ON THE APPLICATION OF THE ADOMIAN DECOMPOSITION METHOD TO SOLVE NON-LINEAR BOUNDARY VALUE PROBLEMS OF A STEADY STATE FLOW OF A LIQUID FILM

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Abstract. This paper shows the reliability of the Adomian Decomposition Method (ADM) for solving a non-linear boundary value problem in a steady state flow of a liquid film. The solutions of the momentum and energy equations are solved through ADM and the results were compared with previously obtained results by the Homotopy Perturbation Method(HPM) and Hermite - Pade Approximation method (HPA). It is observed that solutions obtained by the ADM takes the form of a convergent series that is capable of greatly reducing the size of computation and solve a large class of non-linear equations effectively and accurately. The results of the boundary value problem are presented in tables and graphs.

Keywords: Adomian Decomposition Method (ADM), Variable Viscosity, Homotopy Perturbation Method (HPM) and Hermite - Pade Approximation (HPA).

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1. Introduction

The use of the Adomian Decomposition Method (ADM) has been applied to a wide class of problems in the sciences. The method is useful for obtaining the closed form and numerical approximations of linear or non linear differential equations as in [1-3]. Also, the method has been established to obtain formal solutions of stochastic and deterministic problems in sciences and engineering involving algebraic, differential, integro - differential, differential delay, integral and partial differential equations as in [4]. This method was first introduced by the American Mathematician, George Adomian (1923 - 1996), in search of a solution in the form of a series and on decomposing the non linear operator into a series in which the terms are calculated recursively using Adomian polynomials [5-7]. In addition to that, some modification were introduced recently to extend the reliability of the ADM in solving problems involving differential equations as described in [8-10] where the proposed modification accelerate the rapid convergence of the series solution if compared with the standard Adomian method. Hence, the various applications of the ADM schemes have been established in [11-17].

In industries, issues like productivity and competitiveness require engineering solutions which heavily rely on Mathematical models. These models are used to determine and investigate fluid flow properties like viscous heating, internal heat generation, entropy generation rates, thermal stability and so on as mentioned in literature [10-18]. It is mentioned in [19] that, one of the the major goals in industries is to understand the fluid behaviour and heat transfer accurately in order to predict the flow regime.

In order to have clear understanding of the fluid dynamics in a channel flow between walls, it is extremely important to critically find and choose an approximate as well as accurate method to solutions of different mathematical models under different conditions. However, critical examination into comparative studies relating to the use of the ADM with other methods has been extensively established, for example, [2] compared ADM and Tau Methods, [3] modified ADM to obtain Taylor series, [4] compared ADM with Picard iteration method and [20] made use of the ADM to solve generalized Riccati differential equations.

Meanwhile, there are other methods that have not been compared with the ADM, like the differential transform method [21], the traditional perturbation method [18] and the Homotopy perturbation method [22]. Hence, the purpose of this paper is to compare and show the reliability of ADM in solving a nonlinear boundary value problem in a steady state flow of a liquid film. The present result from ADM shall be compared with the results of [22] where the Homotopy Perturbation Method (HPM) and the Hermite - Pade Approximation (HPA) were previously used.

2. Review of the Adomian Decomposition Method (ADM)

The ADM provides a closed form of the solution where a general functional equation is given as described in [2,17,23]:

$$y - N(y) = f \tag{1}$$

where N is a nonlinear operator, f is a function that is known with the solution y satisfying (1) which is assumed to have a unique solution. We introduce an approximate solution of (1) in the form of

$$y = \sum_{n=0}^{\infty} y_n \tag{2}$$

and decomposing the nonlinear operator N as

$$N(y) = \sum_{n=0}^{\infty} A_n, \tag{3}$$

where A_n are the Adomian Polynomials of (3) as $y_0, y_1, ..., y_n$ that is determined formally from the relation as:

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d^n}}{\mathrm{dz^n}}\left(N\sum_{i=0}^{\infty} z^i y_i\right)\right]_{z=0}. \tag{4}$$

Therefore, we can identify

$$y_0 = f \tag{5}$$

$$y_{n+1} = A_n\left(y_0, y_1, ..., y_n\right), \qquad n = 0, 1, 2, ... \tag{6}$$

3. Application of the Adomian Decomposition Method (ADM)

The dimensionless governing equations of the non-linear boundary value problem in a steady flow of a liquid film together with the boundary conditions in [22] are given as:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \lambda\left(1-y\right)^2 \mathrm{e}^{\beta\theta} = 0 \tag{7}$$

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \left(1-y\right)\mathrm{e}^{\beta\theta} \tag{8}$$

$$\text{with} \qquad u = \theta = 0 \quad \text{on} \quad y = 0 \qquad \text{and} \quad \frac{\mathrm{d}\theta}{\mathrm{d}y} = 0 \quad \text{on} \quad y = 1, \tag{9}$$

where θ is the absolute temperature, u is the velocity, λ is the variable viscosity parameter and β is the Brinkmann number.

For simplicity, we expanded the exponential function in (7) and (8) and take the approximation to be:

$$e^{\beta\theta} \simeq 1 + \beta\theta + \frac{(\beta\theta)^2}{2}. \tag{10}$$

Inserting (10) into (7) and (8) together with the boundary conditions in (9), we obtained the following solution for the energy equation:

$$\begin{aligned} \theta(y) = a_0 y - \lambda\left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2}\right) - \lambda\beta \int_0^y \left(\int_0^y (1-y)^2 \theta(Y) \mathrm{d}Y\right) \mathrm{d}Y \\ - \frac{\lambda\beta^2}{2} \int_0^y \left(\int_0^y (1-y)^2 [\theta(Y)]^2 \mathrm{d}Y\right) \mathrm{d}Y \end{aligned} \tag{11}$$

$$u(y) = y - \frac{y^2}{2} + \beta \int_0^y (1-y)\theta(y) \mathrm{d}Y + \frac{\beta^2}{2} \int_0^y (1-y)(\theta(Y))^2 \mathrm{d}Y, \tag{12}$$

where $a_0 = \frac{\mathrm{d}\theta}{\mathrm{d}y}(0)$ is to be determined by using the boundary condition (9).

We now introduce a series solution of the form

$$\theta(y) = \sum_{n=0}^{\infty} \theta_n(Y) \qquad \text{and} \qquad u(y) = \sum_{n=0}^{\infty} u_n(Y). \tag{13}$$

Substituting (13) into (11) and (12) respectively, then we have:

$$\begin{aligned} \theta(y) = a_0 y - \lambda\left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2}\right) - \lambda\beta \int_0^y \left(\int_0^y (1-y)^2 \sum_{n=0}^{\infty} \theta_n(Y) \mathrm{d}Y\right) \mathrm{d}Y \\ - \frac{\lambda\beta^2}{2} \int_0^y \left(\int_0^y (1-y)^2 \left[\sum_{n=0}^{\infty} \theta_n(Y)\right]^2 \mathrm{d}Y\right) \mathrm{d}Y \end{aligned} \tag{14}$$

$$\begin{aligned} u(y) = y - \frac{y^2}{2} + \beta \int_0^y (1-y)[\sum_{n=0}^{\infty} \theta_n(Y)] \mathrm{d}Y \\ + \frac{\beta^2}{2} \int_0^y (1-y) \left[\sum_{n=0}^{\infty} \theta_n(Y)\right]^2 \mathrm{d}Y. \end{aligned} \tag{15}$$

We let

$$\left[\sum_{n=0}^{\infty} \theta_n(Y)\right]^2 = \sum_{n=0}^{\infty} A_n, \tag{16}$$

where $A_0, A_1, A_2, ...$ are called Adomian polynomials.

Substituting (16) into (14) and (15), we obtain the following:

$$\theta(y) = a_0 y - \lambda\left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2}\right) - \lambda\beta\int_0^y \left(\int_0^y (1-y)^2 \sum_{n=0}^{\infty} \theta_n(Y)\mathrm{d}Y\right)\mathrm{d}Y$$

$$-\frac{\lambda\beta^2}{2}\int_0^y \left(\int_0^y (1-y)^2 \sum_{n=0}^{\infty} A_n \mathrm{d}Y\right)\mathrm{d}Y \tag{17}$$

$$u(y)=y-\frac{y^2}{2}+\beta\int_0^y (1-y)[\sum_{n=0}^{\infty}\theta_n(Y)]\mathrm{d}Y+\frac{\beta^2}{2}\int_0^y \left((1-y)\sum_{n=0}^{\infty} A_n\right)\mathrm{d}Y. \tag{18}$$

The few Adomian polynomials of (16) are given as follows:

$$\begin{aligned} A_0 &= \theta_0(y)^2, \\ A_1 &= 2\theta_0(y)\theta_1(y), \\ A_2 &= 2\theta_0(y)\theta_2(y) + \theta_1(y)^2, \qquad \text{and so on.} \end{aligned} \tag{19}$$

Then, the zeroth component of (17) and (18) following the modification of ADM in [8-10, 15, 16, 23] are given as follows:

$$\theta_0(y) = a_0 y - \lambda\left(\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2}\right) \qquad \text{and} \qquad u_0(y) = y - \frac{y^2}{2} \tag{20}$$

$$\theta_{n+1}(y) = -\lambda\beta\int_0^y \left(\int_0^y (1-y)^2\theta_n(Y)\mathrm{d}Y\right)\mathrm{d}Y$$

$$-\frac{\lambda\beta^2}{2}\int_0^y \left(\int_0^y (1-y)^2 A_n\mathrm{d}Y\right)\mathrm{d}Y \tag{21}$$

$$u_{n+1}(y) = \beta\int_0^y (1-y)\theta_n(Y)\mathrm{d}Y + \frac{\beta^2}{2}\int_0^y ((1-y)A_n)\,\mathrm{d}Y, \text{ for } n \geq 0. \tag{22}$$

Hence, the approximate solutions for temperature and velocity profiles of the boundary value problem using ADM are obtained as follows:

$$\begin{aligned} \theta(y) = a_0(y) &+ \lambda\left(\frac{1}{6}a_0\beta y^4 - \frac{1}{6}a_0\beta y^3 - \frac{y^4}{12} + \frac{y^3}{3} - \frac{y^2}{2}\right) \\ &+ \lambda\left(-\frac{1}{20}a_0\beta y^5 - \frac{1}{120}a_0^2\beta^2(2(y-3)y+5)y^4\right) \\ &+ \lambda^2\left(\frac{\beta y^8}{672} - \frac{\beta y^7}{84} + \frac{\beta y^6}{24} - \frac{\beta y^5}{15} + \frac{\beta y^4}{24}\right) \\ &\frac{+\lambda^2\left(-\left(a_0\beta^2 y^5(y(1344 - 5y(y(7y-54)+180)) - 756)\right)\right)}{30240} + O(\lambda^3) \end{aligned} \tag{23}$$

$$\begin{aligned} u(y) = y &- \frac{y^2}{2} + \frac{1}{504}a_0^2\beta^2(84-63y)y^3 - \frac{1}{3}a_0\beta y^3 + \frac{1}{2}a_0\beta y^2 \\ &+ \lambda\left(\frac{1}{504}a_0\beta^2(y(y(6y-35)+84)-63)y^4 + \frac{\beta y^6}{72} - \frac{\beta y^5}{12} + \frac{5\beta y^4}{24} - \frac{\beta y^3}{6}\right) \\ &- \frac{\beta^2\lambda^2 y^5(y(y(7y((y-10)y+45)-760)+980)-504)}{20160} + O(\lambda^3). \end{aligned} \tag{24}$$

4. Previous works

4.1 The Homotopy Perturbation Method (HPM)

The approximate analytical solutions for temperature and velocity fields from HPM in [22] are given as:

$$\begin{aligned}\theta(y) = \frac{\lambda}{12}\left[1-(1-y)^4\right] + \left(\frac{11\lambda^2\beta}{2016} + \frac{101\lambda^3\beta^2}{1330560}\right) - \frac{\lambda\beta^2}{72}\left(\frac{(1-y)^4}{12} - \frac{(1-y)^8}{56}\right) \\ - \frac{\lambda^3\beta^2}{288}\left(\frac{(1-y)^5}{20} + \frac{(1-y)^12}{132} - \frac{(1-y)^8}{28}\right) \quad \text{and}\end{aligned} \tag{25}$$

$$\begin{aligned}u(y) = \left(y - \frac{y^2}{2}\right) + \frac{\lambda\beta}{72}\left[\left(6y - 3y^2 + (1-y)^6\right) - \left(1 + \frac{7\beta}{120}\right)\right] \\ + \frac{\lambda^2\beta^2}{8640}\left[30y - 15y^2 - 3(1-y)^{10} + 10(1-y)^6\right].\end{aligned} \tag{26}$$

4.2 The Hermite - Pade Approximation (HPA)

The solution for the temperature and velocity fields using HPA in [22] are also given thus:

$$\begin{aligned}\theta(y) &= -\frac{\lambda}{12}y(y-2)(y^2-2y+2) \\ &+ \frac{\lambda^2\beta}{2016}y(y-2)(y^2-2y+2)(3y^4-12y^3+18y^2-12y-8) + O(\lambda^3)\end{aligned} \tag{27}$$

$$\begin{aligned}u(y) &= -\frac{1}{2}y(y-2) + \frac{\lambda\beta}{72}y^2(y-2)^2(y^2-2y+3) \\ &- \frac{\lambda^2\beta^2}{6048}y^2(y-2)^2(3y^6-18y^5+51y^4+7y^2-32y-12) + O(\lambda^3).\end{aligned} \tag{28}$$

5. Results and discussion

The main interest of this study is to establish the reliability and accuracy of the ADM in the solution of a non linear boundary value problem in a steady state and to compare the present result of the ADM with the previously obtained results from the HPM and the HPA in [22].

In tables 1 and 2, the numerical values of solutions from the temperature and velocity profiles obtained from the HPM, HPA and ADM are respectively given with the absolute errors in comparison with the present result from ADM. The numerical solutions showed that the ADM is also another convenient method to get an approximate solution of non linear boundary value problems. The present results from ADM showed the reliability and validity of the method with sizeable number of iterations compared with previously obtained results from HPM and HPA. The absolute errors obtained in each comparison evidently showed that the present ADM results are almost identical with an average difference of order 10^{-4}. This is evident that, results from ADM can conveniently be used as an alternative method to solve non linear problems where exact solutions are not known. Moreso, the assurance of the existence and uniqueness of results from ADM is analysed in [17,24]

Table 1: Comparison of temperature profile between ADM and previous results(HPM and HPA)

$\beta = 1, \lambda = 1$					
y	$\theta_{HPM}(y)$	$\theta_{HPA}(y)$	$\theta_{ADM}(y)$	$\|\theta_{HPM}(y) - \theta_{ADM}(y)\|$	$\|\theta_{HPA}(y) - \theta_{ADM}(y)\|$
0.0	0.00454696	0.00000000	0.00000000	4.54696×10^{-3}	0
0.1	0.03330329	0.03019901	0.03040397	2.89932×10^{-3}	2.04959×10^{-4}
0.2	0.05399523	0.05206157	0.05245361	1.54162×10^{-3}	3.92044×10^{-4}
0.3	0.06828202	0.06719977	0.06774765	5.34382×10^{-4}	5.47873×10^{-4}
0.4	0.07764161	0.07711468	0.07778099	1.39372×10^{-4}	6.66309×10^{-4}
0.5	0.08336393	0.08315313	0.08390098	5.37054×10^{-4}	7.47845×10^{-4}
0.6	0.08654554	0.08647955	0.08727721	7.31679×10^{-4}	7.97668×10^{-4}
0.7	0.08808602	0.08805853	0.08888207	7.96050×10^{-4}	8.23544×10^{-4}
0.8	0.08868590	0.08864524	0.08947920	7.93293×10^{-4}	8.33956×10^{-4}
0.9	0.08884568	0.08878065	0.08961719	7.71510×10^{-4}	8.36535×10^{-4}
1.0	0.08886559	0.08878968	0.08962640	7.60811×10^{-4}	8.36717×10^{-4}

Table 2: Comparison of the velocity profile between ADM and previous results (HPM and HPA)

$\beta = 1, \lambda = 1$					
y	$u_{HPM}(y)$	$u_{HPA}(y)$	$u_{ADM}(y)$	$\|u_{HPM}(y) - u_{ADM}(y)\|$	$\|u_{HPA}(y) - u_{ADM}(y)\|$
0.0	0.00000000	0.00000000	0.00000000	0	0
0.1	0.09642260	0.09649557	0.09650809	8.54821×10^{-5}	1.25197×10^{-5}
0.2	0.18483294	0.18509391	0.18515536	3.22413×10^{-4}	6.14393×10^{-5}
0.3	0.26419672	0.26471145	0.26487414	6.77422×10^{-4}	1.62687×10^{-4}
0.4	0.33377860	0.33456914	0.33489208	1.11348×10^{-3}	3.22944×10^{-4}
0.5	0.39308777	0.39414324	0.39467937	1.59160×10^{-3}	5.36129×10^{-4}
0.6	0.44182085	0.44310865	0.44389284	2.07199×10^{-3}	7.84194×10^{-4}
0.7	0.47980842	0.48128330	0.48232310	2.51468×10^{-3}	1.03980×10^{-3}
0.8	0.50696856	0.50857908	0.50984842	2.87987×10^{-3}	1.26934×10^{-3}
0.9	0.52326969	0.52496199	0.52639763	3.12794×10^{-3}	1.43564×10^{-3}
1.0	0.52870370	0.53042328	0.53192311	3.21941×10^{-3}	1.49983×10^{-3}

The effects of the Brinkmann number (λ) and the viscosity parameter (β) on the temperature profiles are displayed in figures 1 and 2. The result is approximately the same obtained in [22]. The results showed that the fluid temperature increases as both Brinkmann number (λ) and the viscosity parameter (β) increase. Similarly, figures 3 and 4 displayed the velocity profile. The results also showed that the maximum velocity are obtained as both Brinkmann number (λ) and the viscosity parameter (β) increase in values. Both temperature and

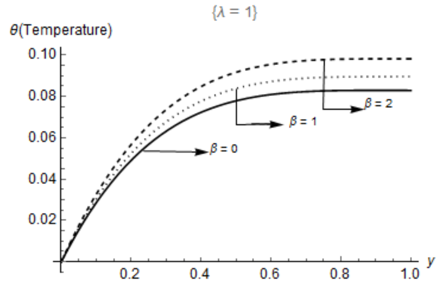

Figure 1: Effect of β on temperature profile

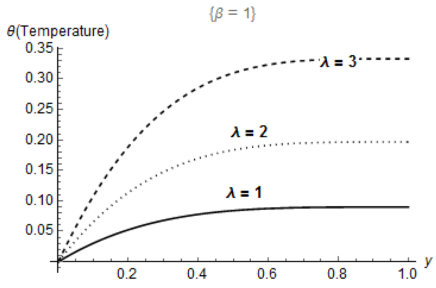

Figure 2: Effect of λ on temperature profile

velocity profiles agreed with the previously obtained results where HPM and HAM were used.

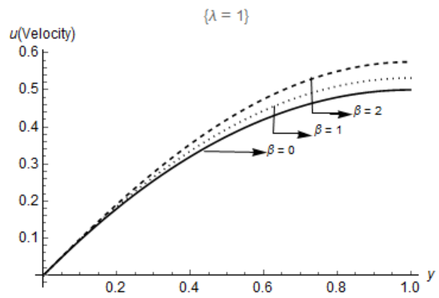

Figure 3: Effect of β on velocity profile

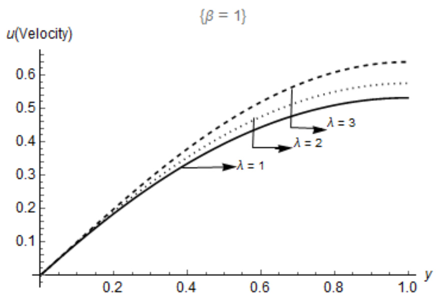

Figure 4: Effect of λ on velocity profile

6. Conclusion

In this study, the ADM has been applied to obtain the solution of a non linear boundary value problem in a steady state flow of a liquid film. The solutions of both temperature and velocity from the ADM were compared with the previously obtained results from [22] where the HPM and the HPA were formerly used. The results showed that the ADM can also be used as an alternative method of getting an approximate and reliable solutions to linear and nonlinear differential equations, and therefore can be applied to wide range of problems in the fields of science and engineering.

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