SOLUTION OF LINEAR AND NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS USING LEGENDRE WAVELET METHOD

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Abstract. In this paper, we utilize a robust and precise method for solving both linear and nonlinear singular initial or boundary value problems. We use Legendre wavelets to construct operational matrix of integration and product operational matrix to solve the problems. This method reduces the problems into algebraic equations and gives a fast convergent series of easily computable components. Illustrative examples are incorporated to show the productivity and exactness of the technique. The outcomes obtained by the utilized method demonstrate that the proposed way is entirely sensible when compared with exact solution.

Keywords: Legendre wavelets, operational matrix of integration, product operational matrix, singular value problems, MATLAB.

1. Introduction

The investigation of singular initial as well as boundary value problems of the second order ordinary differential equations (ODEs) have attracted a special attention of many mathematicians and physicists. One of the equations describing

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this type is formulated as

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t), \quad a < t < b, \tag{1.1}$$

with the initial conditions

$$y(a) = \alpha_1, \quad y'(a) = \alpha_2$$

or boundary conditions

$$y(a) = \alpha_1, \quad y(b) = \alpha_2,$$

where a, b, α_1 and α_2 are finite constants, whereas the coefficients $p(t), q(t)$ and $f(t)$ are given continuous functions. Eq.(1.1) with singularities appears to many phenomena in mathematical physics, astrophysics, chemistry and mechanics such as in the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere, and theory of thermionic currents. While making the model in these fields, singularity typically occurs at the end of the interval of integration which causes the computed solution in losing its accuracy in the vicinity of the singular points. A systematic study of the formulation of these models and the physical structure of the solutions can be found in [5, 7, 10, 12]. Due to the presence of singularity, it is not easy to derive the analytical or exact solutions of most of the singular boundary value problems, so that it requires an efficient and accurate numerical method for solving these kind of problems. A number of mathematicians and physicists have attempted to solve these sort of problems which are available in the literature. Biazar et al. [3] used He's variational iteration method for solving linear cum nonlinear ordinary differential equations. Lu [18] employed the variational iteration method for solving a nonlinear system of second-order boundary value problems. Rashidinia et al. [22] suggested a convergence of cubic-spline technique for solving boundary-value problems, while B-spline approach has been used by Caglar et al. [4]. Later on, Tatari et al. [26] generalized He's variational iteration method for solving the same systems of singular initial value problems. Homotopy perturbation with reproducing kernel has been applied by Geng and Cui [11] whereas Akinfenwa et al. [1] have employed the continuous block backward differentiation formula for solving the same behaviour Stiff differential equations. Again Dehghan and Nikpour [9] showed their interest to solve singular initial value problem by differential quadrature collocation method based on local basis functions. Moreover, Kadalbajoo and Aggarwal [14] have also employed a method based on Chebyshev polynomials coupled with B-spline for solving singular boundary value problems in which they have suggested the economized expansion procedure to avoid the singularity from the singular BVPs. Although all these methods has successfully solved these type of problems but still small convergence rate and high computational efforts compel us to use a high accuracy and low computational method. For more details, refer [16] and the references therein.

In recent years, wavelet theory [8] have received a special attention by researchers from both theoretical and practical point of view in different fields of science and engineering. It possesses several beautiful properties like orthogonality, compact support, accurate representation of concern polynomial at a certain degree and ability to represent different type of functions at different levels of resolution. In addition, wavelet method has made a connection with fast numerical algorithms [2]. From the last decades many wavelet methods such as Haar wavelet method [6, 7, 17], Daubechies wavelet method [8, 20], Haar wavelet collocation method [25], Chebyshev wavelet method [13, 15], Legendre wavelet method [23, 24, 28] and Laguerre wavelet method [29] were used to solve differential equations but among them Legendre wavelet method attracted much attention. The key strength of Legendre wavelet method (LWM) is to convert the differential equations into a system of algebraic equations by the operational matrices of integral.

In this paper, we use Legendre wavelet method to solve the singular initial or boundary value problems by using operational matrix of integration along with product operational matrix. This paper is organized as follows. In Section 2, we present the Legendre wavelets, their approximation and the operational matrix of integration. The derivation of product operational matrix and the convergence analysis of Legendre wavelets is discussed in Section 3. Several numerical examples are included in Section 4 to show the efficiency and accuracy of our proposed method. In the end, conclusions are appended.

2. Definition and properties of Legendre wavelets

In this section, we state some necessary definitions and preliminaries of Legendre wavelet theory which are required for establishing our results.

2.1 Wavelets and Legendre wavelets

For last two decades, wavelets have found their ways towards recent fields of science and technology. Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet $\psi(t)$ as:

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right) \quad a,\, b \in \mathbb{R},\ a \neq 0.$$

If we choose $a = a^{-k}$ and $b = nba^{-k}$ where $a > 1$, $b > 0$ and $n, k \in \mathbb{Z}^+$, then we get the following family of discrete wavelets as

$$\psi_{k,n}(t) = |a|^{\frac{k}{2}}\, \psi(a^k t - nb),$$

where $\psi_{k,n}(t)$ forms a basis for $L^2(\mathbb{R})$. In particular, when $a = 2$ and $b = 1$ then $\psi_{k,n}(t)$ forms an orthonormal basis.

Legendre wavelets $\psi_{n,m} = \psi(k, \acute{n}, m, t)$ have four arguments; $k = 2, 3, \ldots,$ $\acute{n} = 2n-1$, $n = 1, 2, 3, \ldots, 2^{k-1}$, m is the order of Legendre polynomials and t denotes the time. Legendre wavelets defined in [24] are as follows:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{(m+\frac{1}{2})}2^{\frac{k}{2}}T_m(2^kt-2n+1), & t \in [\frac{2n-2}{2^k}, \frac{2n}{2^k}], \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $m = 0, 1, 2, \ldots, M-1$. The coefficient $\sqrt{(m+\frac{1}{2})}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = \acute{n}2^{-k}$. Here, $T_m(x)$ are well-known Legendre polynomials of order m which are orthogonal with respect to the weight function $w(t) = 1$ on the interval [-1,1] and can be determined with the aid of the following recurrence formulae:

$$T_0(t) = 1, T_1(t) = t,$$

$$T_{m+1}(t) = \left(\frac{2m+1}{m+1}\right)tT_m(t) - \left(\frac{m}{m+1}\right)T_{m-1}(t), \quad m = 1, 2, 3, \ldots.$$

The set of Legendre wavelets $\psi_{n,m}(t)$ also forms an orthonormal set with respect to the weight function $w(t) = 1$.

2.2 Function approximation

Any function f defined over $[0, 1)$ can be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t), \tag{2.2}$$

where

$$c_{n,m} = \langle f(t), \psi_{n,m}(t)\rangle = \int_0^t f(t)\psi_{n,m}(t)dt. \tag{2.3}$$

The series in Eq.(2.2) is truncated, which can then be written as

$$f(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) = C^T\Psi(t), \tag{2.4}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices are given by

$$\begin{aligned} C = [&c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, c_{21}, \ldots, c_{2M-1}, \ldots, \\ &c_{2^{k-1}0}, c_{2^{k-1}1}, \ldots, c_{2^{k-1}M-1}]^T, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \Psi(t) = [&\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M-1}(t), \psi_{20}(t), \psi_{21}(t), \ldots, \\ &\psi_{2M-1}(t), \ldots, \psi_{2^{k-1}0}(t), \psi_{2^{k-1}1}(t), \ldots, \psi_{2^{k-1}M-1}(t)]^T. \end{aligned} \tag{2.6}$$

The suitable collocation points depends on resolution are as follows:

$$t_i = \frac{2i-1}{2^kM}, \quad i = 1, 2, \ldots, 2^{k-1}M. \tag{2.7}$$

2.3 Operational matrix of integration

The integration of $\Psi(t)$ defined in Eq.(2.6) can be approximated by Legendre wavelet series with Legendre wavelet coefficient matrix P as

$$\int_0^t \Psi(t)dt = P\Psi(t). \tag{2.8}$$

Now we construct the structure of operational matrix of integration for $\psi(t)$ which is defined in Eq.(2.1). To illustrate the calculation procedure, we choose $M = 3$, $k = 2$. On the basis of chosen values we get the six basis function which are given by

$$\Psi_1 = \psi_{1,0} = \begin{cases} 2^{1/2}, & \text{for } t \in [0, \frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases}, \ \Psi_2 = \psi_{1,1} = \begin{cases} 6^{1/2}(4t-1), & \text{for } t \in [0, \frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases}$$

$$\Psi_3 = \psi_{1,2} = \begin{cases} (10)^{1/2}\left[\frac{3}{2}(4t-1)^2 - \frac{1}{2}\right], & \text{for } t \in [0, \frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases},$$

$$\Psi_4 = \psi_{2,0} = \begin{cases} 2^{1/2}, & \text{for } t \in [\frac{1}{2}, 1) \\ 0, & \text{otherwise.} \end{cases}$$

$$\Psi_5 = \psi_{2,1} = \begin{cases} 6^{1/2}(4t-3), & \text{for } t \in [\frac{1}{2}, 1) \\ 0, & \text{otherwise.} \end{cases},$$

$$\Psi_6 = \psi_{2,2} = \begin{cases} (10)^{1/2}\left[\frac{3}{2}(4t-3)^2 - \frac{1}{2}\right], & \text{for } t \in [\frac{1}{2}, 1) \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\Psi_{6\times 6} = \begin{bmatrix} \sqrt{2} & \frac{-2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} & 0 & 0 & 0 \\ \sqrt{2} & 0 & \frac{\sqrt{10}}{2} & 0 & 0 & 0 \\ \sqrt{2} & \frac{2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \frac{-2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} \\ 0 & 0 & 0 & \sqrt{2} & 0 & \frac{\sqrt{10}}{2} \\ 0 & 0 & 0 & \sqrt{2} & \frac{2}{3}\sqrt{6} & \frac{\sqrt{10}}{6} \end{bmatrix}. \tag{2.9}$$

So, integrating the above defined basis $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6$ from 0 to t and using wavelet coefficients, we obtain

$$\int_0^t \Psi_1(t)dt = \begin{cases} 2^{1/2}t, & t \in [0, \frac{1}{2}) \\ \frac{2^{1/2}}{2}, & t \in [\frac{1}{2}, 1) \end{cases} = \left[\frac{1}{4}, \frac{2^{1/2}}{4 \times 6^{1/2}}, 0, \frac{1}{2}, 0, 0\right]^T \Psi_6(t),$$

$$\int_0^t \Psi_2(t)dt = \begin{cases} 2 \times 6^{1/2}t^2 - 6^{1/2}t, & t \in [0, \frac{1}{2}) \\ 0, & t \in [\frac{1}{2}, 1) \end{cases}$$
$$= \left[-\frac{3^{1/2}}{12}, 0, \frac{3^{1/2}}{12 \times 5^{1/2}}, 0, 0, 0\right]^T \Psi_6(t),$$

Similarly, we have

$$\int_0^t \Psi_3(t)dt = \left[0, -\frac{5^{1/2}}{20 \times 3^{1/2}}, 0, 0, 0, 0\right]^T \Psi_6(t),$$
$$\int_0^t \Psi_4(t)dt = \left[0, 0, 0, \frac{1}{4}, \frac{2^{1/2}}{4 \times 5^{1/2}}, 0\right]^T \Psi_6(t),$$
$$\int_0^t \Psi_5(t)dt = \left[0, 0, 0, \frac{-3^{1/2}}{12}, 0, \frac{3^{1/2}}{12 \times 5^{1/2}}\right]^T \Psi_6(t),$$
$$\int_0^t \Psi_6(t)dt = \left[0, 0, 0, 0, \frac{-5^{1/2}}{20 \times 3^{1/2}}, 0,\right]^T \Psi_6(t).$$

Due to the support of $\Psi_i,\ i = 1, 2, \ldots 6$, it is obvious that we have matrix P in [24] as

$$P_{6\times 6} = \frac{1}{4}\begin{bmatrix} 1 & \frac{2^{1/2}}{6^{1/2}} & 0 & 2 & 0 & 0 \\ \frac{-3^{1/2}}{3} & 0 & \frac{3^{1/2}}{3 \times 5^{1/2}} & 0 & 0 & 0 \\ 0 & \frac{-5^{1/2}}{5 \times 3^{1/2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{2^{1/2}}{6^{1/2}} & 0 \\ 0 & 0 & 0 & \frac{-3^{1/2}}{3} & 0 & \frac{3^{1/2}}{3 \times 5^{1/2}} \\ 0 & 0 & 0 & 0 & \frac{-5^{1/2}}{5 \times 3^{1/2}} & 0 \end{bmatrix}. \tag{2.10}$$

For general case, we have

$$P_{2^{k-1}M \times 2^{k-1}M} = \frac{1}{2^k}\begin{bmatrix} L & F & F & \ldots & F \\ O & L & F & \ddots & \vdots \\ O & O & L & \ddots & F \\ \vdots & \ddots & \ddots & \ddots & F \\ O & \ldots & O & O & L \end{bmatrix}, \tag{2.11}$$

where L, F and O are $M \times M$ matrices given by

$$L = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} & 0 & \dots & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \dots & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{-\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & \ddots & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & \dots & \dots & 0 & \frac{-\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

and

$$O = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

3. The product operational matrix

The following properties of product of two Legendre wavelet functions are also used for solving differential as well as integral equations:

$$C^T \Psi(t) \Psi^T(t) \approx \tilde{C} \Psi^T(t), \tag{3.1}$$

where C and $\Psi(t)$ are given in Eq.(2.5) and Eq.(2.6), respectively, and $\tilde{C}$ is $(2^{k-1}M) \times (2^{k-1}M)$ product operational matrix. To illustrate the calculation procedure, we choose $M = 3$, $k = 2$ and using $\Psi(t)$ as defined in Eq.(2.6), we have

$$\Psi(t)\Psi^T(t) = \begin{bmatrix} \psi_{10}\psi_{10} & \psi_{10}\psi_{11} & \psi_{10}\psi_{12} & \psi_{10}\psi_{20} & \psi_{10}\psi_{21} & \psi_{10}\psi_{22} \\ \psi_{11}\psi_{10} & \psi_{11}\psi_{11} & \psi_{11}\psi_{12} & \psi_{11}\psi_{20} & \psi_{11}\psi_{21} & \psi_{11}\psi_{22} \\ \psi_{12}\psi_{10} & \psi_{12}\psi_{11} & \psi_{12}\psi_{12} & \psi_{12}\psi_{20} & \psi_{12}\psi_{21} & \psi_{12}\psi_{22} \\ \psi_{20}\psi_{10} & \psi_{20}\psi_{11} & \psi_{20}\psi_{12} & \psi_{20}\psi_{20} & \psi_{20}\psi_{21} & \psi_{20}\psi_{22} \\ \psi_{21}\psi_{10} & \psi_{21}\psi_{11} & \psi_{21}\psi_{12} & \psi_{21}\psi_{20} & \psi_{21}\psi_{21} & \psi_{21}\psi_{22} \\ \psi_{22}\psi_{10} & \psi_{22}\psi_{11} & \psi_{22}\psi_{12} & \psi_{22}\psi_{20} & \psi_{22}\psi_{21} & \psi_{22}\psi_{22} \end{bmatrix}. \tag{3.2}$$

As we know, the support of $\psi_{m,n}$, the entries of vector $\Psi(t)$, are the intervals $\left[\frac{2n-2}{2^k}, \frac{2n}{2^k}\right]$, therefore $\psi_{ij}\psi_{kl} = 0$ if $i \neq k$. We also have $\psi_{i0}\psi_{ij} = 2^{1/2}\psi_{ij}$, $\psi_{i1}\psi_{i1} = \dfrac{4}{10^{1/2}}\psi_{i2} + 2^{1/2}\psi_{i0}$.

If we retain only the elements of $\Psi(t)$, then we have

$$\Psi(t)\Psi^T(t) = \begin{bmatrix} 2^{1/2}\psi_{10} & 2^{1/2}\psi_{11} & 2^{1/2}\psi_{12} & 0 & 0 & 0 \\ 2^{1/2}\psi_{11} & 2^{1/2}\psi_{10} + \frac{4}{10^{1/2}}\psi_{12} & \frac{4}{10^{1/2}}\psi_{11} & 0 & 0 & 0 \\ 2^{1/2}\psi_{12} & \frac{4}{10^{1/2}}\psi_{11} & 2^{1/2}\psi_{10} + \frac{20}{7\times 10^{1/2}}\psi_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2^{1/2}\psi_{20} & 2^{1/2}\psi_{21} & 2^{1/2}\psi_{22} \\ 0 & 0 & 0 & 2^{1/2}\psi_{21} & 2^{1/2}\psi_{20} + \frac{4}{10^{1/2}}\psi_{22} & \frac{4}{10^{1/2}}\psi_{21} \\ 0 & 0 & 0 & 2^{1/2}\psi_{22} & \frac{4}{10^{1/2}}\psi_{21} & 2^{1/2}\psi_{20} + \frac{20}{7\times 10^{1/2}}\psi_{22} \end{bmatrix}.$$

Therefore the 6×6 matrix $\tilde{C}$ in Eq.(3.1) can be written as

$$\tilde{C} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \tag{3.3}$$

where $B_i,\ i = 1, 2$, are 3×3 matrices given by

$$B_i = \begin{bmatrix} 2^{1/2}c_{i0} & 2^{1/2}c_{i1} & 2^{1/2}c_{i2} \\ 2^{1/2}c_{i1} & 2^{1/2}c_{i0} + \frac{4}{10^{1/2}}c_{i2} & \frac{4}{10^{1/2}}c_{i1} \\ 2^{1/2}c_{i2} & \frac{4}{10^{1/2}}c_{i1} & 2^{1/2}c_{i0} + \frac{20}{7\times 10^{1/2}}c_{i2} \end{bmatrix}$$

where $c_{i,d}$, $d = 0, 1, 2$ are taken from Eq.(2.5), for more information, one can refer to [24].

Now, we state the following result regarding the convergence of Legendre wavelet method for expansion of any continuous function $f(t) \in L^2(\mathbb{R})$.

Theorem 3.1 ([27]). *The series solution defined in Eq.* (2.4) *of Eq.* (1.1) *using Legendre wavelet method converges to* $y(t)$.

4. Numerical experiment and discussion

In order to show the effectiveness of Legendre wavelet method (LWM), we implement LWM to many linear and nonlinear singular ordinary differential equations. All the numerical experiments were carried out with MATLAB R2010b and MAPLE 14 codes.

Example 4.1 ([19]). Consider the linear singular initial value problem

$$y''(t) + \frac{2}{t}y'(t) - 10y(t) = 12 - 10t^4, \quad 0 < t < 1, \tag{4.1}$$

with initial conditions as

$$y(0) = 0,\ \ y'(0) = 0.$$

First we assume that the unknown function $y''(t)$ is given by

$$y''(t) = C^T\Psi(t). \tag{4.2}$$

Integrating Eq.(4.2) from 0 to t and using boundary conditions, we have

$$y'(t) = C^T P \Psi(t) + y'(0). \tag{4.3}$$

Again integrating Eq.(4.3) from 0 to t, we obtain

$$y(t) = C^T P^2 \Psi(t) + t y'(0) + y(0). \tag{4.4}$$

We can express $\frac{2}{t}$ and $12 - 10t^4$ as

$$\begin{aligned} \frac{2}{t} &= \left[\frac{32}{5}\sqrt{2}, \frac{-12}{5}\sqrt{6}, \frac{24}{25}\sqrt{10}, \frac{320}{231}\sqrt{2}, \frac{-12}{77}\sqrt{6}, \frac{8}{382}\sqrt{10}\right]^T \Psi(t) \\ &= H_1^T \Psi(t), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} 12 - 10t^4 &= \left[\frac{41047}{6912}\sqrt{2}, \frac{-65}{1728}\sqrt{6}, \frac{-29}{1728}\sqrt{10}, \frac{28087}{6912}\sqrt{2}, \frac{-425}{576}\sqrt{6}, \frac{-245}{1728}\sqrt{10}\right]^T \Psi(t) \\ &= H_2^T \Psi(t). \end{aligned} \tag{4.6}$$

Now substituting Eq.(4.2) to Eq.(4.6) in Eq.(4.1), we get

$$C^T \Psi(t) + H_1^T \Psi(t) C^T P \Psi(t) - 10 C^T P^2 \Psi(t) = H_2^T \Psi(t).$$

Thus with the orthonormality of Legendre wavelet and Eq.(3.1), we have

$$C I_{6\times 6} + \tilde{H}_1 P C - 10 P^{2T} C = H_2^T. \tag{4.7}$$

where $\tilde{H}_1$ can be calculated similarly to Eq.(3.3) and I is the identity matrix. Eq.(4.7) is a set of algebraic equations which can be solved for C. Substituting the value of C in Eq.(4.4), we get the solution of Eq.(4.1). The obtained numerical solution of Eq.(4.1) is presented in comparison with the ADM [19] and exact solution $y(t) = 2t^2 + t^4$ in Table 4.1 and graphically shown in Figure 4.1.

t	LWM Result	Ref.[19]	Exact
0.1	0.02020000	0.02009999	0.02010000
0.2	0.08162990	0.08159992	0.08160000
0.3	0.18811922	0.18809781	0.18810000
0.4	0.34557010	0.34557801	0.34560000
0.5	0.56260300	0.56236789	0.56250000
0.6	0.84980100	0.84902639	0.84960000
0.7	1.22051000	1.21810873	1.22010000
0.8	1.68980100	1.68372918	1.68960000
0.9	2.27620000	2.26081676	2.27610000

Table 4.1 Comparison of the approximate solution of Example 4.1 against the exact and ADM [19] solutions for $M = 3, k = 2$.

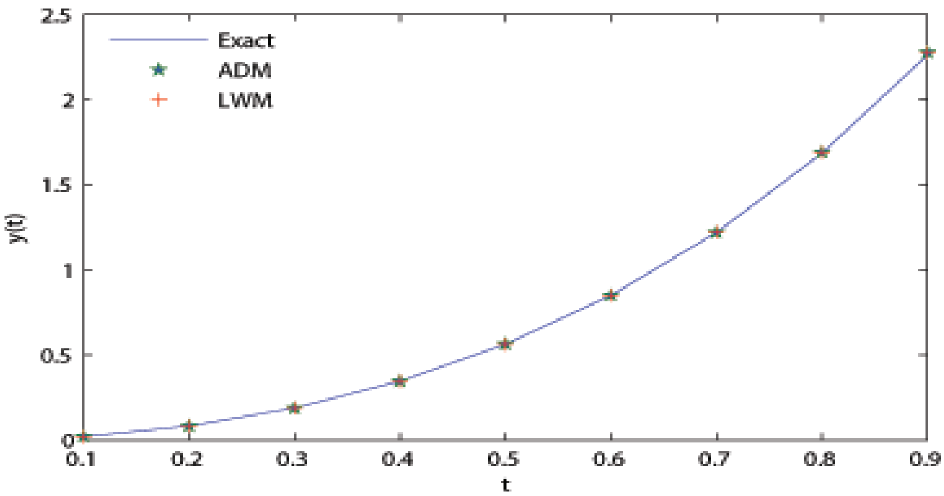

Figure 4.1 Comparison of numerical and exact solution.

Example 4.2 ([19])**.** Consider the linear singular initial value problem

$$y''(t) + \frac{4}{t}y'(t) + \frac{2}{t^2}y(t) = 12, \quad 0 < t < 1, \tag{4.8}$$

with initial conditions as

$$y(0) = 0, \;\; y'(0) = 0.$$

We can express $\frac{4}{t}$ and $\frac{2}{t^2}$ as

$$\begin{aligned} \frac{4}{t} &= \left[\frac{64}{5}\sqrt{2}, \frac{-24}{5}\sqrt{6}, \frac{48}{25}\sqrt{10}, \frac{640}{231}\sqrt{2}, \frac{-24}{77}\sqrt{6}, \frac{16}{385}\sqrt{10}\right]^T \Psi(t) \\ &= H_3^T\Psi(t). \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \frac{2}{t^2} &= \left[\frac{1504}{25}\sqrt{2}, \frac{-864}{25}\sqrt{6}, \frac{2208}{125}\sqrt{10}, \frac{106336}{53361}\sqrt{2}, \frac{-2592}{5929}\sqrt{6}, \frac{7648}{88935}\sqrt{10}\right]^T \Psi(t) \\ &= H_4^T\Psi(t). \end{aligned} \tag{4.10}$$

We follow the procedure outlined in Example 4.1 and obtain the algebraic equation as

$$C^T\Psi(t) + H_3^T\Psi(t)C^TP\Psi(t) + H_4^TC^TP^2\Psi(t) = 12.$$

Thus using the orthonormality of Legendre wavelet and Eq.(3.1),we have

$$CI_{6\times 6} + \tilde{H}_3PC + \tilde{H}_4P^{2T}C = 12. \tag{4.11}$$

Where $\tilde{H}_3$ and $\tilde{H}_4$ can be calculated similarly to Eq.(3.3) and I is the identity matrix. Eq.(4.11) is a set of algebraic equations which can be solved for C. Substituting the value of C in Eq.(4.4), we get the solution of Eq.(4.8). The obtained numerical solution of Eq.(4.8) is compared with the ADM [19] and exact solution $y(t) = t^2$ in Table 4.2 and graphically shown in Figure 4.2.

t	LWM Result	Ref.[19]	Exact
0.1	0.010000	0.010000	0.010000
0.2	0.040000	0.040000	0.040000
0.3	0.090000	0.090000	0.090000
0.4	0.160000	0.160000	0.160000
0.5	0.250000	0.250000	0.250000
0.6	0.360000	0.360000	0.360000
0.7	0.490000	0.490000	0.490000
0.8	0.490000	0.490000	0.490000
0.9	0.810000	0.810000	0.810000

Table 4.2 Comparison of the approximate solution of Example 4.2 against the exact and ADM [19] solutions for $M = 3, k = 2$.

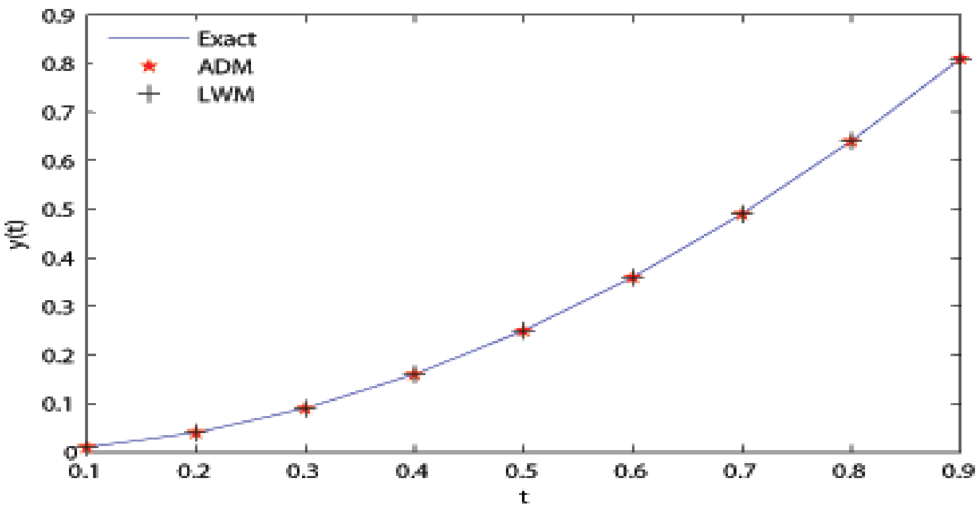

Figure 4.2 Comparison of numerical and exact solution.

Example 4.3 ([19]). Consider the linear singular initial value problem

$$y''(t) + \frac{4}{t}y'(t) + \left(\frac{2}{t^2} + t\right) y(t) = 20t + t^4, \quad 0 < t < 1, \tag{4.12}$$

with initial conditions as

$$y(0) = 0, \;\; y'(0) = 0.$$

We can express $\frac{4}{t}$ and $\left(\frac{2}{t^2}+t\right)$ as

$$\frac{2}{t^2+t} = \Big[60.2850\sqrt{2}, 34.5183\sqrt{6}, 17.6640\sqrt{10}, 2.3678\sqrt{2}, 0.3955\sqrt{6},$$
$$0.0860\sqrt{10}\Big]\,\Psi(t) = H_5^T\Psi(t). \tag{4.13}$$

$$\left(20t+t^4\right) = \Big[2.5061\sqrt{2}, 0.8371\sqrt{6}, 0.0017\sqrt{10}, 7.6936\sqrt{2},$$
$$0.9071\sqrt{6}, 0.0142\sqrt{10}\Big]\,\Psi(t) = H_6^T\Psi(t). \tag{4.14}$$

The given algebraic equation follows from the procedure configured in Example 4.1 as

$$C^T\Psi(t) + H_3^T\Psi(t)C^TP\Psi(t) + H_5^TC^TP^2\Psi(t) = H_6^T\Psi(t).$$

Applying the orthonormality condition of Legendre wavelet and Eq.(3.1),we have

$$CI_{6\times 6} + \tilde{H}_3PC + \tilde{H}_5P^{2T}C = H_6^T. \tag{4.15}$$

Where $\tilde{H}_3$ and $\tilde{H}_5$ can be calculated similarly to Eq.(3.3) and I is the identity matrix. Eq.(4.15) is a set of algebraic equations which can be solved for C. Substituting the value of C in Eq.(4.4), we get the solution of Eq.(4.12). The obtained numerical solution of Eq.(4.12) is presented in comparison with the ADM [19] and exact solution $y(t) = t^3$ in Table 4.3 and graphically in Figure 4.3.

t	LWM Results	Ref.[19]	Exact
0.1	0.001000	0.001000	0.001000
0.2	0.008003	0.008001	0.008000
0.3	0.027008	0.027013	0.027000
0.4	0.064045	0.064073	0.064000
0.5	0.125131	0.125279	0.125000
0.6	0.216534	0.216834	0.216000
0.7	0.345017	0.345107	0.343000
0.8	0.513002	0.516702	0.512000
0.9	0.730000	0.738553	0.729000

Table 4.3 Comparison of the approximate solution of Example 4.3 against the exact and ADM [19] solutions for $M = 3, k = 2$.

Example 4.4 ([21])**.** Consider the linear singular initial value problem

$$y''(t) + \frac{1}{t}y'(t) = \left(\frac{8}{8-t^2}\right)^2, \quad 0 < t < 1, \tag{4.16}$$

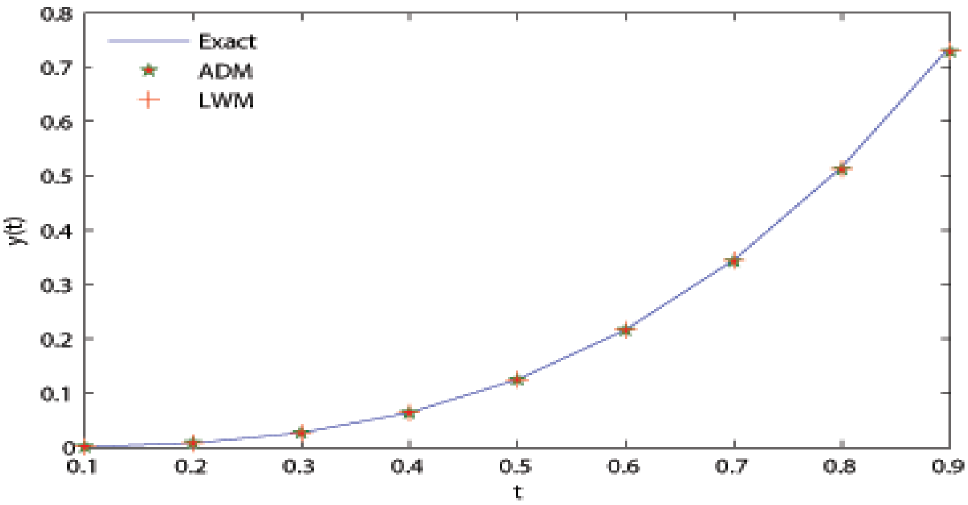

Figure 4.3 Comparison of numerical and exact solution.

with initial conditions as

$$y(0) = 0, \;\; y'(0) = 0.$$

We can express $\frac{1}{t}$ and $(\frac{8}{8-t^2})^2$ as

$$\begin{aligned} \frac{1}{t} &= \left[\frac{16}{5}\sqrt{2}, -\frac{6}{5}\sqrt{6}, \frac{12}{25}\sqrt{10}, \frac{160}{231}\sqrt{2}, -\frac{6}{77}\sqrt{6}, \frac{4}{385}\sqrt{10}\right]\Psi(t) \\ &= H_7^T\Psi(t). \end{aligned} \tag{4.17}$$

$$\begin{aligned} \left(\frac{8}{8-t^2}\right)^2 = \Big[&0.5107\sqrt{2}, 0.0054\sqrt{6}, 0.0011\sqrt{10}, 0.5832\sqrt{2}, \\ &0.0197\sqrt{6}, 0.0019\sqrt{10}\Big]\Psi(t) = H_8^T\Psi(t). \end{aligned} \tag{4.18}$$

We follow the strategy defined in Example 4.1 and obtain the algebraic equation as

$$C^T\Psi(t) + H_7^T\Psi(t)C^TP\Psi(t) = H_8^T\Psi(t).$$

Eq.(3.1) and orthonormality of Legendre wavelet together give putting, we have

$$CI_{6\times 6} + \tilde{H}_7PC = H_8^T. \tag{4.19}$$

Where $\tilde{H}_7$ can be calculated similarly to Eq.(3.3) and I is the identity matrix. Eq.(4.19) is a set of algebraic equations which can be solved for C. Substituting the value of C in Eq.(4.4), we get the solution of Eq.(4.16). The obtained numerical solution of Eq.(4.16) is presented in comparison with the exact solution $y(t) = 2\log(\frac{7}{8-t^2})$ in Table 4.4 and graphically shown in Figure 4.4.

t	LWM Results	Ref.[21]	Exact
0.1	-0.26456123	-0.26456122	-0.26456122
0.2	-0.25703772	-0.25703770	-0.25703770
0.3	-0.24443526	-0.24443526	-0.24443526
0.4	-0.22665738	-0.22665736	-0.22665737
0.5	-0.20356540	-0.20356538	-0.20356538
0.6	-0.17497491	-0.17497491	-0.17497490
0.7	-0.14065064	-0.14065063	-0.14065063
0.8	-0.10029958	-0.10029956	-0.10029956
0.9	-0.53562045	-0.53562045	-0.53562045

Table 4.4 Comparison of the approximate solution of Example 4.4 against the exact and Chebyshev wavelet [21] solutions for $M = 3, k = 2$.

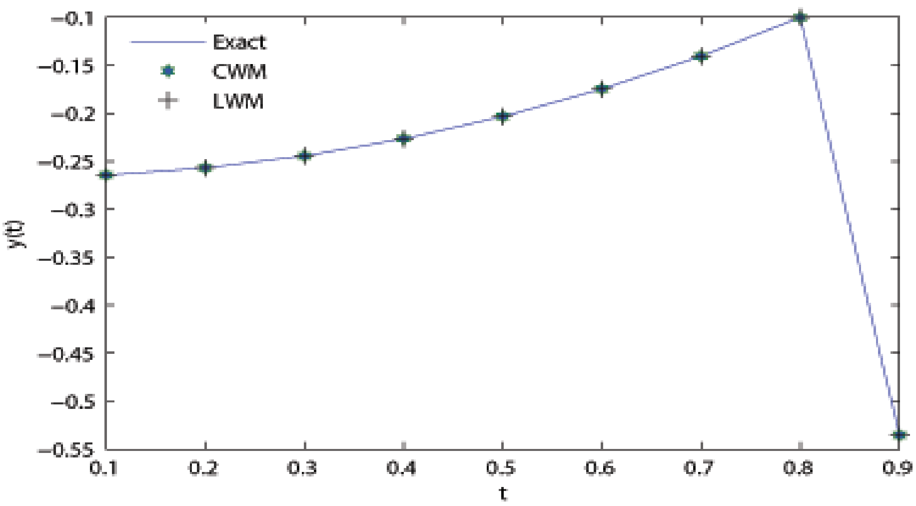

Figure 4.4 Comparison of numerical and exact solution.

Example 4.5 ([21])**.** Consider the nonlinear singular boundary value problem

$$y''(t) + \frac{2}{t}y'(t) + y^5(t) = f(t), \quad 0 < t \leq 1, \tag{4.20}$$

with boundary conditions as $y'(0) = 0,\ y(1) = \frac{\sqrt{3}}{2}$ where $f(t) = 0$ and the exact solution is $y(t) = \frac{1}{\sqrt{1+\frac{t^2}{2}}}$ which explains the equilibrium of isothermal gas sphere.

For this case, we also follow the same procedure as we did in previous examples to approximate $y''(t),\ y'(t)$ and $y(t)$ with the given boundary conditions using basis functions $\Psi(t)$. We collocate the obtained algebraic equation at suitable collocation points in Eq.(2.7). So, we have a nonlinear system of algebraic equations with the same number of unknowns which can be easily solved by classical Newton method to obtain the vector C. Now we substitute the wavelet coefficient C into $y(t)$ to get the approximation solution for the Eq.(4.20) which is shown numerically in Table 4.5 and graphically in Figure 4.5. We observe from Table 4.5 and Figure 4.5 that the proposed technique is in full agreement with the exact and Chebyshev wavelet method [21] solutions.

t	LWM Results	Ref.[21]	Exact
0.1	0.99503719	0.99503720	0.99503719
0.2	0.98058069	0.98058067	0.98058067
0.3	0.95782631	0.95782628	0.95782628
0.4	0.92847672	0.92847670	0.92847669
0.5	0.89442718	0.89442718	0.89442719
0.6	0.85749294	0.85749292	0.85749292
0.7	0.81923195	0.81923192	0.81923192
0.8	0.78086881	0.78086880	0.78086880
0.9	0.74329414	0.74329414	0.74329414

Table 4.5 Comparison of the approximate solution of Example 4.5 against the exact and CWM [21] solutions for $M = 3, k = 2$.

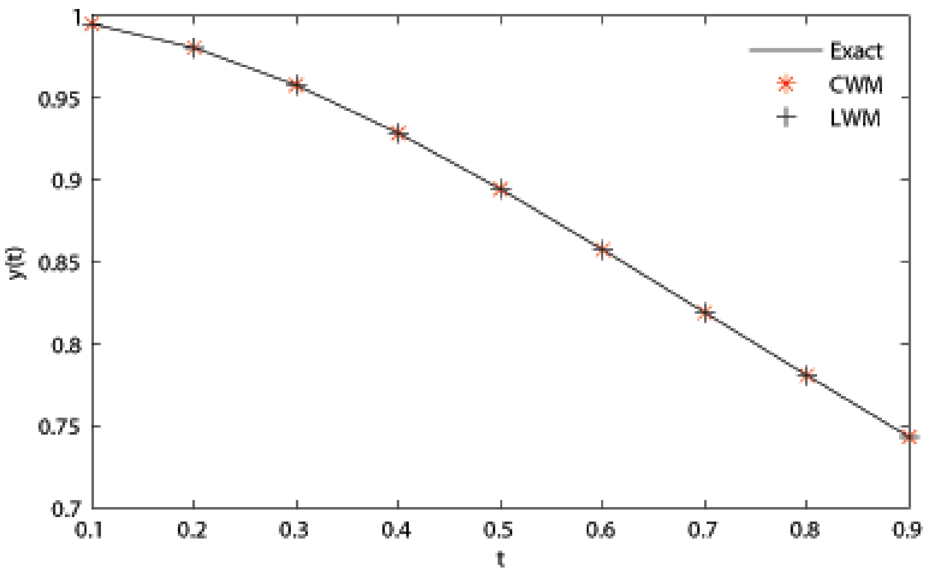

Figure 4.5 Comparison of numerical and exact solution.

5. Conclusion

In this study, we have utilized Legendre wavelet method and its properties for solving linear and nonlinear singular ordinary differential equations. This method also reduces the problem to the solution of algebraic equations which can be solved easily by classical numerical methods. The advantage of this method is that the values of k and M are adjustable as well to yield more accurate numerical solutions. The given numerical examples support the claim that only a small number of Legendre wavelets are needed to accomplish a satisfactory result. A symbolic and numerical calculation software package like MATLAB R2010a and MAPLE 14 are used. The results corroborate our belief that the proposed method is a reliable technique to handle these types of problems. It can therefore be concluded that this method is quite suitable, accurate, and efficient in comparison to other classical methods.

References

[1] O.A. Akinfenwa, S.N. Jator and N.M. Yao, *Continuous block backward differentiation formula for solving stiff ordinary differential equations*, Comput. Math. Appl., 65 (2013), 996-1005.

[2] G. Beylkin, R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms*, Commun. Pure Appl. Math., 4 (1991), 141-183.

[3] J. Biazar and H. Ghazvini, *He's variational iteration method for solving linear and non-linear systems of ordinary differential equations*, Appl. Math. Comput., 191 (2007), 287-297.

[4] N. Caglar and H. Caglar, *B-spline method for solving linear system of second-order boundary value problems*, Comput. Math. Appl., 57 (2009), 757-762.

[5] C. Canuto et al., *Spectral methods in fluid dynamics*, Springer-Verlag, Berlin, 1988.

[6] F.C. Chen and C.H. Hsiao, *Haar wavelet method for solving lumped and distributed-parameter systems*, IEE Proc. Control Theory Appl., 144 (1997), 87-94.

[7] C.F. Chen and C.H. Hsiao, *A Walsh series direct method for solving variational problems*, J. Franklin Inst., 30 (1997), 265-280.

[8] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, 61, SIAM, Philadelphia, 1992.

[9] M. Dehghan and A. Nikpour, *Numerical solution of the system of second-order boundary value problems using the local radial basis functions based*

differential quadrature collocation method, Appl. Math. Model., 37 (2013), 8578-8599.

[10] L. Fox and I.B. Parker, *Chebyshev polynomials in numerical analysis*, Clarendon Press, Oxford, 1968.

[11] F. Geng and M. Cui, *Homotopy perturbation-reproducing kernel method for nonlinear systems of second order boundary value problems*, J. Comput. Appl. Math., 235 (2011), 2405-2411.

[12] D. Gottelieb and S. Orszag, *Numerical analysis of spectral methods, Theory and Applications*, SIAM, Philadelphia, PA, 1977.

[13] J. Iqbal and R. Abass, *Numerical Solution of Klein/Sine-Gordon Equations by Spectral Method Coupled with Chebyshev Wavelets*, Applied Mathematics, 7 (2016), 2097-2109.

[14] M.K. Kadalbajoo and V.K. Aggarwal, *Numerical solution of singular boundary value problems via Chebyshev polynomial and B-spline*, Appl. Math. Comput., 160 (2005), 851-863.

[15] M. T. Kajania, A. H. Vencheha and M. Ghasemi, *The Chebyshev wavelets operational matrix of integration and product operation matrix*, Int. J. Comp. Math., 86 (2009), 1118-1125.

[16] M. Kumar and N. Singh, *A collection of computational techniques for solving singular boundary-value problems*, Adv. Eng. Softw., 40 (2009), 288-297.

[17] U. Lepik, *Haar wavelet method for solving stiff differential equations*, Math. Modell. Anal., 14 (2009), 467-481.

[18] J. Lu, *Variational iteration method for solving a nonlinear system of second-order boundary value problems*, Comput. Math. Appl., 54 (2007), 1133-1138.

[19] M. Mahmoudi and H. Jafari, *Modified Laplace decomposition method for singular IVPs in the second-order ordinary differential equations*, Casp. J. Math. Sci., 3 (2014), 105-113.

[20] K. Maleknejad and H. Derili, *The collocation method for Hammerstein equations by Daubechies wavelets*, Appl. Math. Comput., 172 (2006), 846-864.

[21] A. K. Nasab et al., *Wavelet analysis method for solving linear and nonlinear singular boundary value problems*, Appl. Math. Model., 37 (2013), 5876-5886.

[22] J. Rashidinia et al., *Convergence of cubic-spline approach to the solution of a system of boundary-value problems*, Appl. Math. Comput., 192 (2007), 319-331.

[23] M. Razzaghi and S. Yousefi, *Legendre wavelets method for the solution of nonlinear problems in the calculus of variations*, Math. Comput. Model., 34 (2001), 45-54.

[24] M. Razzaghi and S. Yousefi, *The Legendre wavelets operational matrix of integration*, Int. J. Syst. Sci., 32 (2001), 495-502.

[25] F. A. Shah, R. Abass and J. Iqbal, *Numerical Solution of Singularly Perturbed Problems using Haar Wavelet Collocation Method*, Cogent Mathematics, 3 (2016), 1-13.

[26] M. Tatari and M. Dehghan, *Improvement of He's variational iteration method for solving systems of differential equations*, Comput. Math. Appl., 58 (2009), 2160-2166.

[27] S.G. Venkatesh, S.K. Ayyaswamy and S. R. Balachandar, *The Legendre wavelet method for solving initial value problems of Bratu-type*, Comput. Math. Appl., 63 (2012), 1287-1295.

[28] S. Yousefi and M. Razzaghi, *Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations*, Math. Comput. Simul., 70 (2005), 1-8.

[29] F. Zhou and X. Xu, *Numerical solutions for the linear and nonlinear singular boundary value problems using Laguerre wavelets*, Adv. Diff. Eqs., 17 (2016), 1-17.

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