COMMUTATIVE NEUTROSOPHIC TRIPLET GROUP AND NEUTRO-HOMOMORPHISM BASIC THEOREM

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Abstract. Recently, the notions of neutrosophic triplet and neutrosophic triplet group are introduced by Florentin Smarandache and Mumtaz Ali. The neutrosophic triplet is a group of three elements that satisfy certain properties with some binary operations. The neutrosophic triplet group is completely different from the classical group in the structural properties. In this paper, we further study neutrosophic triplet group. First, to avoid confusion, some new symbols are introduced, and several basic properties of neutrosophic triplet group are rigorously proved (because the original proof is flawed),

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and a result about neutrosophic triplet subgroup is revised. Second, some new properties of commutative neutrosophic triplet group are funded, and a new equivalent relation is established. Third, based on the previous results, the following important propositions are proved: from any commutative neutrosophic triplet group, an Abel group can be constructed; from any commutative neutrosophic triplet group, a BCI-algebra can be constructed. Moreover, some important examples are given. Finally, by using any neutrosophic triplet subgroup of a commutative neutrosophic triplet group, a new congruence relation is established, and then the quotient structure induced by neutrosophic triplet subgroup is constructed and the neutro-homomorphism basic theorem is proved.

Keywords: neutrosophic triplet, neutrosophic triplet group, Abel group, BCI-algebra, neutro-homomorphism basic theorem.

1. Introduction

From a philosophical point of view, Florentin Smarandache introduced the concept of a neutrosophic set (see [12, 13, 14]). The neutrosophic set theory is applied to many scientific fields and also applied to algebraic structures (see [1, 3, 7, 10, 11, 15, 17, 19]). Recently, Florentin Smarandache and Mumtaz Ali in [16], for the first time, introduced the notions of neutrosophic triplet and neutrosophic triplet group. The neutrosophic triplet is a group of three elements that satisfy certain properties with some binary operation. The neutrosophic triplet group is completely different from the classical group in the structural properties. In 2017, Florentin Smarandache has written the monograph [15] which is present the last developments in neutrodophic theories (including neutrosophic triplet and neutrosophic triplet group).

In this paper, we further study neutrosophic triplet group. We discuss some new properties of commutative neutrosophic triplet group, and investigate the relationships among commutative neutrosophic triplet group, Abel group (that is, commutative group) and BCI-algebra. Moreover, we establish the quotient structure and neutro-homomorphism basic theorem.

As a guide, it is necessary to give a brief overview of the basic aspects of BCI-algebra and related algebraic systems. In 1966, K. Iseki introduced the concept of BCI-algebra as an algebraic counterpart of the BCI-logic (see [5, 24]). The algebraic structures closely related to BCI algebra are BCK-algebra, BCC-algebra, BZ-algebra, BE-algebra, and so on (see [2, 8, 20, 21, 22, 25]). As a generalization of BCI-algebra, W. A. Dudek and Y. B. Jun [4] introduced the notion of pseudo-BCI algebras. Moreover, pseudo-BCI algebra is also as a generalization of pseudo-BCK algebra (which is close connection with various non-commutative fuzzy logic formal systems, see [18, 22, 23, 24]). Recently, some articles related filter theory of pseudo-BCI algebras are published (see [26, 27, 28, 29]). As non-classical logic algebras, BCI-algebras are closely related to Abel groups (see [9]); similarly, BZ-algebras (pseudo-BCI algebras) are closely related general groups (see [20, 26]), and some results in [9, 20] will be applied in this paper.

2. Some basic concepts

2.1 On neutrosophic triplet group

Definition 2.1 ([16]). Let N be a set together with a binary operation $*$. Then, N is called a *neutrosophic triplet set* if for any $a \in N$, there exist a neutral of "a" called $neut(a)$, different from the classical algebraic unitary element, and an opposite of "a" called $anti(a)$, with $neut(a)$ and $anti(a)$ belonging to N, such that:

$$a * neut(a) = neut(a) * a = a;$$
$$a * anti(a) = anti(a) * a = neut(a).$$

The elements a, $neut(a)$ and $anti(a)$ are collectively called as neutrosophic triplet, and we denote it by $(a, neut(a), anti(a))$. By $neut(a)$, we mean neutral of a and apparently, a is just the first coordinate of a neutrosophic triplet and not a neutrosophic triplet. For the same element "a" in N, there may be more neutrals to it $neut(a)$ and more opposites of it $anti(a)$.

Definition 2.2 ([16]). The element b in $(N, *)$ is the second component, denoted as $neut(\cdot)$, of a neutrosophic triplet, if there exist other elements a and c in N such that $a*b = b*a = a$ and $a*c = c*a = b$. The formed neutrosophic triplet is (a, b, c).

Definition 2.3 ([16]). The element c in $(N, *)$ is the third component, denoted as $anti(\cdot)$, of a neutrosophic triplet, if there exist other elements a and b in N such that $a*b = b*a = a$ and $a*c = c*a = b$. The formed neutrosophic triplet is (a, b, c).

Definition 2.4 ([16]). Let $(N, *)$ be a neutrosophic triplet set. Then, N is called a *neutrosophic triplet group*, if the following conditions are satisfied:

(1) If $(N, *)$ is well-defined, i.e. for any $a, b \in N$, one has $a * b \in N$.

(2) If $(N, *)$ is associative, i.e. $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

Definition 2.5 ([16]). Let $(N, *)$ be a neutrosophic triplet group. Then, N is called a *commutative neutrosophic triplet group* if for all $a, b \in N$, we have $a * b = b * a$.

Definition 2.6 ([16]). Let $(N, *)$ be a neutrosophic triplet group under $*$, and let H be a subset of N. Then, H is called a neutrosophic triplet subgroup of N if H itself is a *neutrosophic triplet group* with respect to $*$.

Remark 2.7. In order to include richer structure, the original concept of neutrosophic triplet is generalized to neutrosophic extended triplet by Florentin Smarandache. A neutrosophic extended triplet is a neutrosophic triplet, defined as above, but where the neutral of x (called "extended neutral") is allowed to also be equal to the classical algebraic unitary element (if any). Therefore, the restriction "different from the classical algebraic unitary element if any" is released. As a consequence, the "extended opposite" of x, is also allowed to be equal to the classical inverse element from a classical group. Thus, a neutrosophic extended triplet is an object of the form $(x, neut(x), anti(x))$, for $x \in N$, where $neut(x) \in N$ is the extended neutral of x, which can be equal or different from the classical algebraic unitary element if any, such that: $x * neut(x) = neut(x) * x = x$, and $anti(x) \in N$ is the extended opposite of x such that: $x * anti(x) = anti(x) * x = neut(x)$. In this paper, "neutrosophic triplet" means that "neutrosophic extended triplet".

2.2 On BCI-algebras

Definition 2.8 ([5, 23])**.** A BCI-algebra is an algebra $(X; \to, 1)$ of type $(2, 0)$ in which the following axioms are satisfied:
(i) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,
(ii) $x \to x = 1$,
(iii) $1 \to x = x$,
(iv) if $x \to y = y \to x = 1$, then $x = y$.

In any BCI-algebra $(X; \to, 1)$ one can define a relation $\leq$ by putting $x \leq y$ if and only if $x \to y = 1$, then $\leq$ is a partial order on X.

Definition 2.9 ([9, 26])**.** Let $(X; \to, 1)$ be a BCI-algebra. The set $\{x | x \leq 1\}$ is called the *p-radical* (or BCK-part) of X. A BCI-algebra X is called *p-semisimple* if its p-radical is equal to $\{1\}$.

Proposition 2.10 ([9])**.** *Let $(X; \to, 1)$ be a BCI-algebra. Then the following are equivalent:*
(i) X is p-semisimple,
(ii) $x \to 1 = 1 \Rightarrow x = 1$,
(iii) $(x \to 1) \to 1 = x$, $\forall x \in X$,
(iv) $(x \to 1) \to y = (y \to 1) \to x$ for all $x, y \in X$.

Proposition 2.11 ([26])**.** *Let $(X; \to, 1)$ be a BCI-algebra. Then the following are equivalent:*
(S1) X is p-semisimple,
(S2) $x \to y = 1 \Rightarrow x = y$ for all $x, y \in X$,
(S3) $(x \to y) \to (z \to y) = z \to x$ for all $x, y, z \in X$,
(S4) $(x \to y) \to 1 = y \to x$ for all $x, y \in X$,
(S5) $(x \to y) \to (a \to b) = (x \to a) \to (y \to b)$ for all $x, y, a, b \in X$.

Proposition 2.12 ([9, 26])**.** *Let $(X; \to, 1)$ be p-semisimple BCI-algebra; define $+$ and $-$ as follows: for all $x, y \in X$,*

$$x + y \stackrel{def}{=} (x \to 1) \to y, \quad -x \stackrel{def}{=} x \to 1.$$

Then $(X; +, -, 1)$ is an Abel group.

Proposition 2.13 ([9, 26]). *Let $(X;+,-,1)$ be an Abel group. Define $(X;\leq, \to, 1)$, where*

$$x \to y = -x + y,\ x \leq y \text{ if and only if } -x + y = 1,\ \forall x, y \in X.$$

Then, $(X;\leq,\to,1)$ is a BCI-algebra.

3. Some properties of neutrosophic triplet group

As mentioned earlier, for a neutrosophic triplet group $(N,*)$, if $a \in N$, then $neut(a)$ may not be unique, and $anti(a)$ may not be unique. Thus, the symbolic $neut(a)$ sometimes means one and sometimes more than one, which is ambiguous. To this end, this paper introduces the following notations to distinguish:

$neut(a)$: denote any certain one of neutral of a;

$\{neut(a)\}$: denote the set of all neutral of a.

Similarly,

$anti(a)$: denote any certain one of opposite of a;

$\{anti(a)\}$: denote the set of all opposite of a.

Remark 3.1. In order not to cause confusion, we always assume that: (1) for the same a, when multiple $neut(a)$ (or $anti(a)$) are present in the same expression, they are always are consistent. Of course, if they are neutral (or opposite) of different elements, they refer to different objects (for example, in general, $neut(a)$ is different from $neut(b)$). (2) if $neut(a)$ and $anti(a)$ are present in the same expression, then they are match each other.

Proposition 3.2. *Let $(N,*)$ be a neutrosophic triplet group with respect to $*$ and $a \in N$. Then*

$$neut(a) * neut(a) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$a * neut(a) = a,\ neut(a) * a = a.$$

From this, using associative law, we can get

$$a * (neut(a) * neut(a)) = (neut(a) * neut(a)) * a = a.$$

By Definition 2.1, it follows that $(neut(a) * neut(a))$ is a neutral of a. That is, $neut(a) * neut(a) \in \{neut(a)\}$. □

Remark 3.3. This proposition is a revised version of Theorem 3.21(1) in [16]. If $neut(a)$ is unique, then they are same. But, if $neut(a)$ is not unique, they are different. For example, assume $\{neut(a)\} = \{s,t\}$, then $neut(a)$ denote any one of s,t. Thus $neut(a) * neut(a)$ represents one of $s*s$, and $t*t$. Moreover, Proposition 3.2 means that $s*s,\ t*t \in \{neut(a)\} = \{s,t\}$, that is,

$$s*s = s,\ \text{or } s*s = t;\quad t*t = s,\ \text{or } t*t = t.$$

And, in this case, the equation $neut(a)*neut(a) = neut(a)$ means that $s*s = s$, $t*t = t$. So, they are different.

Proposition 3.4. *Let $(N, *)$ be a neutrosophic triplet group with respect to $*$ and $a \in N$. If*

$$neut(a) * neut(a) = neut(a).$$

Then

$$neut(a) * anti(a) \in \{anti(a)\};$$
$$anti(a) * neut(a) \in \{anti(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$a * neut(a) = neut(a) * a = a;$$
$$a * anti(a) = anti(a) * a = neut(a).$$

From this, using associative law, we can get

$$a * (neut(a) * anti(a)) = (a * neut(a)) * anti(a) = a * anti(a) = neut(a).$$

And,

$$(neut(a) * anti(a)) * a = neut(a) * (anti(a) * a) = neut(a) * neut(a) = neut(a).$$

By Definition 2.1, it follows that $(neut(a) * anti(a))$ is a opposite of a. That is, $neut(a) * anti(a) \in \{anti(a)\}$. In the same way, we can get $anti(a) * neut(a) \in \{anti(a)\}$. □

Proposition 3.5. *Let $(N, *)$ be a neutrosophic triplet group with respect to $*$ and let $a, b, c \in N$. Then*

(1) *$a * b = a * c$ if and only if $neut(a) * b = neut(a) * c$.*

(2) *$b * a = c * a$ if and only if $b * neut(a) = c * neut(a)$.*

Proof. Assume $a * b = a * c$. Then $anti(a) * (a * b) = anti(a) * (a * c)$. By associative law, we have

$$(anti(a) * a) * b = (anti(a) * a) * c.$$

Using Definition 2.1 we get $neut(a) * b = neut(a) * c$.

Conversely, assume $neut(a) * b = neut(a) * c$. Then $a * (neut(a) * b) = a * (neut(a) * c)$. By associative law, we have

$$(a * neut(a)) * b = (a * neut(a)) * c.$$

Using Definition 2.1 we get $a * b = a * c$. That is, (1) holds.

Similarly, we can prove that (2) holds. □

Proposition 3.6. *Let $(N, *)$ be a neutrosophic triplet group with respect to $*$ and let $a, b, c \in N$.*

(1) *If $anti(a) * b = anti(a) * c$, then $neut(a) * b = neut(a) * c$.*

(2) *If* $b * anti(a) = c * anti(a)$, *then* $b * neut(a) = c * neut(a)$.

Proof. Assume $anti(a)*b = anti(a)*c$. Then $a*(anti(a)*b) = a*(anti(a)*c)$. By associative law, we have

$$(a * anti(a)) * b = (a * anti(a)) * c.$$

Using Definition 2.1 we get $neut(a) * b = neut(a) * c$. It follows that (1) holds.

Similarly, we can prove that $b * anti(a) = c * anti(a) \Rightarrow b * neut(a) = c * neut(a)$. □

Theorem 3.7. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a \in N$. *Then*

$$neut(neut(a)) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$neut(a) * neut(neut(a)) = neut(a);$$
$$neut(neut(a)) * neut(a) = neut(a).$$

Then

$$a * (neut(a) * neut(neut(a))) = a * neut(a);$$
$$(neut(neut(a)) * neut(a)) * a = neut(a) * a.$$

By associative law and Definition 2.1, we have

$$a * neut(neut(a)) = a;$$
$$neut(neut(a)) * a = a.$$

From this, by Definition 2.1, $neut(neut(a)) \in \{neut(a)\}$. □

Theorem 3.8. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a \in N$. *Then*

$$neut(anti(a)) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$anti(a) * neut(anti(a)) = anti(a);$$
$$neut(anti(a)) * anti(a) = anti(a).$$

Then

$$a * (anti(a) * neut(anti(a))) = a * anti(a);$$
$$(neut(anti(a)) * anti(a)) * a = anti(a) * a.$$

Using associative law and Definition 2.1,

$$neut(a) * neut(anti(a)) = neut(a);$$
$$neut(anti(a)) * neut(a) = neut(a).$$

It follows that $a*neut(anti(a)) = a$, $neut(anti(a))*a = a$. That is, $neut(anti(a)) \in \{neut(a)\}$. □

Theorem 3.9. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a \in N$. *Then*

$$neut(a) * anti(anti(a)) = a.$$

where, $neut(a) \in \{neut(a)\}$, $anti(a) \in \{anti(a)\}$, *and* $neut(a)$ *matches* $anti(a)$, *that is,* $a * anti(a) = anti(a) * a = neut(a)$.

Proof. For any $a \in N$, by Definition 2.1 we have

$$anti(a) * anti(anti(a)) = neut(anti(a)).$$

Then

$$a * (anti(a) * anti(anti(a))) = a * neut(anti(a)).$$
$$(a * anti(a)) * anti(anti(a)) = a * neut(anti(a)).$$
$$neut(a) * anti(anti(a)) = a * neut(anti(a)).$$

On the other hand, by Theorem 3.8, $neut(anti(a)) \in \{neut(a)\}$. By Definition 2.1, it follows that $a*neut(anti(a))=a$. Therefore, $neut(a)*anti(anti(a))=a$. □

Theorem 3.10. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a \in N$. *Then*

$$anti(neut(a)) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$neut(a) * anti(neut(a)) = neut(neut(a));$$
$$anti(neut(a)) * neut(a) = neut(neut(a)).$$

Thus

$$a * (neut(a) * anti(neut(a))) = a * neut(neut(a));$$
$$(anti(neut(a)) * neut(a)) * a = neut(neut(a)) * a.$$

Applying associative law and Definition 2.1,

$$a * anti(neut(a)) = a * neut(neut(a));$$
$$anti(neut(a)) * a = neut(neut(a)) * a.$$

On the other hand, by Theorem 3.7, $neut(neut(a)) \in \{neut(a)\}$. It follows that

$$a * neut(neut(a)) = neut(neut(a)) * a = a.$$

Therefore,

$$a * anti(neut(a))) = anti(neut(a)) * a = a.$$

This means that $anti(neut(a)) \in \{neut(a)\}$. □

Theorem 3.11. *Let* $(N, *)$ *be a neutrosophic triplet group with respect to* $*$ *and* $a, b \in N$. *Then*

$$neut(a * a) \in \{neut(a)\}.$$

Proof. For any $a \in N$, by Definition 2.1 we have

$$(a * a) * neut(a * a) = a * a.$$

From this and applying the associativity of operation $*$ and Definition 2.1 we get

$$(anti(a) * a) * a * neut(a * a) = (anti(a) * a) * a.$$
$$neut(a) * a * neut(a * a) = neut(a) * a.$$
$$a * neut(a * a) = a.$$

Similarly, we can prove $neut(a * a) * a = a$. This means that $neut(a * a) \in \{neut(a)\}$. $\square$

Now, we note that Proposition 3.18 in [16] is not true.

Example 3.12. Consider $(Z_{10}, \sharp)$, where $\sharp$ is defined as $a \sharp b = 3ab(mod10)$. Then, $(Z_{10}, \sharp)$ is a neutrosophic triplet group under the binary operation $\sharp$ with Table 1.

Table 1 Cayley table of neutrosophic triplet group $(Z_{10}, \sharp)$

$\sharp$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	3	6	9	2	5	8	1	4	7
2	0	6	2	8	4	0	6	2	8	4
3	0	9	8	7	6	5	4	3	2	1
4	0	2	4	6	8	0	2	4	6	8
5	0	5	0	5	0	5	0	5	0	5
6	0	8	6	4	2	0	8	6	4	2
7	0	1	2	3	4	5	6	7	8	9
8	0	4	8	2	6	0	4	8	2	6
9	0	7	4	1	8	5	2	9	6	3

For each $a \in Z_{10}$, we have $neut(a)$ in Z_{10}. That is,

$$neut(0) = 0, neut(1) = 7, neut(2) = 2, neut(3) = 7, neut(4) = 2,$$
$$neut(5) = 5, neut(6) = 2, neut(7) = 7, neut(8) = 2, neut(9) = 7.$$

Let $H = \{0, 2, 5, 7\}$, then $(H, \sharp)$ is a neutrosophic triplet subgroup of $(Z_{10}, \sharp)$, but

$$anti(5) \in \{1, 3, 5, 7, 9\} \not\subset H,$$
$$anti(0) \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \not\subset H.$$

Therefore, Proposition 3.18 in [16] should be revised to the following form.

Proposition 3.13. *Let $(N, *)$ be a neutrosophic triplet group and H be a subset of N. Then H is a neutrosophic triplet subgroup of N if and only if the following conditions hold:*

(1) *$a * b \in H$ for all $a, b \in H$.*

(2) *there exists $neut(a) \in H$ for all $a \in H$.*

(3) *there exists $anti(a) \in H$ for all $a \in H$.*

4. New properties of commutative neutrosophic triplet group

Theorem 4.1. *Let $(N,)$ be a commutative neutrosophic triplet group with respect to $*$ and $a, b \in N$. Then*

$$\{neut(a)\} * \{neut(b)\} \subseteq \{neut(a * b)\}.$$

Proof. For any $a, b \in N$, by Definition 2.1 and 2.4 we have

$$a * neut(a) * neut(b) * b = (a * neut(a)) * (neut(b) * b) = a * b.$$

From this and applying the commutativity and associativity of operation $*$ we get

$$(neut(a) * neut(b)) * (a * b) = (a * b) * (neut(a) * neut(b)) = a * b.$$

This means that $neut(a)*neut(b) \in \{neut(a*b)\}$, that is, $\{neut(a)\}*\{neut(b)\} \subseteq \{neut(a * b)\}$. □

Proposition 4.2. *Let $(N, *)$ be a commutative neutrosophic triplet group with respect to $*$ and $H = \{neut(a) \mid a \in N\}$. Then H is a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $unit(h) \in H$ for any $h \in N$.*

Proof. For any $h_1, h_2 \in N$, by the definition of H, there exists $a, b \in N$ such that $h_1 = neut(a)$, $h_2 = neut(b)$. Then, by Theorem 4.1 we have

$$h_1 * h_2 = neut(a) * neut(b) \in \{neut(a * b)\} \subseteq H.$$

Moreover, applying Theorem 3.7 and 3.10,

$$neut(h_1) = neut(neut(a)) \in \{neut(a)\} \subseteq H.$$
$$anti(h_1) = anti(neut(a)) \in \{neut(a)\} \subseteq H.$$

Using Proposition 3.13 we know that H is a neutrosophic triplet subgroup of N, and it satisfies

$$(\forall a \in N)\ neut(a) \in H, \text{and } unit(h) \in H \text{ for any } h \in N.$$

□

Theorem 4.3. *Let $(N, *)$ be a commutative neutrosophic triplet group with respect to $*$ and $a, b \in N$. Then*

$$\{anti(a)\} * \{anti(b)\} \subseteq \{anti(a * b)\}.$$

Proof. For any $a, b \in N$, by Definition 2.1 and 2.4 we have

$$a * anti(a) * anti(b) * b = (a * anti(a)) * (anti(b) * b) = neut(a) * neut(b).$$

From this and applying the commutativity and associativity of operation $*$ we get

$$(anti(a) * anti(b))(a * b) = (a * b) * (anti(a) * anti(b)) = neut(a) * neut(b).$$

Applying Theorem 4.1, $neut(a) * neut(b) \in \{neut(a * b)\}$. Hence, by Definition 2.1, $anti(a) * anti(b) \in \{anti(a * b)\}$, that is, $\{anti(a)\} * \{anti(b)\} \subseteq \{anti(a * b)\}$. □

Theorem 4.4. *Let $(N, *)$ be a commutative neutrosophic triplet group with respect to $*$. Define binary relation $\approx_{neut}$ on N as following:*

$\forall a, b \in N$, $a \approx_{neut} b$ iff there exists $anti(b) \in \{anti(b)\}$, and $p, q \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in \{neut(q)\}.$$

Then $\approx_{neut}$ is reflexive and symmetric.

Proof. (1) For any $a \in N$, by Proposition 3.2, $neut(a) * neut(a) \in \{neut(a)\}$. Using Definition 2.1 we get

$$a * anti(a) * neut(a) = neut(a) * neut(a) \in \{neut(a)\}.$$

Then, $a \approx_{neut} a$.

(2) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$$a * anti(b) * neut(p) = neut(q). \tag{C1}$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in neut(p)$, $neut(q) \in \{neut(q)\}$. Using Theorem 3.10, $anti(neut(p)) \in \{neut(p)\}$. So, we denote $anti(neut(p)) = x \in \{neut(p)\}$. Thus,

$$\begin{aligned}
& b * anti(a) * x = b * anti(a) * anti(neut(p)) \\
& = anti(a) * b * anti(neut(p)) && (by\ Definition\ 2.5) \\
& = anti(a) * (neut(b) * anti(anti(b))) * anti(neut(p)) && (by\ Theorem\ 3.9) \\
& = (anti(a) * anti(anti(b)) * anti(neut(p))) * neut(b) && (by\ Definition\ 2.4 and\ 2.5) \\
& \in \{anti(a * anti(b) * neut(p))\} * neut(b) && (by\ Theorem\ 4.3) \\
& \subseteq \{anti(neut(q))\} * neut(b) && (by\ the\ above\ result\ (C1)) \\
& \subseteq \{neut(q)\} * neut(b) && (by\ Theorem\ 3.10) \\
& \subseteq \{neut(q * b)\} && (by\ Theorem\ 4.1)
\end{aligned}$$

This means that $b \approx_{neut} a$. □

Definition 4.5. Let $(N, *)$ be a neutrosophic triplet group. Then, N is called a neutrosophic triplet group with condition (AN) if for all $a, b \in N$, we have
(AN) $\quad \{anti(a * b)\} \subseteq \{anti(a)\} * \{anti(b)\}$.

Proposition 4.6. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN) and $a, b \in N$. Then*

$$neut(a * b) \in \{neut(a)\} * \{neut(b)\}.$$

Proof. For any $a, b \in N$, by Definition 4.5, there exists $anti(a) \in \{anti(a)\}$, $anti(b) \in \{anti(b)\}$ such that

$$anti(a * b) = anti(a) * anti(b).$$

Then

$$\begin{aligned} neut(a * b) &= (a * b) * anti(a * b) = (a * b) * (anti(a) * anti(b)) \\ &= (a * anti(a)) * (b * anti(b)) = neut(a) * neut(b). \end{aligned}$$

This means that $neut(a * b) \in \{neut(a)\} * \{neut(b)\}$. □

Lemma 4.7. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN) and $a, b \in N$. If there exists $anti(b) \in \{anti(b)\}$, $p, q \in N$, $neut(p) \in \{neut(p)\}$ and $neut(q) \in \{neut(q)\}$ such that*

$$a * anti(b) * neut(p) = neut(q).$$

Then for any $x \in \{anti(b)\}$, there exists $p_1, q_1 \in N$, $neut(p_1) \in \{neut(p_1)\}$ and $neut(q_1) \in \{neut(q_1)\}$ such that

$$a * x * neut(p_1) = neut(q_1).$$

Proof. For any $x \in \{anti(b)\}$, there exists $y \in \{neut(b)\}$ such that $b * x = x * b = y$. Thus, from $a * anti(b) * neut(p) = neut(q)$ we get

$$\begin{aligned} & a * x * (neut(b) * neut(p)) \\ & = a * x * (anti(b) * b) * neut(p) \\ & = (a * anti(b) * neut(p)) * (x * b) \\ & = neut(q) * y \\ & \in neut(q) * \{neut(b)\} \\ & \subseteq \{neut(q * b)\} \qquad\qquad (\text{by Theorem } 4.1) \end{aligned}$$

Therefore, there exists $p_1, q_1 \in N$, $neut(p_1) \in \{neut(p_1)\}$ and $neut(q_1) \in \{neut(q_1)\}$ such that $a * x * neut(p_1) = neut(q_1)$. □

Theorem 4.8. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN). Define binary relation $\approx_{neut}$ on N as following:*

$\forall a, b \in N$, $a \approx_{neut} b$ iff there exists $anti(b) \in \{anti(b)\}$, $p, q \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in \{neut(q)\}.$$

Then $\approx_{neut}$ is an equivalent relation on N.

Proof. By Theorem 4.4, we only prove that $\approx_{neut}$ is transitive. Assume that $a \approx_{neut} b$ and $b \approx_{neut} c$, then there exists $p, q, r, s \in N$ such that

(C1) $$a * anti(b) * neut(p) = neut(q).$$

(C2) $$b * anti(c) * neut(r) = neut(s).$$

where $anti(b) \in \{anti(b)\}$, $anti(c) \in \{anti(c)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$, $neut(r) \in \{neut(r)\}$, $neut(s) \in \{neut(s)\}$. Using Theorem 3.10 and Theorem 4.1, we have

$$neut(p)*neut(c)*anti(neut(s)) \in \{neut(p)\}*\{neut(c)\}*\{neut(s)\} \subseteq \{neut(p*s*c)\}.$$

Denote $y = neut(p) * neut(c) * anti(neut(s)) \in \{neut(p * s * c)\}$, then

$a * anti(c) * y = a * anti(c) * neut(p) * neut(c) * anti(neut(s))$
$= a * anti(c) * neut(p) * anti(neut(s)) * neut(c)$ (by Definition 2.5)
$= a * anti(c) * neut(p) * anti(b * anti(c) * neut(r)) * neut(c)$
(by the above result (C2))
$\in a * anti(c) * neut(p) * \{anti(b) * anti(anti(c)) * anti(neut(r))\} * neut(c)$
(by Definition 4.5)
$\subseteq a * anti(c) * neut(p) * \{anti(b) * c * anti(neut(r))\}$
(by Definition 2.4, 2.5 and Theorem 3.9)
$\subseteq a * neut(p) * \{anti(b) * neut(r) * (anti(c) * c)\}$
(by Theorem 3.10, Definition 2.4 and 2.5)
$= a * neut(p) * \{anti(b) * neut(r) * neut(c)\}$ (by Definition 2.1)
$\subseteq \{(a * anti(b) * neut(p)) * neut(r) * neut(c)\}$ (by Definition 2.1)
$\subseteq \{neut(q_1)*neut(r)*neut(c)\}$ (by the above result (C1) and Lemma 4.7)
$\subseteq \{neut(q_1 * r * c)\}$ (by Theorem 4.1)

This means that $a \approx_{neut} c$. □

5. Commutative neutrosophic triplet group and Abel group with BCI-algebra

Theorem 5.1. *Let $(N, *)$ be a commutative neutrosophic triplet group condition (AN). Define binary relation $\approx_{neut}$ on N as Theorem 4.8. Then the following statements are hold:*

(1) $a, b, c \in N$, $a \approx_{neut} b \Rightarrow a * c \approx_{neut} b * c$.

(2) $a \approx_{neut} b \Rightarrow neut(a) \approx_{neut} neut(b)$.

(3) $a \approx_{neut} b \Rightarrow anti(a) \approx_{neut} anti(b)$.

(4) $a, b \in N$, $neut(a) \approx_{neut} neut(b)$.

Proof. (1) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$(C1)$ $$a * anti(b) * neut(p) = neut(q),$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$. Thus,

$$\begin{aligned}
&(a*c)*anti(b*c)*neut(p)\\
&\in (a*c)*\{anti(b)\}*\{anti(c)\}*neut(p) \qquad (by\ Definition\ 4.5)\\
&\subseteq \{a*anti(b)*neut(p)\}*\{c*anti(c)\}\ (by\ Definition\ 2.4\ and\ 2.5)\\
&= \{a*anti(b)*neut(p)\}*\{neut(c)\} \qquad (by\ Definition\ 2.1)\\
&\subseteq \{neut(q_1)\}*\{neut(c)\}(by\ the\ above\ result\ (C1)\ and\ Lemma\ 4.7)\\
&\subseteq \{neut(q_1*c)\} \qquad (by\ Theorem\ 4.1)
\end{aligned}$$

It follows that $a*c \approx_{neut} b*c$.

(2) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$$a*anti(b)*neut(p) = neut(q).$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$. Then, applying Theorem 3.8 and Theorem 4.1 we have

$$neut(a)*anti(neut(b))*neut(p) \in \{neut(a)\}*\{neut(b)\}*\{neut(p)\} \subseteq \{neut(a*b*p)\}.$$

This means that $neut(a) \approx_{neut} neut(b)$.

(3) Assume $a \approx_{neut} b$, then there exists $p, q \in N$ such that

$$a*anti(b)*neut(p) = neut(q).$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$, $neut(q) \in \{neut(q)\}$. Using Theorem 3.10,

$$anti(neut(p)) \in \{neut(p)\},\ anti(neut(q)) \in \{neut(q)\}.$$

Applying Theorem 4.3 we have

$$\begin{aligned}
anti(a)*anti(anti(b))*anti(neut(p)) &\in \{anti(a*anti(b)*neut(p))\}\\
&\subseteq \{anti(neut(q))\} \subseteq \{neut(q)\}.
\end{aligned}$$

It follows that $anti(a) \approx_{neut} anti(b)$.

(4) $\forall a, b \in N$, since

$$\begin{aligned}
&neut(a)*anti(neut(b))*neut(a)\\
&\in neut(a)*\{neut(b)\}*neut(a) \qquad (by\ Theorem\ 3.10)\\
&\subseteq \{neut(a*b*a)\} \qquad (by\ Theorem\ 4.1)
\end{aligned}$$

This means that $neut(a) \approx_{neut} neut(b)$. □

Theorem 5.2. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN). Define binary relation $\approx_{neut}$ on N as Theorem 4.8. Then the quotient $N/\approx_{neut}$ is an Abel group with respect to the following operation:*

$$\forall\ a, b \in N,\ [a]_{neut} \bullet [b]_{neut} = [a*b]_{neut}.$$

where $[a]_{neut}$ is the equivalent class of a, the unit elment of $(N/\approx_{neut}, \bullet)$ is $1_{neut} = [neut(a)]_{neut}$, $\forall a \in N$, $neut(a) \in \{neut(a)\}$.

Proof. By Theorem 5.1 (1) $\sim$ (3) we know that the operation "$\bullet$" is well definition. Obviously, $(N/\approx_{neut}, \bullet)$ is a commutative neutrosophic triplet group.

Moreover, by Theorem 5.1 (4) we get

$$\forall a, b \in N,\ [neut(a)]_{neut} = [neut(b)]_{neut}.$$
$$\forall a, b \in N,\ neut([a]_{neut}) = neut([b]_{neut}).$$

This means that $neut(\cdot)$ is unique. Denote

$$1_{neut} = [neut(a)]_{neut},\ \forall\ a \in N,\ neut(a) \in \{neut(a)\}.$$

Then 1_{neut} is the unit element of $(N/\approx_{neut}, \bullet)$. Moreover, by Theorem 5.1 (3) we get that $anti([a]_{neut})$ is unique, $\forall a \in N$. Therefore, $(N/\approx_{neut}, \bullet)$ is an Abel group. $\square$

Theorem 5.3. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN). Define binary relation $\approx_{neut}$ on N as Theorem 4.8. If define a new operation "$\rightarrow$" on the quotient $N/\approx_{neut}$ as following:*

$$\forall a, b \in N,\ [a]_{neut} \rightarrow [b]_{neut} = [a]_{neut} \bullet anti([b]_{neut}).$$

Then $(N/\approx_{neut}, \rightarrow, 1_{neut})$ is a BCI-algebra, where 1_{neut}=$[neut(a)]_{neut}$, $\forall a$∈N.

Proof. By Theorem 5.2 and Proposition 2.13 we can get the result. $\square$

Example 5.4. Let $N = \{1, 2, 3, 4, 6, 7, 8, 9\}$. The operation $*$ on N is defined as Tables 2. Then, $(N, *)$ is a neutrosophic triplet group with condition (AN). For each $a \in N$, we have $neut(a)$ in N. That is,

$$neut(1) = 7,\ neut(2) = 2,\ neut(3) = 7,\ neut(4) = 2,$$
$$neut(6) = 2,\ neut(7) = 7,\ neut(8) = 2,\ neut(9) = 7.$$

Moreover, for each $a \in N$, $anti(a)$ in N. That is,

$$anti(1) = 9,\ anti(2) \in \{2, 7\},\ anti(3) = 3,\ anti(4) \in \{1, 6\},$$
$$anti(6) \in \{4, 9\},\ anti(7) = 7,\ anti(8) \in \{3, 8\},\ anti(9) = 1.$$

It is easy to verify that $N/\approx_{neut}$= $\{[2]_{neut}, [1]_{neut}, [3]_{neut}, [4]_{neut}\}$ and $(N/\approx_{neut}, \bullet)$ is isomorphism to $(Z_4, +)$, where

$$[2]_{neut} = \{2, 7\},\ [1]_{neut} = \{1, 6\},\ [3]_{neut} = \{3, 8\},\ [4]_{neut} = \{4, 9\}.$$

Table 2 Cayley table of neutrosophic triplet group $(N, *)$

$*$	1	2	3	4	6	7	8	9
1	3	6	9	2	8	1	4	7
2	6	2	8	4	6	2	8	4
3	9	8	7	6	4	3	2	1
4	2	4	6	8	2	4	6	8
6	8	6	4	2	8	6	4	2
7	1	2	3	4	6	7	8	9
8	4	8	2	6	4	8	2	6
9	7	4	1	8	2	9	6	3

Table 3 Cayley table of Abel group $((N/\approx_{neut}, \bullet)$

$\bullet$	$[2]_{neut}$	$[1]_{neut}$	$[3]_{neut}$	$[4]_{neut}$
$[2]_{neut}$	$[2]_{neut}$	$[1]_{neut}$	$[3]_{neut}$	$[4]_{neut}$
$[1]_{neut}$	$[1]_{neut}$	$[3]_{neut}$	$[4]_{neut}$	$[2]_{neut}$
$[3]_{neut}$	$[3]_{neut}$	$[4]_{neut}$	$[2]_{neut}$	$[1]_{neut}$
$[4]_{neut}$	$[4]_{neut}$	$[2]_{neut}$	$[1]_{neut}$	$[3]_{neut}$

Table 4 Cayley table of Abel group $(Z_4, +)$

$+$	0	1	3	4
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Example 5.5. Consider $(Z_{10}, \sharp)$, where $\sharp$ is defined as $a \sharp b = 3ab(mod 10)$. Then, $(Z_{10}, \sharp)$ is a neutrosophic triplet group with condition (AN), the binary operation $\sharp$ is defined in Table 1. For each $\in Z_{10}$, we have $neut(a)$ in Z_{10}. That is,

$$neut(0) = 0,\ neut(1) = 7,\ neut(2) = 2,\ neut(3) = 7,\ neut(4) = 2,$$
$$neut(5) = 5,\ neut(6) = 2,\ neut(7) = 7,\ neut(8) = 2,\ neut(9) = 7.$$

Moreover, for each $a \in Z_{10}$, $anti(a)$ in Z_{10}. That is,

$$anti(0) \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},\ anti(1) = 9,\ anti(2) \in \{2, 7\},$$
$$anti(3) = 3,\ anti(4) \in \{1, 6\},\ anti(5) \in \{1, 3, 5, 7, 9\},$$
$$anti(6) \in \{4, 9\},\ anti(7) = 7,\ anti(8) \in \{3, 8\},\ anti(9) = 1.$$

It is easy to verify that $N/\approx_{neut} = \{1_{neut} = [0]_{neut}\}$ and $(N/\approx_{neut}, \bullet)$ is isomorphism to $\{1\}$,where

$$[0]_{neut} = 1_{neut} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

6. Quotient structure and neutro-homomorphism basic theorem

Definition 6.1 ([16])**.** Let $(N_1, *_1)$ and $(N_2, *_2)$ be two neutrosophic triplet groups. Let $f: N_1 \to N_2$ be a mapping. Then, f is called *neutro-homomorphism* if for all $a, b \in N_1$, we have:

(1) $f(a *_1 b) = f(a) *_2 f(b)$;
(2) $f(neut(a)) = neut(f(a))$;
(3) $f(anti(a)) = anti(f(a))$.

Theorem 6.2. *Let $(N, *)$ be a commutative neutrosophic triplet group with respect to $*$, H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as following:*

$\forall a, b \in N$, $a \approx_H b$ iff there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in H.$$

Then $\approx_H$ is reflexive and symmetric.

Proof. (1) For any $a \in N$, by Proposition 3.2 and the hypothesis ($neut(a) \in H$ for any $a \in N$), we have

$$neut(a) * neut(a) \in \{neut(a)\} \subseteq H.$$

By Definition 2.1 we get

$$a * anti(a) * neut(a) = neut(a) * neut(a) \in H.$$

Then, $a \approx_H a$.

(2) Assume $a \approx_H b$, then there exists $p \in N$ such that

$$(C2) \qquad a * anti(b) * neut(p) \in H.$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. Moreover, by the hypothesis ($anti(a) \in H$ for any $a \in H$), we have

$$(C3) \qquad anti(a * anti(b) * neut(p)) \in H.$$

Using Theorem 3.10, $anti(neut(p)) \in \{neut(p)\}$. So, we denote $anti(neut(p)) = x \in \{neut(p)\}$. Thus,

$b * anti(a) * x$
$= b * anti(a) * anti(neut(p))$
$= anti(a) * b * anti(neut(p))$ (by Definition 2.5)
$= anti(a) * (neut(b) * anti(anti(b))) * anti(neut(p))$ (by Theorem 3.9)
$= (anti(a) * anti(anti(b)) * anti(neut(p))) * neut(b)$ (by Definition 2.4 and 2.5)
$\in \{anti(a * anti(b) * neut(p))\} * neut(b)$ (by Theorem 4.3)
$\subseteq H$ (by (C3), the hypothesis and Proposition 3.13 (1))

This means that $b \approx_H a$. □

Lemma 6.3. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), $a, b \in N$, and H be a neutrosophic triplet subgroup of N such that ($\forall a \in N$) $neut(a) \in H$ and ($\forall a \in H$) $anti(a) \in H$. If there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that*

$$a * anti(b) * neut(p) \in H.$$

Then for any $x \in \{anti(b)\}$, there exists $p_1 \in N$, and $neut(p_1) \in \{neut(p_1)\}$ such that

$$a * x * neut(p_1) \in H.$$

Proof. For any $x \in \{anti(b)\}$, there exists $y \in \{neut(b)\}$ such that $b*x = x*b = y$. Since $(\forall a \in N)\ neut(a) \in H$, then $y \in H$. Thus, from $a*anti(b)*neut(p) \in H$ we get

$$\begin{aligned}
& a * x * (neut(b) * neut(p)) \\
& = a * x * (anti(b) * b) * neut(p) \\
& = (a * anti(b) * neut(p)) * (x * b) \\
& = (a * anti(b * neut(p)) * y \\
& \in H \qquad\qquad\qquad\qquad (by\ Proposition\ 3.13)
\end{aligned}$$

□

Theorem 6.4. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as following:*

$\forall a, b \in N$, $a \approx_H b$ iff there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in H.$$

Then $\approx_H$ is an equivalent relation on N.

Proof. By Theorem 6.2, we only prove that $\approx_H$ is transitive. Assume that $a \approx_H b$ and $b \approx_H c$, then there exists $p, r \in N$ and $q, s \in N$ such that

(C3) $$a * anti(b) * neut(p) = q \in H.$$

(C4) $$b * anti(c) * neut(r) = s \in H.$$

where $anti(b) \in \{anti(b)\}$, $anti(c) \in \{anti(c)\}$, $neut(p) \in \{neut(p)\}$, $neut(r) \in \{neut(r)\}$. Using Theorem 4.1 and the hypothesis ($neut(a) \in H$ for any $a \in N$), we have

$$neut(p) * neut(s) * neut(c) \in neut(p * s * c) \subseteq H.$$

Denote $y = neut(p) * neut(s) * neut(c) \in neut(p * s * c)$, then

$$\begin{aligned}
& a * anti(c) * y \\
& = a * anti(c) * neut(p) * neut(s) * neut(c) \\
& = a * anti(c) * neut(p) * (s * anti(s)) * neut(c) \qquad \text{(by Definition 2.1)} \\
& = a * anti(c) * neut(p) * s * anti(b * anti(c) * neut(r)) * neut(c) \\
& \qquad\qquad \text{(by the above result (C4))} \\
& \in a*anti(c)*neut(p)*s*\{anti(b)\}*\{anti(anti(c))\}*\{anti(neut(r))\}neut(c) \\
& \qquad\qquad \text{(by Definition 4.5)} \\
& = a * anti(c) * neut(p) * s * \{anti(b)\} * c * \{anti(neut(r))\} \ \text{(by Theorem 3.9)} \\
& \subseteq a * anti(c) * neut(p) * s * \{anti(b)\} * c * \{neut(r)\} \qquad \text{(by Theorem 3.10)} \\
& \subseteq \{a * anti(b) * neut(p)\} * s * (anti(c) * c) * \{neut(r)\} \ \text{(by Definition 2.4 and 2.5)} \\
& \subseteq H * s * neut(c) * \{neut(r)\}
\end{aligned}$$

(by Definition 2.1, the above result (C3) and Lemma 6.3)
$\subseteq H$ (by (C4), the hypothesis and Proposition 3.13 (1))
It follows that $a \approx_H c$. □

Theorem 6.5. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as following:*

$\forall a, b \in N$, $a \approx_H b$ iff there exists $anti(b) \in \{anti(b)\}$, $p \in N$, and $neut(p) \in \{neut(p)\}$ such that

$$a * anti(b) * neut(p) \in H.$$

Then the following statements are hold:
(1) $a, b, c \in N$, $a \approx_H b \Rightarrow a * c \approx_H b * c$.
(2) $a \approx_H b \Rightarrow neut(a) \approx_H neut(b)$.
(3) $a \approx_H b \Rightarrow anti(a) \approx_H anti(b)$.

Proof. (1) Assume $a \approx_H b$, then there exists $p \in N$ such that

$$(C2) \qquad a * anti(b) * neut(p) \in H.$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. We have
$(a * c) * anti(b * c) * neut(p)$
$\in (a * c) * \{anti(b)\} * \{anti(c)\} * neut(p)$ (by Definition 4.5)
$\subseteq \{a * anti(b) * neut(p)\} * \{c * anti(c)\}$ (by Definition 2.4 and 2.5)
$= \{a * anti(b) * neut(p)\} * neut(c)$ (by Definition 2.1)
$\in H$. (by (C2), the hypothesis, Lemma 6.3 and Proposition 3.13 (1))
It follows that $a * c \approx_H b * c$.

(2) Assume $a \approx_H b$, then there exists $p \in N$ such that $a * anti(b) * neut(p) \in H$, where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. Applying Theorem 3.8 and Theorem 4.1 we have
$neut(a) * anti(neut(b)) * neut(p) \in neut(a) * \{neut(b)\} * neut(p)$
$\subseteq \{neut(a{*}b{*}p)\} \subseteq H$. (by the hypothesis, $neut(a) \in H$ for any $a \in N$)
It follows that $neut(a) \approx_H neut(b)$.

Assume $a \approx_H b$, then there exists $p \in N$ such that

$$a * anti(b) * neut(p) \in H.$$

where $anti(b) \in \{anti(b)\}$, $neut(p) \in \{neut(p)\}$. Applying the hypothesis (($\forall a \in N$) $neut(a) \in H$ and ($\forall a \in H$) $anti(a) \in H$) and Theorem 3.10,

$$anti(a * anti(b) * neut(p)) \in H.$$
$$anti(neut(p)) \in \{neut(p)\} \subseteq H.$$

Moreover, by Theorem 4.3 we have

$$anti(a) * anti(anti(b)) * anti(neut(p)) \in \{anti(a * anti(b) * neut(p))\} \subseteq H.$$

Hence, $anti(a) \approx_H anti(b)$. □

Theorem 6.6. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as Theorem 6.5. Then the quotient $N/\approx_H$ is a commutative neutrosophic triplet group with respect to the following operation:*

$$\forall a, b \in N,\ [a]_H \bullet [b]_H = [a * b]_H.$$

where $[a]_H$ is the equivalent class of a with respect to $\approx_H$. Moreover, $(N, *)$ is neutron-homomorphism to $(N/\approx_H, \bullet)$ with respect to the following mapping:

$$f:\ N \to N/\approx_H;\ \text{and}\ \forall a \in N,\ f(a) = [a]_H.$$

Proof. By Theorem 6.5 we know that the operation "$\bullet$" is well definition. Obviously, $(N/\approx_H, \bullet)$ is a commutative neutrosophic triplet group.

By the definitions of operation "$\bullet$" and mapping f we have

$$\forall a, b \in N,\ f(a * b) = [a * b]_H = [a]_H \bullet [b]_H = f(a) \bullet f(b).$$

Moreover, by Theorem 6.5 (2) and (3) we get

$$\forall a \in N, f(neut(a)) = [neut(a)]_H = neut([a]_H) = neut(f(a)).$$
$$\forall a \in N, f(anti(a)) = [anti(a)]_H = anti([a]_H) = anti(f(a)).$$

Therefore, $(N, *)$ is neutron-homomorphism to $(N/\approx_H, \bullet)$ with respect to the mapping f. □

Theorem 6.7. *Let $(N, *)$ be a commutative neutrosophic triplet group with condition (AN), H be a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Define binary relation $\approx_H$ on N as Theorem 6.5. If define a new operation "$\to$" on the quotient $N/\approx_H$ as following: $\forall a, b \in N$, $[a]_H \to [b]_H = [a]_H \bullet anti([b]_H)$. Then $(N/\approx_H, \to, 1_H)$ is a BCI-algebra, where $1_H = [neut(a)]_H$, $\forall a \in N$.*

Proof. By Theorem 6.7 and Proposition 2.13 we can get the result. □

Example 6.8. Let $N = \{1, 2, 3, 4, 6, 7, 8, 9\}$. The operation $*$ on N is defined as Tables 2. Then, $(N, *)$ is a neutrosophic triplet group with condition (AN). We can get the following equation

$$neut(1) = 7,\ neut(2) = 2,\ neut(3) = 7,\ neut(4) = 2,$$
$$neut(6) = 2,\ neut(7) = 7,\ neut(8) = 2,\ neut(9) = 7;$$
$$anti(1) = 9,\ anti(2) \in \{2, 7\},\ anti(3) = 3,\ anti(4) \in \{1, 6\},$$
$$anti(6) \in \{4, 9\},\ anti(7) = 7,\ anti(8) \in \{3, 8\},\ anti(9) = 1.$$

Denote $H = \{2, 3, 7, 8\}$, it is easy to verify that H is a neutrosophic triplet subgroup of N such that $(\forall a \in N)$ $neut(a) \in H$ and $(\forall a \in H)$ $anti(a) \in H$. Moreover, $N/\approx_H = \{H = [2]_H, [1]_H\}$ and $(N/\approx_H, \bullet)$ is isomorphism to $(Z_2, +)$, where

$$[2]_H = \{2, 3, 7, 8\},\ [1]_H = \{1, 4, 6, 9\}.$$

Table 5 Cayley table of Abel group $(N/\approx_H, \bullet)$

$\bullet$	$[2]_H$	$[1]_H$
$[2]_H$	$[2]_H$	$[1]_H$
$[1]_H$	$[1]_H$	$[2]_H$

Table 6 Cayley table of Abel group $(Z_2, +)$

$+$	0	1
0	0	1
1	1	0

The following example shows that the basic theorem of neutro-homomorphism (Theorem 6.7) is a natural and substantial generalization of the basic theorem of group-homomorphism.

Example 6.9. Let $(N, *)$ be a commutative group. Then, $(N, *)$ is a neutrosophic triplet group with condition (AN). Obviously, if H is a subgroup of N, then binary relation $\approx_H$ on N is the relation induced by subgroup H, that is,

$$\forall a, b \in N,\ a \approx_H b \text{ if and only if } a * b^{-1} \in H.$$

Thus, $(N, *)$ is group-homomorphism to $(N/\approx_H, \bullet) = (N/H, \bullet)$.

7. Conclusion

This paper is focus on neutrosophic triplet group. We proved some new properties of (commutative) neutrosophic triplet group, and constructed a new equivalent relation on any commutative neutrosophic triplet group with condition (AN). Based on these results, for the first time, we have described the inner link between commutative neutrosophic triplet group with condition (AN) and Abel group with BCI-algebra. Furthermore, we establish the quotient structure by neutrosophic triplet subgroup, and prove the basic theorem of neutro-homomorphism, which is a natural and substantial generalization of the basic theorem of group-homomorphism. Obviously, these results will play an important role in the further study of neutrosophic triplet group.

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