# ENERGY OF A BIPOLAR FUZZY GRAPH AND ITS APPLICATION IN DECISION MAKING

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**Abstract.** In many domains of information processing, bipolarity is a core feature to be considered: positive information represents what is possible or preferred, while negative information represents what is forbidden or surely false. If the information is moreover endowed with vagueness and imprecision, then bipolar fuzzy sets (BFSs) constitute an appropriate knowledge representation framework. In this paper, we introduce the novel concepts of energy of a graph in the context of a bipolar fuzzy environment and investigate some of their properties. We show that if $\mathcal{G}$ is a bipolar fuzzy graph (BFG) on $n$ vertices, then $E(\mathcal{G}) \leq \frac{n}{2}(1+\sqrt{n})$ must hold. Moreover, we introduce the concept of energy of bipolar fuzzy digraphs (BFDGs) along with its application in decision making problem.
**Keywords:** bipolar fuzzy graph, spectrum, energy, decision making.

## 1. Introduction

Zhang [21] introduced the concept of BFS characterized by a positive membership function and a negative membership function as an extension of traditional fuzzy set [20] whose basic component is only a membership function. This domain has recently motivated work in several directions, for instance for

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applications in preference modeling, knowledge representation, argumentation, cooperative games and multi-criteria decision analysis. The range of membership degree of BFSs is $[-1, 1]$. In a BFS, the positive membership degree (0, 1] of an element indicates that the element somewhat satisfies the corresponding property, the negative membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property and the membership degree 0 of an element means that the element is irrelevant to the property [11].

In real life, many situations can be simply abstracted as the graphics problems containing points and connection. For instance, in the Internet, a router can be represented as a vertex and an edge connects two routers with optical fiber. The theory of graphs was first introduced in 1736, when Euler published his paper on graph theory, and solved the problem of the Konigsberg's bridges, which gave birth to a new branch of mathematics. The energy of a graph was originally investigated by Gutman in 1978 [8] and has a wide rang of applications in different fields, including, computer science, physics, chemistry and other branches of mathematics. Fuzzy graphs are designed to represent structures of relationships between objects such that the existence of a concrete object (vertex) and relationship between two objects (edge) are matters of degree. The concept of fuzzy graphs was initiated by Kaufmann [10], based on Zadeh's fuzzy relations. Later, another elaborated definition of fuzzy graph with fuzzy vertex and fuzzy edges was introduced by Rosenfeld [17] and obtaining analogs of several graph theoretical concepts such as paths, cycles and connectedness etc, he developed the structure of fuzzy graphs. Energy of a fuzzy graph was investigated in [5] by Anjali and Mathew. Akram et al. originally proposed the concept of BFGs, and made a lot of studies on this extension of fuzzy graphs [1, 2, 3, 4, 18]. Naz et al. put forward some new concepts concerning the extended structures of fuzzy graphs and provided their applications in decision-making [6, 13, 14, 15]. Borzooei and Rashmanlou [7] defined the energy of a vague graph. However, to the best of our knowledge, no work addressing the energy in bipolar fuzzy setting is in literature. So, the main purpose of this paper is to introduce the concept of energy of a BFG and BFDG.

The paper is structured as follows: Section 2 contains a brief background about BFSs and BFGs. Section 3 mainly proposes the concept of the energy of a BFG, and investigates its properties. Section 4 introduces the concept of energy of BFDGs along with its application in decision making problem, and finally conclusions are given in Section 5.

Throughout this paper, $\mathcal{V}$ represents a crisp universe of generic elements, $G$ stands for the crisp (undirected, simple) graph and $\mathcal{G}$ is the BFG.

## 2. Preliminaries

In the following, some basic concepts on BFSs and BFGs are reviewed to facilitate next sections.

A graph $G = (V,\ E)$ is a mathematical structure consisting of a set of vertices $V$ and a set of edges $E$, where each edge is an unordered pair of distinct vertices. If $G$ is a graph with $n$ vertices and $m$ edges, its adjacency matrix $A(G)$ is the $n \times n$ matrix whose $ij$-th entry is the number of edges joining vertices $i$ and $j$. The eigenvalues $\lambda_i, i = 1, 2, \ldots, n$, of the adjacency matrix of $G$ are the eigenvalues of $G$. The spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of the adjacency matrix of $G$ is the spectrum, Spec$(G)$, of $G$. The eigenvalues of a graph satisfy the following relations:

$$\sum_{i=1}^{n} \lambda_i = 0, \sum_{i=1}^{n} \lambda_i^2 = 2m.$$

**Definition 2.1** ([8, 9])**.** The energy of a graph $G$, denoted by $E(G)$, is defined as the sum of the absolute values of the eigenvalues of $G$, i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. A graph with all isolated vertices $K_n^c$ has zero energy while the complete graph $K_n$ with $n$ vertices has energy $2(n-1)$.

**Definition 2.2** ([16])**.** The energy of a digraph $D$, denoted by $E(D)$, is defined as the sum of the absolute values of the real part of eigenvalues of $D$, i.e., $E(D) = \sum_{i=1}^{n} |\mathrm{Re}(z_i)|$.

In 1965, Zadeh [20] originally introduced the fuzzy set, characterized by a membership function in $[0, 1]$, which is very useful in dealing with uncertainty, imprecision and vagueness.

**Definition 2.3** ([20])**.** A fuzzy set $\upsilon$ on a set $\mathcal{V}$ is defined through its membership function $\upsilon : \mathcal{V} \to [0, 1]$, where $\upsilon(x)$ represents the degree to which point $x \in \mathcal{V}$ belongs to the fuzzy set. The smallest element and the largest element are the function constantly equal to 0 and 1, respectively.

**Definition 2.4** ( [19] )**.** A fuzzy preference relation $R$ on a set of alternatives $\mathcal{V} = \{x_1, x_2, \ldots, x_n\}$ is characterized by a membership function $\eta_R : \mathcal{V} \times \mathcal{V} \to [0, 1]$. A fuzzy preference relation can be conveniently represented by the $n \times n$ matrix $R = (r_{ij})_{n \times n}$, where $r_{ij}$ indicates the degree of preference of alternative $x_i$ over $x_j$ with $r_{ij} \in [0, 1],\ r_{ij} + r_{ji} = 1,\ r_{ij} = 0.5$ for all $i, j = 1, 2, \ldots, n$.

**Definition 2.5** ( [21])**.** A BFS $X$ in a non-empty set $\mathcal{V}$ is an object having the following form $X = \{(x, \eta_X^P(x), \eta_X^N(x)) \mid x \in \mathcal{V}\}$ which is characterized by a positive membership function $\eta_X^P$ and a negative membership function $\eta_X^N$, where $\eta_X^P : \mathcal{V} \to [0, 1],\ x \in \mathcal{V} \to \eta_X^P(x) \in [0, 1],\ \eta_X^N : \mathcal{V} \to [-1, 0],\ x \in \mathcal{V} \to \eta_X^N(x) \in [-1, 0]$. If $\eta_X^P(x) \neq 0$ and $\eta_X^N(x) = 0$, then $x$ is regarded as having only positive satisfaction for $X$. If $\eta_X^P(x) = 0$ and $\eta_X^N(x) \neq 0$, then $x$ does not satisfy the property of $X$ but somewhat satisfies the counter property of $X$. Finally, if $\eta_X^P(x) \neq 0$ and $\eta_X^N(x) \neq 0$, then the membership function of the property overlaps that of its counter property over some portion of $\mathcal{V}$.

By introducing the concept of BFSs into the theory of graphs, Akram [1] put forward the notion of the BFGs as follows.

**Definition 2.6** ([1]). A BFG with a finite set $\mathcal{V}$ as the underlying set is a pair $\mathcal{G} = (X, Y)$, where $X = (\eta_X^P, \eta_X^N)$ is a BFS on $\mathcal{V}$ and $Y = (\eta_Y^P, \eta_Y^N)$ is a bipolar fuzzy relation on $\mathcal{V}$ such that $\eta_Y^P(xy) \leq \min\{\eta_X^P(x), \eta_X^P(y)\}$ and $\eta_Y^N(xy) \geq \max\{\eta_X^N(x), \eta_X^N(y)\}$ for all $x, y \in \mathcal{V}$, We call $X$ the bipolar fuzzy vertex set of $\mathcal{G}$ and $Y$ the bipolar fuzzy edge set of $\mathcal{G}$.

## 3. Energy of a bipolar fuzzy graph

In this section, based on the extension of the energy of a fuzzy graph [5], we define the concept of energy of a BFG, which can be used in real scientific and engineering applications.

**Definition 3.1.** The adjacency matrix $A(\mathcal{G})$ of a BFG $\mathcal{G} = (X, Y)$ is defined as a square matrix $A(\mathcal{G}) = [a_{ij}]$, $a_{ij} = (\eta_Y^P(u_iu_j), \eta_Y^N(u_iu_j))$, where $\eta_Y^P(u_iu_j)$ and $\eta_Y^N(u_iu_j)$ represent the strength of positive relationship and strength of negative relationship between $u_i$ and $u_j$, respectively.

**Example 3.1.** Consider a graph $G = (V, E)$, where $V = \{u_1, u_2, u_3, u_4, u_5\}$ and $E = \{u_1u_2, u_1u_3, u_1u_4, u_1u_5, u_2u_3, u_3u_4, u_4u_5\}$. Let $\mathcal{G} = (X, Y)$ be a BFG of a graph $G$, as shown in Fig. 1. Tabular representation of a BFG is given in Table 1. The adjacency matrix of a BFG given in Fig. 1, is

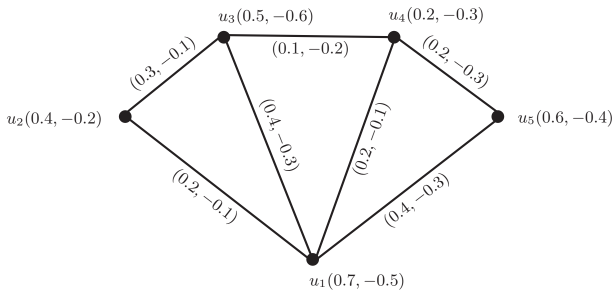


Figure 1: Bipolar fuzzy graph.

Table 1: Tabular representation of a BFG.

| $X$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ |
|---|---|---|---|---|---|
| $\eta_X^P$ | 0.7 | 0.4 | 0.5 | 0.2 | 0.6 |
| $\eta_X^N$ | $-0.5$ | $-0.2$ | $-0.6$ | $-0.3$ | $-0.4$ |

| $Y$ | $u_1u_2$ | $u_1u_3$ | $u_1u_4$ | $u_1u_5$ | $u_2u_3$ | $u_3u_4$ | $u_4u_5$ |
|---|---|---|---|---|---|---|---|
| $\eta_Y^P$ | 0.2 | 0.4 | 0.2 | 0.4 | 0.3 | 0.1 | 0.2 |
| $\eta_Y^N$ | $-0.1$ | $-0.3$ | $-0.1$ | $-0.3$ | $-0.1$ | $-0.2$ | $-0.3$ |

$$A(\mathcal{G}) = \begin{bmatrix} (0,0) & (0.2,-0.1) & (0.4,-0.3) & (0.2,-0.1) & (0.4,-0.3) \\ (0.2,-0.1) & (0,0) & (0.3,-0.1) & (0,0) & (0,0) \\ (0.4,-0.3) & (0.3,-0.1) & (0,0) & (0.1,-0.2) & (0,0) \\ (0.2,-0.1) & (0,0) & (0.1,-0.2) & (0,0) & (0.2,-0.3) \\ (0.4,-0.3) & (0,0) & (0,0) & (0.2,-0.3) & (0,0) \end{bmatrix}.$$

**Definition 3.2.** The spectrum of adjacency matrix of a BFG $A(\mathcal{G})$ is defined as $(S,T)$, where $S$ and $T$ are the sets of eigenvalues of $A(\eta_Y^P(u_iu_j))$ and $A(\eta_Y^N(u_iu_j))$, respectively.

**Definition 3.3.** The energy of a BFG $\mathcal{G}=(X,Y)$ is defined as

$$E(\mathcal{G}) = \big(E(\eta_Y^P(u_iu_j)), E(\eta_Y^N(u_iu_j))\big) = \left(\sum_{\substack{i=1\\ \lambda_i\in S}}^{n} |\lambda_i|, \sum_{\substack{i=1\\ \delta_i\in T}}^{n} |\delta_i|\right).$$

**Theorem 3.1.** *Let $\mathcal{G}=(X,Y)$ be a BFG and $A(\mathcal{G})$ be its adjacency matrix. If $\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_n$ and $\delta_1\ge\delta_2\ge\ldots\ge\delta_n$ are the eigenvalues of $A(\eta_Y^P(u_iu_j))$ and $A(\eta_Y^N(u_iu_j))$, respectively. Then*

1. $\sum_{\substack{i=1\\ \lambda_i\in S}}^{n} \lambda_i = 0$ and $\sum_{\substack{i=1\\ \delta_i\in T}}^{n} \delta_i = 0$.
2. $\sum_{\substack{i=1\\ \lambda_i\in S}}^{n} \lambda_i^2=2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2$ *and* $\sum_{\substack{i=1\\ \delta_i\in T}}^{n} \delta_i^2=2\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2$.

**Proof.** 1. Since $A(\mathcal{G})$ is a symmetric matrix with zero trace, so its eigenvalues are real with sum equal to zero.

2. By trace properties of a matrix, we have $tr((A(\eta_Y^P(u_iu_j)))^2) = \sum_{\substack{i=1\\ \lambda_i\in S}}^{n} \lambda_i^2$, where

$$\begin{aligned} tr((A(\eta_Y^P(u_iu_j)))^2) &= \big(0+(\eta_Y^P(u_1u_2))^2+\ldots+(\eta_Y^P(u_1u_n))^2\big)\\ &\quad +\big((\eta_Y^P(u_2u_1))^2+0+\ldots+(\eta_Y^P(u_2u_n))^2\big)\\ &\quad \vdots\\ &\quad +\big((\eta_Y^P(u_nu_1))^2+(\eta_Y^P(u_nu_2))^2+\ldots+0\big)\\ &= 2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2. \end{aligned}$$

Hence $\sum_{\substack{i=1\\ \lambda_i\in S}}^{n} \lambda_i^2 = 2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2$.

Similarly, we can show that $\sum_{\substack{i=1\\ \delta_i\in T}}^{n} \delta_i^2 = 2\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2$. □

**Example 3.2.** The spectrum and the energy of a BFG $\mathcal{G}$, given in Fig. 1 are as follows: $Spec(\mathcal{G}) = \{(-0.5661,-0.6219),(-0.2767,-0.1029),(-0.1504,0.0814),(0.2075,0.1361),(0.7857,0.5074)\}$, $E(\mathcal{G})=(1.9864,1.4498)$.

Further, $\sum_{\substack{i=1\\ \lambda_i\in S}}^{5}\lambda_i=-0.5661-0.2767-0.1504+0.2075+0.7857=0, \sum_{\substack{i=1\\ \delta_i\in T}}^{5}\delta_i=-0.6219-0.1029+0.0814+0.1361+0.5074=0. \sum_{\substack{i=1\\ \lambda_i\in S}}^{5}\lambda_i^2=1.0800=2(0.54)=2\sum_{1\le i<j\le 5}(\eta_Y^P(u_iu_j))^2, \sum_{\substack{i=1\\ \delta_i\in T}}^{5}\delta_i^2=0.6800=2(0.34)=2\sum_{1\le i<j\le 5}(\eta_Y^N(u_iu_j))^2$.

We now find upper and lower bounds of the energy of a BFG $\mathcal{G}$, in terms of the number of vertices and the sum of squares of positive membership and negative membership values of edges.

**Theorem 3.2.** *Let $\mathcal{G}=(X,Y)$ be a BFG on $n$ vertices and $A(\mathcal{G})=(A(\eta_Y^P(u_iu_j)), A(\eta_Y^N(u_iu_j)))$ be the adjacency matrix of $\mathcal{G}$. Then*

**(i)** $\sqrt{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2+n(n-1)|\det(A(\eta_Y^P(u_iu_j)))|^{\frac{2}{n}}}$
$\le E(\eta_Y^P(u_iu_j))\le\sqrt{2n\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}$;

**(ii)** $\sqrt{2\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2+n(n-1)|\det(A(\eta_Y^N(u_iu_j)))|^{\frac{2}{n}}}$
$\le E(\eta_Y^N(u_iu_j))\le\sqrt{2n\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2}$.

**Proof.** $(i)$ Upper bound: Applying Cauchy-Schwarz inequality to the vectors $(1,1,\dots,1)$ and $(|\lambda_1|,|\lambda_2|,\dots,|\lambda_n|)$ with $n$ entries, we get

$$\sum_{i=1}^{n}|\lambda_i|\le\sqrt{n}\sqrt{\sum_{i=1}^{n}|\lambda_i|^2} \tag{3.1}$$

$$\left(\sum_{i=1}^{n}\lambda_i\right)^2=\sum_{i=1}^{n}|\lambda_i|^2+2\sum_{1\le i<j\le n}\lambda_i\lambda_j \tag{3.2}$$

By comparing the coefficients of $\lambda^{n-2}$ in the characteristic polynomial

$$\prod_{i=1}^{n}(\lambda-\lambda_i)=|A(\mathcal{G})-\lambda I|,$$

we have

$$\sum_{1\le i<j\le n}\lambda_i\lambda_j=-\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2. \tag{3.3}$$

Substituting (3.3) in (3.2), we obtain

$$\sum_{i=1}^{n}|\lambda_i|^2=2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2. \tag{3.4}$$

Substituting (3.4) in (3.1), we obtain

$$\sum_{i=1}^{n}|\lambda_i| \leq \sqrt{n}\sqrt{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2} = \sqrt{2n\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2}.$$

Therefore,

$$E(\eta_Y^P(u_iu_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2}$$

Lower bound:

$$\begin{aligned}(E(\eta_Y^P(u_iu_j)))^2 &= \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{1\leq i<j\leq n}|\lambda_i\lambda_j| \\ &= 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 + \frac{2n(n-1)}{2}AM\{|\lambda_i\lambda_j|\}\end{aligned}$$

Since $AM\{|\lambda_i\lambda_j|\} \geq GM\{|\lambda_i\lambda_j|\}$, $1 \leq i < j \leq n$, so,

$$E(\eta_Y^P(u_iu_j)) \geq \sqrt{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 + n(n-1)GM\{|\lambda_i\lambda_j|\}}$$

also since

$$\begin{aligned}GM\{|\lambda_i\lambda_j|\} = \left(\prod_{1\leq i<j\leq n}|\lambda_i\lambda_j|\right)^{\frac{2}{n(n-1)}} &= \left(\prod_{i=1}^{n}|\lambda_i|^{n-1}\right)^{\frac{2}{n(n-1)}} \\ &= \left(\prod_{i=1}^{n}|\lambda_i|\right)^{\frac{2}{n}} = |\det(A(\eta_Y^P(u_iu_j)))|^{\frac{2}{n}}\end{aligned}$$

so,

$$E(\eta_Y^P(u_iu_j)) \geq \sqrt{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 + n(n-1)|\det(A(\eta_Y^P(u_iu_j)))|^{\frac{2}{n}}}.$$

Thus, $\sqrt{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 + n(n-1)|\det(A(\eta_Y^P(u_iu_j)))|^{\frac{2}{n}}}$
$\leq E(\eta_Y^P(u_iu_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2}$.

Similarly, we can show that
$\sqrt{2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2 + n(n-1)|\det(A(\eta_Y^N(u_iu_j)))|^{\frac{2}{n}}} \leq E(\eta_Y^N(u_iu_j))$
$\leq \sqrt{2n\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2}$. □

The following result gives us upper bound of the energy of a BFG, with the conditions $n \leq 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2$ and $n \leq 2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2$.

**Theorem 3.3.** *Let* $\mathcal{G} = (X, Y)$ *be a BFG on* $n$ *vertices and* $A(\mathcal{G}) = (A(\eta_Y^P(u_iu_j)), A(\eta_Y^N(u_iu_j)))$ *be the adjacency matrix of* $\mathcal{G}$. *If* $n \leq 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2$ *and* $n \leq 2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2$. *Then*

**(i)** $E(\eta_Y^P(u_iu_j)) \leq \frac{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 - \left(\frac{2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2}{n}\right)^2\right\}}$;

**(ii)** $E(\eta_Y^N(u_iu_j)) \leq \frac{2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2 - \left(\frac{2\sum_{1\leq i<j\leq n}(\eta_Y^N(u_iu_j))^2}{n}\right)^2\right\}}$.

**Proof.** If $A = [a_{ij}]_{n\times n}$ is a symmetric matrix with zero trace. Then $\lambda_{\max} \geq \frac{2\sum_{1\leq i<j\leq n} a_{ij}}{n}$, where, $\lambda_{\max}$ is the maximum eigenvalue of $A$. If $A(\mathcal{G})$ is the adjacency matrix of a BFG $\mathcal{G}$, then $\lambda_1 \geq \frac{2\sum_{1\leq i<j\leq n}\eta_Y^P(u_iu_j)}{n}$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Moreover, since

$$\sum_{i=1}^{n}\lambda_i^2 = 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2$$

$$\sum_{i=2}^{n}\lambda_i^2 = 2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 - \lambda_1^2 \tag{3.5}$$

Applying Cauchy-Schwarz inequality to the vectors $(1, 1, \ldots, 1)$ and $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|)$ with $n-1$ entries, we get

$$E(\eta_Y^P(u_iu_j)) - \lambda_1 = \sum_{i=2}^{n}|\lambda_i| \leq \sqrt{(n-1)\sum_{i=2}^{n}|\lambda_i|^2}. \tag{3.6}$$

Substituting (3.5) in (3.6), we must have

$$E(\eta_Y^P(u_iu_j)) - \lambda_1 \leq \sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 - \lambda_1^2\right)}$$

$$E(\eta_Y^P(u_iu_j)) \leq \lambda_1 + \sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(\eta_Y^P(u_iu_j))^2 - \lambda_1^2\right)}. \tag{3.7}$$

Now, since the function $F(x) = x + \sqrt{(n-1)(2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2 - x^2)}$ decreases on the interval $(\sqrt{\frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n}}, \sqrt{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2})$. Also $n \le 2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2$, $1 \le \frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n}$. So,
$\sqrt{\frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n}} \le \frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n} \le \frac{2\sum_{1\le i<j\le n}\eta_Y^P(u_iu_j)}{n} \le \lambda_1$
$\le \sqrt{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}$. Therefore, (3.7) implies $E(\eta_Y^P(u_iu_j))$
$\le \frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n}$
$+ \sqrt{(n-1)\{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2 - (\frac{2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2}{n})^2\}}$.
Similarly, $E(\eta_Y^N(u_iu_j)) \le \frac{2\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2}{n}$
$+ \sqrt{(n-1)\{2\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2 - (\frac{2\sum_{1\le i<j\le n}(\eta_Y^N(u_iu_j))^2}{n})^2\}}$. □

**Theorem 3.4.** *Let $\mathcal{G} = (X, Y)$ be a BFG on $n$ vertices. Then $E(\mathcal{G}) \le \frac{n}{2}(1 + \sqrt{n})$.*

**Proof.** Suppose that $\mathcal{G} = (X, Y)$ is a BFG with $n$ vertices.

If $n \le 2\sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2 = 2y$, then by routine calculus, it is easy to show that $f(y) = \frac{2y}{n} + \sqrt{(n-1)(2y - (\frac{2y}{n})^2)}$ is maximized when $y = \frac{n^2+n\sqrt{n}}{4}$. Substituting this value of $y$ in place of $y = \sum_{1\le i<j\le n}(\eta_Y^P(u_iu_j))^2$ in Theorem 3.3, we must have $E(\eta_Y^P(u_iu_j)) \le \frac{n}{2}(1+\sqrt{n})$. Similarly, it is easy to show that $E(\eta_Y^N(u_iu_j)) \le \frac{n}{2}(1+\sqrt{n})$. Hence $E(\mathcal{G}) \le \frac{n}{2}(1+\sqrt{n})$. □

## 4. Energy of a bipolar fuzzy digraph

In this section, we generalize the concept of energy to BFDGs.

**Definition 4.1.** Let $\mathcal{D} = (X, \overrightarrow{Y})$ be a BFDG on $n$ vertices. The energy of $\mathcal{D}$ is defined as

$$E(\mathcal{D}) = \Big(E(\eta_{\overrightarrow{Y}}^P(u_iu_j)), E(\eta_{\overrightarrow{Y}}^N(u_iu_j))\Big) = \left(\sum_{\substack{i=1\\ z_i\in S}}^{n} |\mathrm{Re}(z_i)|, \sum_{\substack{i=1\\ w_i\in T}}^{n} |\mathrm{Re}(w_i)|\right),$$

where $\mathrm{Re}(z_i)$ and $\mathrm{Re}(w_i)$ represent the real part of eigenvalues $z_i$ and $w_i$, respectively.

**Example 4.1.** Consider a digraph $D = (V, \overrightarrow{E})$, where $V = \{u_1, u_2, u_3, u_4, u_5\}$ and $\overrightarrow{E} = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_3u_1, u_5u_4\}$. Let $\mathcal{D} = (X, \overrightarrow{Y})$ be a BFDG of (crisp) digraph $D$, as given in Fig. 2.
The corresponding adjacent matrix $R$ is as follows:

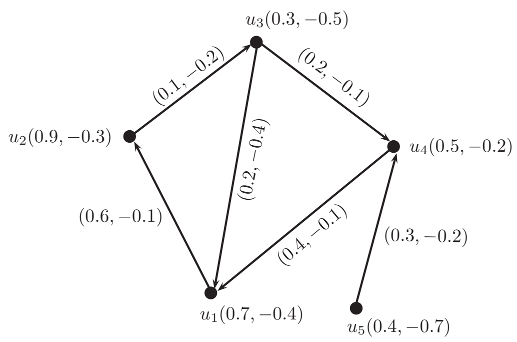


Figure 2: Bipolar fuzzy digraph.

$$R = \begin{bmatrix} (0,0) & (0.6,-0.1) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0.1,-0.2) & (0,0) & (0,0) \\ (0.2,-0.4) & (0,0) & (0,0) & (0.2,-0.1) & (0,0) \\ (0.4,-0.1) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0.3,-0.2) & (0,0) \end{bmatrix}$$

The spectrum and the energy of a BFDG $\mathcal{D}$, given in Fig. 2 are $Spec(\mathcal{G}) = \{(0,0),(0.3031,-0.2077),(-0.0432+0.2669i, 0.0914+0.1739i),(-0.0432-0.2669i, 0.0914-0.1739i),(-0.2166,0.0250)\}$ and $E(\mathcal{G}) = (0.6061, 0.4154)$, respectively.

**Definition 4.2.** A bipolar fuzzy preference relation $R$ on a set of alternatives $\mathcal{V} = \{x_1, x_2, \ldots, x_n\}$ is defined as a matrix $R = (b_{ij})_{n\times n} \subset \mathcal{V}\times\mathcal{V}$ where $b_{ij} = (\eta^P(x_ix_j), \eta^N(x_ix_j))$ for all $i, j = 1, 2, ..., n$. Let $b_{ij} = (\eta^P_{ij}, \eta^N_{ij})$ is a bipolar fuzzy value, composed by the certainty degree $\eta^P_{ij}$ to which $x_i$ is positively preferred to $x_j$ and the certainty degree $\eta^N_{ij}$ to which $x_i$ is negatively preferred to $x_j$ with $0 \le \eta^P_{ij} \le 1, -1 \le \eta^N_{ij} \le 0, \eta^P_{ij} + \eta^P_{ji} = 1, \eta^N_{ij} + \eta^N_{ji} = -1$ and $b_{ii} = (0.5, -0.5)$ for all $i, j = 1, 2, \ldots, n$.

### 4.1 Application of energy of a BFDG in decision making problem

In modern warfare, it is very important to maintain the communication smoothly. Thus, the performance of the communication equipment plays a key role in campaign victory and defeat. It is necessary for communication units to keep regular communication drills. Suppose that the headquarters are drawing up a plan of communication drill next round. According to the consultations with different simulation environments, there are four possible training venues (alternatives) $x_i (i = 1, 2, 3, 4)$ to choose from. The leaders of the communication unit invite a decision group which contains six experts $e_k (k = 1, 2, \ldots, 6)$ to evaluate all venues so as to make the most reasonable choice. Based on their experiences, the experts compare each pair of alternatives and give individual judgments using the following bipolar fuzzy preference relations $R_k = (r^{P(k)}_{ij}, r^{N(k)}_{ij})_{4\times 4}$ $(k = 1, 2, \ldots, 6)$:

$$R_1 = \begin{bmatrix} (0.50,-0.50) & (0.23,-0.17) & (0.57,-0.67) & (0.63,-0.5) \\ (0.77,-0.83) & (0.50,-0.50) & (0.17,-1) & (0.84,-0.67) \\ (0.43,-0.33) & (0.83,0) & (0.50,-0.50) & (0.42,-0.17) \\ (0.37,-0.50) & (0.16,-0.33) & (0.58,-0.83) & (0.50,-0.50) \end{bmatrix}$$

$$R_2 = \begin{bmatrix} (0.50,-0.50) & (0.38,-0.38) & (0.51,-0.58) & (0.27,-0.84) \\ (0.62,-0.62) & (0.50,-0.50) & (0.60,-0.69) & (0.75,-0.90) \\ (0.49,-0.42) & (0.40,-0.31) & (0.50,-0.50) & (0.14,-0.80) \\ (0.73,-0.16) & (0.25,-0.10) & (0.86,-0.20) & (0.50,-0.50) \end{bmatrix}$$

$$R_3 = \begin{bmatrix} (0.50,-0.50) & (0.57,-0.10) & (0.40,-0.60) & (0.46,-0.70) \\ (0.43,-0.90) & (0.50,-0.50) & (0.61,-0.80) & (0.19,-0.40) \\ (0.60,-0.40) & (0.39,-0.20) & (0.50,-0.50) & (0.80,-0.90) \\ (0.54,-0.30) & (0.81,-0.60) & (0.20,-0.10) & (0.50,-0.50) \end{bmatrix}$$

$$R_4 = \begin{bmatrix} (0.50,-0.50) & (0.30,-0.33) & (0.42,-0.17) & (0.26,-0.67) \\ (0.70,-0.67) & (0.50,-0.50) & (0.90,-0.33) & (0.72,-0.17) \\ (0.58,-0.83) & (0.10,-0.67) & (0.50,-0.50) & (0.81,-1) \\ (0.74,-0.33) & (0.28,-0.83) & (0.19,0) & (0.50,-0.50) \end{bmatrix}$$

$$R_5 = \begin{bmatrix} (0.50,-0.50) & (0.70,-0.34) & (0.16,-0.20) & (0.41,-0.96) \\ (0.30,-0.66) & (0.50,-0.50) & (0.80,-0.33) & (0.29,-0.98) \\ (0.84,-0.80) & (0.20,-0.67) & (0.50,-0.50) & (0.63,-0.99) \\ (0.59,-0.04) & (0.71,-0.02) & (0.37,-0.01) & (0.50,-0.50) \end{bmatrix}$$

$$R_6 = \begin{bmatrix} (0.50,-0.50) & (0.23,-0.50) & (1.0,-0.70) & (0.30,-1.0) \\ (0.77,-0.50) & (0.50,-0.50) & (0.60,-0.80) & (0.36,-0.60) \\ (0,-0.30) & (0.40,-0.20) & (0.50,-0.50) & (0.72,-0.80) \\ (0.70,0) & (0.64,-0.40) & (0.28,-0.20) & (0.50,-0.50) \end{bmatrix}$$

The BFDGs $\mathcal{D}_i$ corresponding to bipolar fuzzy preference relations given in matrices $R_i$ are shown in Fig. 3. The energy of each BFDG is $E(R_1) = (2.9357, 2.3961), E(R_2) = (2.8289, 2.5249), E(R_3) = (2.9602, 2.8495), E(R_4) = (2.7304, 2.6413), E(R_5) = (2.9699, 1.9252), E(R_6) = (2.9510, 2.5897)$. Then the weights can be calculated as:

$$w_k = (w_k^P, w_k^N) = \left( \frac{E(R_k^P)}{\sum_{l=1}^{m} E(R_l^P)}, \frac{E(R_k^N)}{\sum_{l=1}^{m} E(R_l^N)} \right), \qquad k = 1, 2, \ldots, m,$$

Here $w_1 = (0.1690, 0.1605), w_2 = (0.1628, 0.1692), w_3 = (0.1704, 0.1909), w_4 = (0.1571, 0.177), w_5 = (0.1709, 0.1290), w_6 = (0.1698, 0.1735)$.

The collective bipolar fuzzy preference relation aggregated from the six bipolar fuzzy preference relations is determined as:

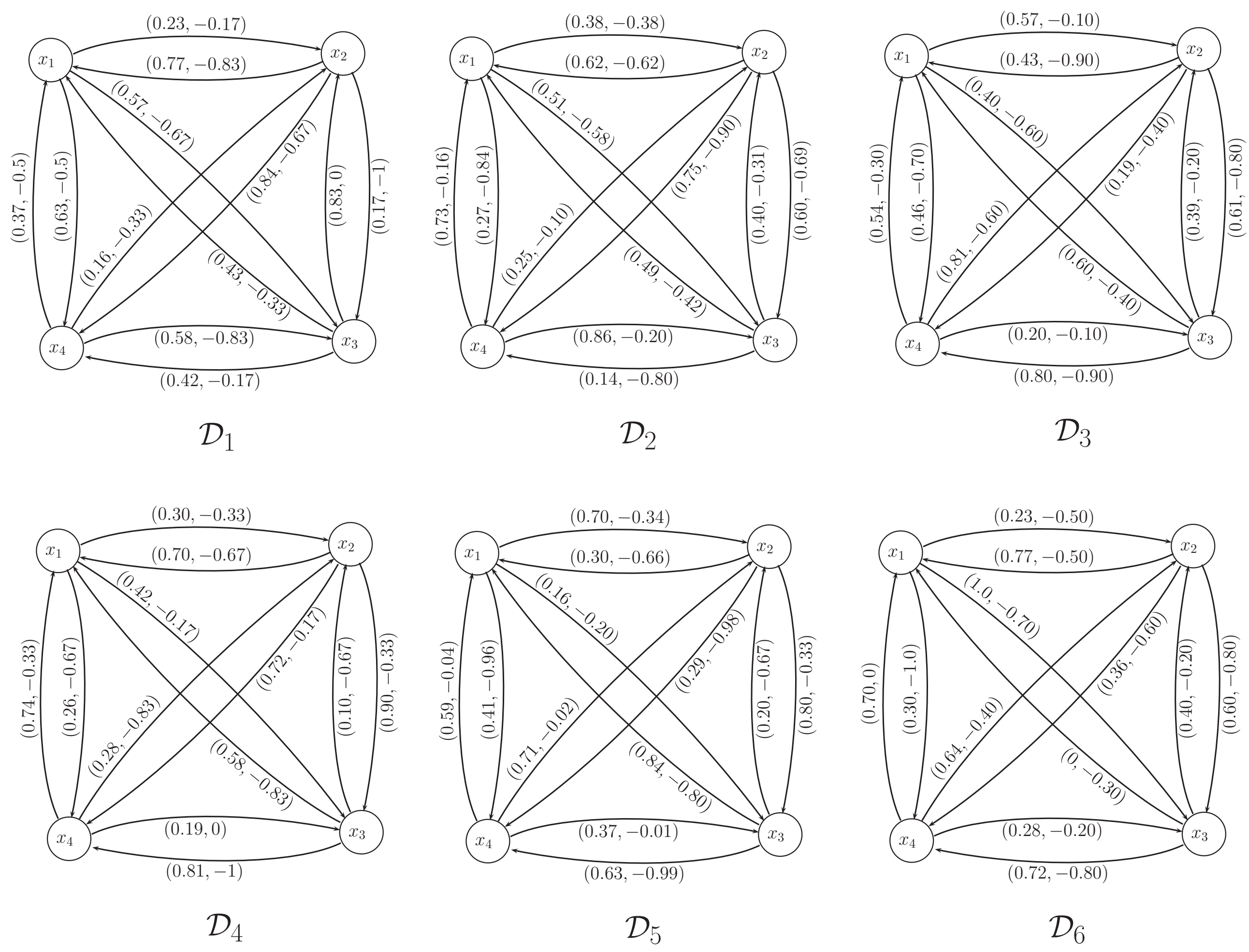


Figure 3: Bipolar fuzzy digraphs

$$R = \sum_{k=1}^{6} w_k R_k =$$
$$\begin{bmatrix} (0.5, -0.5) & (0.4037, -0.2997) & (0.5106, -0.4976) & (0.3907, -0.7719) \\ (0.5963, -0.7004) & (0.5, -0.5) & (0.6103, -0.6697) & (0.5202, -0.5968) \\ (0.4894, -0.5025) & (0.3897, -0.3304) & (0.5, -0.5) & (0.5873, -0.7780) \\ (0.6093, -0.2282) & (0.4798, -0.4033) & (0.4127, -0.2221) & (0.5, -0.5) \end{bmatrix}.$$

Calculate their scores using the score function $s_{ij} = \eta_{ij}^- + \eta_{ij}^+$ [12]:

$$R = \begin{bmatrix} 0 & 0.1040 & 0.0130 & -0.3812 \\ -0.1041 & 0 & -0.0594 & -0.0766 \\ -0.0131 & 0.0593 & 0 & -0.1907 \\ 0.3811 & 0.0765 & 0.1906 & 0 \end{bmatrix}.$$

The net flow of $x_i$, i.e., the net degree of preference of $x_i$ over the other alternatives is

$$\phi(x_i) = \sum_{k=1}^{m} w_k \left( \sum_{j=1 j\neq i}^{n} (r_{ij}^{(k)} - r_{ji}^{(k)}) \right), \ i = 1, 2, \ldots, n.$$

So, the net flows of the four alternatives are $\phi(x_1) = -0.5281, \phi(x_2) = -0.4799$, $\phi(x_3) = -0.2887, \phi(x_4) = 1.2967$, which give the ranking of $x_4 \succ x_3 \succ x_2 \succ x_1$. Thus, the best choice is $x_4$.

## 5. Conclusions

A bipolar fuzzy model provides more precision, flexibility, and compatibility to the system as compared to the classical and fuzzy model. In this paper, we have introduced the concept of energy of a graph in bipolar fuzzy setting and investigated its properties. We have derived the maximal energy of BFGs. We have also introduced the concept of energy of a BFDG along with its application in decision making problem. In further work, it is necessary and meaningful to extend the energy of BFGs to (1) Pythagorean fuzzy graphs, (2) Interval-valued Pythagorean fuzzy graphs, (3) Hesitant fuzzy graphs, and (4) Hesitant Pythagorean fuzzy graphs.

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