-homomorphism on Lie ideals (without the assumption of square-closeness) of 2-torsion free prime ring.

Recently, Dhara et al. [17] obtained the following result: *Let R be a prime ring, I a nonzero ideal of R and $F : R \to R$ be a multiplicative (generalized)-derivation associated with the map $d : R \to R$. For some $0 \neq a \in R$, suppose that $a(F(xy) \pm F(x)F(y)) = 0$ for each $x, y \in I$. Then one of the following hold:*

1. *$d(R) = 0$ and $aF(R) = 0$.*
2. *$d(R) = 0$ and $F(r) = \mp r$, where $r \in R$.*

Following this line of investigation, in this section we studied the situations $a(F(xy) \pm F(x)F(y)) \in Z(R)$ and $a(F(xy) \pm F(y)F(x)) = 0$.

Theorem 7. *Let $(0, I_d \neq)F : R \to R$ be a multiplicative (generalized)-derivation of R together with a mapping $d : R \to R$. If for some $0 \neq a \in R$, $a(F(xy) \pm F(x)F(y)) \in Z(R)$ for all $x, y \in I$, then $[ad(z), z] = 0$ for all $z \in I$.*

Furthermore, if R is prime and d is a derivation on R, then R is commutative.

Proof. For each $x, y \in I$, we consider

$$a(F(xy) \pm F(x)F(y)) \in Z(R). \tag{3.37}$$

Replace y by yt in (3.37), where $t \in I$, we find $a(F(xy) \pm F(x)F(y))t + a(xyd(t) \pm F(x)yd(t)) \in Z(R)$. On commuting with t and using (3.37), we get

$$[a(xyd(t) \pm F(x)yd(t)), t] = 0. \tag{3.38}$$

Put $y = ty$ in (3.38) and we obtain

$$[a(xtyd(t) \pm F(x)tyd(t)), t] = 0. \tag{3.39}$$

On replacing x by xt in (3.38), we get

$$[a(xtyd(t) \pm F(x)tyd(t)), t] \pm [a(xd(t)yd(t)), t] = 0. \tag{3.40}$$

Subtracting (3.39) from (3.40), we get $[a(xd(t)yd(t)), t] = 0$. Substituting $d(t)x$ in place of x, we obtain $[ad(t)xd(t)yd(t), t] = 0$. That is,

$$ad(t)xd(t)yd(t)t - tad(t)xd(t)yd(t) = 0. \tag{3.41}$$

Replacing x by $xad(t)z$ in (3.41), where $z \in I$, we find

$$ad(t)xad(t)zd(t)yd(t)t - tad(t)xad(t)zd(t)yd(t) = 0.$$

Making use of (3.41), we get

$$ad(t)xtad(t)zd(t)yd(t) - ad(t)xad(t)zd(t)tyd(t) = 0.$$

$$ad(t)x[ad(t)zd(t), t]yd(t) = 0. \tag{3.42}$$

Putting $x = zd(t)x$ in (3.42) in order to get

$$ad(t)zd(t)x[ad(t)zd(t), t]yd(t) = 0. \tag{3.43}$$

Replacing x by tx in (3.43), we get

$$ad(t)zd(t)tx[ad(t)zd(t), t]yd(t) = 0. \tag{3.44}$$

Left multiply (3.43) by t and subtract it from (3.44), we left with

$$[ad(t)zd(t), t]x[ad(t)zd(t), t]yd(t) = 0.$$

In this way, we obtain

$$[ad(t)zd(t), t]x[ad(t)zd(t), t]y[ad(t)zd(t), t] = 0.$$

That is, for each $z, t \in I$ we have$(I[ad(t)zd(t), t])^3 = (0)$. Semiprimeness of R forces that $I[ad(t)zd(t), t] = (0)$. Hence, for each $t, z \in I$, we get $[ad(t)zd(t), t] = 0$. That is,

$$ad(t)zd(t)t - tad(t)zd(t) = 0. \tag{3.45}$$

Substitute $zad(t)w$ for z in (3.45), where $w \in I$, we get

$$ad(t)zad(t)wd(t)t - tad(t)zad(t)wd(t) = 0. \tag{3.46}$$

By using (3.45), equation (3.46) can be written as

$$\begin{aligned} 0 =& ad(t)ztad(t)wd(t) - ad(t)zad(t)twd(t) \\ =& ad(t)z[t, ad(t)]wd(t). \end{aligned} \tag{3.47}$$

Replacing z by tz and w by wt in (3.47) in order to get

$$ad(t)tz[t, ad(t)]wtd(t) = 0. \tag{3.48}$$

Multiply t on both sides of (3.47), we have

$$tad(t)z[t, ad(t)]wd(t)t = 0. \tag{3.49}$$

Subtracting (3.48) and (3.49) to obtain $[ad(t), t]z[ad(t), t]w[ad(t), t] = 0$. That means, $(I[ad(t), t])^3 = (0)$. Hence, by the same reasons we obtain $[ad(t), t] = 0$ for any $t \in I$, as desired.

Further, if R is a prime ring and d is derivation of R, then by Lemma 3, either $d = 0$ or R is commutative. If $d = 0$ then our hypothesis gives,

$$aF(x)(y - F(y)) \in Z(R). \tag{3.50}$$

Replacing y by yk in (3.50), where $k \in I$, we get

$$aF(x)(y - F(y))k \in Z(R). \tag{3.51}$$

On commuting both sides by $j \in I$, we find

$$aF(x)(y - F(y))kj - jaF(x)(y - F(y))k = 0. \tag{3.52}$$

By using (3.51) and (3.52), we get $j(aF(x)(y - F(y)))k \in Z(R)$. Put $r = aF(x)(y-F(y))$ and our assumption implies that $r \neq 0$ so, we have $IrI \subseteq Z(R)$. That means R contains a nonzero central ideal. Hence, by Lemma 1, R is commutative. $\square$

Example 1. Consider $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$, be a ring over integers modulo 2 and let $I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z}_2 \right\}$, be an ideal of R. We define maps $F, d : R \to R$ by $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & nb \\ 0 & 0 \end{pmatrix}$, $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & (n-1)b \\ 0 & 0 \end{pmatrix}$, where n is any positive integer. Clearly,F is a multiplicative (generalized)-derivation associated with the map d and for any $0 \neq a \in R$ it is easy to see that the identities $a(F(xy) + F(x)F(y)) \in Z(R)$ and $a(F(xy) - F(x)F(y)) \in Z(R)$ hold for each $x, y \in I$. Here R is not semiprime ring because $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0)$. But neither $[ad(z), z] \neq (0)$ for all $z \in I$ nor R is commutative. Hence, the condition of semiprimeness and primeness in Theorem 7 is not superfluous.

Recently, in [11] Camci and Aydin proved that: *If F is a multiplicative (generalized)-derivation of a semiprime (prime) ring R together with a map f, then f must be multiplicative derivation of R.* In the following theorem, we are taking f as a left multiplier instead of a multiplicative derivation.

Theorem 8. *Let R be a non-commutative prime ring and I be a nonzero ideal of R. Let $F : R \to R$ be a mapping (not necessarily additive) of R such that $F(xy) = F(x)y + xd(y)$, where d is a left-multiplier of R. If for some $0 \neq a \in R$, $a(F(xy) \pm F(y)F(x)) = 0$ for all $x, y \in I$, then either $aF(R) = (0)$ or $F : R \to Z(R)$.*

Proof. For each $x, y \in I$, we consider

$$a(F(xy) - F(y)F(x)) = 0. \tag{3.53}$$

Replacing x by xy in (3.53), we get $a(F(xy)y + xyd(y) - F(y)F(x)y - F(y)xd(y)) = 0$ for all $x, y \in I$. Our hypothesis forces that

$$axyd(y) = aF(y)xd(y). \tag{3.54}$$

Putting ax in place of x in (3.54), we find

$$a^2xyd(y) = aF(y)axd(y). \tag{3.55}$$

Left multiply (3.54) by a and subtract from (3.55) and we get $a[F(y), a]xd(y) = 0$. Primeness of R implies that either $d(I) = (0)$ or $a[F(I), a] = (0)$.

We assume that

$$a[F(y), a] = 0, \text{ for all } y \in I. \tag{3.56}$$

On substitution of yx for y in (3.56), where $x \in I$, we have $aF(y)[x, a] + a[yd(x), a] = 0$. Replacing x by zx, where $z \in I$, in this expression and using it, we obtain $aF(yz)[x, a] = 0$. For some $r \in R$, substitute xr in place of x to get $aF(yz)x[r, a] = 0$. Replace x by px, where $p \in R$ and we get $aF(yz)Rx[r, a] = (0)$. Therefore, either $aF(I^2) = 0$ or $I[r, a] = (0)$. If $I[r, a] = (0)$, then $a \in Z(R)$.

Since we know that center of a prime ring contains no zero divisor, so Eq. (3.54) gives that $(xy - F(y)x)d(y) = 0$ for each $x, y \in I$. For some $t \in I$, replacing x by tx, we find

$$(txy - F(y)tx)d(y) = 0 \tag{3.57}$$

On pre-multiplying the expression $(xy - F(y)x)d(y) = 0$ by t, we obtain

$$(txy - tF(y)x)d(y) = 0 \tag{3.58}$$

Now we combine (3.57) and (3.58) in order to get $[F(y), t]xd(y) = 0$. It implies that either $d(I) = (0)$ or $[F(I), I] = (0)$. Further, if for any $x, y \in I$, $[F(x), y] = 0$. Putting $x = xy$, we find $x[d(y), y] + [x, y]d(y) = 0$ for any $x, y \in I$. Again we put wx instead of x in the last relation, for all $w \in I$, we obtain $[w, y]xd(y) = 0$. For some $r \in R$, we replace x by rx and obtain $[w, y]Rxd(y) = (0)$ where $x, y, w \in I$. It implies that either $[I, I] = (0)$ or $d(I) = (0)$. Since R is assumed to be non-commutative, by Lemma 1, $[I, I] \neq (0)$, so we have $d(I) = (0)$. On the other side, if $aF(yz) = 0$ for each $y, z \in I$, then for some $t \in I$, substitution of zt for z yields that $ayzd(t) = 0$. Since $a \neq 0$ and I a nonzero ideal of the prime ring R, we have $d(I) = (0)$. Therefore, each of our case gives $d(I) = 0$.

Next, we see effect of this outcome $d(I) = (0)$ on the behavior of the mapping F. We consider, $d(I) = (0)$ our hypothesis implies

$$aF(x)y = aF(y)F(x). \tag{3.59}$$

Replacing y by yt in (3.59) where $t \in I$, we get

$$aF(x)yt = aF(y)tF(x). \tag{3.60}$$

Right multiply (3.59) by t and subtract form (3.60) in order to get $aF(y)[F(x), t] = 0$. Put $y = ry$ in the last expression, where $r \in R$, we find $aF(r)y[F(x), t] = 0$. For some $s \in R$ again we replace y be sy in order to get $aF(R)RI[F(I), I] = (0)$. Primeness of R implies that either $aF(R) = (0)$ or $I[F(I), I] = (0)$. Assume that $I[F(I), I] = (0)$. That means for each $x, t \in I$, we have

$$[F(x), t] = 0. \tag{3.61}$$

Putting $x = rx$ where $r \in R$, in the above relation to obtain $[F(r), t]x + F(r)[x, t] = 0$. In particular, we obtain $[F(r), t]t = 0$. Linearizing the last relation w.r.t.t and we get

$$[F(r), t]y + [F(r), y]t = 0. \tag{3.62}$$

Substitute ys for y in (3.62), where $s \in R$, we obtain

$$[F(r), t]ys + [F(r), y]st + y[F(r), s]t = 0. \tag{3.63}$$

Combining (3.62) and (3.63) and we have

$$[F(r), y][s, t] + y[F(r), s]t = 0. \tag{3.64}$$

Replace t by tz in (3.64), where $z \in I$, we get $[F(r), y][s, t]z + [F(r), y]t[s, z] + y[F(r), s]tz = 0$. Eq. (3.64) reduces it to $[F(r), y]t[s, z] = 0$, for all $y, t, z \in I$ and $r, s \in R$. In particular, putting $s = F(r)$ and $y = z$, we obtain $[F(r), z]I[F(r), z] = (0)$. Thus, by primeness of I for each $r \in R$ and $z \in I$, we have $[F(r), z] = 0$. Evidently, $[F(r), s] = 0$, where $r, s \in R$ i.e. $F(R) \subseteq Z(R)$. Hence, F sends R into $Z(R)$.

On replacing F by $-F$ and d by $-d$ in the proof given above, we can get the same conclusions for the situation $a(F(xy) + F(y)F(x)) = 0$. Hence, it proves the theorem. □

We conclude with the following example, which is showing that the Theorem 8 can't be extended to multiplicative (generalized)-derivations.

Example 2. Consider $R = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} : m, n, p, q \in \mathbb{Z}_2 \right\}$, be a ring over integers modulo 2. Since a matrix ring over an integral domain is a prime ring, so R is a non-commutative prime ring. Let $I = \left\{ \begin{pmatrix} m & n \\ 0 & 0 \end{pmatrix} : m, n \in \mathbb{Z}_2 \right\}$, be an ideal of R. We define maps $F, d : R \to R$ by $F\begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} m & 0 \\ p & 0 \end{pmatrix}$, $d\begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} 0 & n \\ p & 0 \end{pmatrix}$. Note that F is a multiplicative (generalized)-derivation associated with the map d. For any $0 \neq a \in R$, it is easy to verify that the identities $a(F(xy) + F(y)F(x)) = 0$ and $a(F(xy) - F(y)F(x)) = 0$ are satisfied on I, but neither $aF(R) = (0)$ nor $F(R) \subseteq Z(R)$. Hence, the restrictions imposed in Theorem 8 are crucial.

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