ON CHARACTERIZATION OF MONOIDS BY PROPERTIES OF GENERATORS II

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Abstract. Kilp and Knauer in (Communications In Algebra, 20(7), 1841-1856, 1992) gave a characterizations of monoids when all generators in the category of right S-acts(S is a monoid) satisfy properties such as freeness, projectivity, strong flatness, Condition (P), principal weak flatness, principal weak injectivity, weak injectivity, injectivity, divisibility, strong faithfulness and torsion freeness. Sedaghatjoo in (Semigroup Forum, 87: 653-662, 2013) gave a characterizations of monoids when all generators in the category of right S-acts satisfy properties such as weak flatness, Condition (E) and regularity. Continuing this study the authors (On characterization of monoids by properties of generators, submitted) investigated the corresponding problem for (finitely generated, cyclic, monocyclic) right acts. To our knowledge the problem has not been yet studied for properties such as GP-flatness, strongly (P)-cyclic, (P)-regularity and Conditions $(EP), (E'P), (E'), (P_E), (PWP), (PWP_E), WPF, WKF, PWKF, TKF, (WP)$ etc. In this article we answer the question corresponding to these properties.
Keywords: generator, GP-flat, strongly (P)-cyclic, (P)-regular, condition (PWP).

1. Introduction

For a monoid S, a non-empty set A is called a right S-act, usually denoted A_S, if S acts on A unitarily from the right, that is, there exists a mapping $A \times S \longrightarrow A$,

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$(a, s) \longmapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A_S$ and all $s, t \in S$. Throughout this article, S will always stand for a monoid and A_S is a right S-act. For basic definitions and terminology relating semigroups and acts over monoids, we refer the reader to [11] and [13].

Let $\mathbf{C}$ be a category. An object $G \in \mathbf{C}$ is called a *generator* in $\mathbf{C}$ if the functor $Mor_{\mathbf{C}}(G, -)$ is faithful, i.e, for any $X, Y \in \mathbf{C}$ and any $f, g \in Mor_{\mathbf{C}}(X, Y)$ with $f \neq g$ there exists $\alpha \in Mor_{\mathbf{C}}(G, X)$ such that $f\alpha \neq g\alpha$.

We recall from [13, II, 3.16] that G_S is a generator if and only if there exists an epimorphism $\pi : G_S \longrightarrow S_S$. Hence S_S is a generator in **Act**-S. Recall from [13], [8] and [7] that:

An S-act A_S satisfies *Condition* (P), if for all $a, a' \in A_S, s, s' \in S$, $as = a's' \Rightarrow (\exists a'' \in A_S)(\exists u, u' \in S)(a = a''u,\ a' = a''u'$ and $us = u's')$. A_S satisfies *Condition* (E), if for all $a \in A_S, s, s' \in S$, $as = as' \Rightarrow (\exists a' \in A_S)(\exists u \in S)(a = a'u$ and $us = us')$.

A_S satisfies *Condition* (EP), if for all $a \in A_S, s, s' \in S$, $as = as' \Rightarrow (\exists a' \in A_S)(\exists u, u' \in S)(a = a'u = a'u'$ and $us = u's')$.

A_S satisfies *Condition* $(E'P)$, if for all $a \in A_S, s, s', z \in S$, $as = as', sz = s'z \Rightarrow (\exists a' \in A_S)(\exists u, u' \in S)(a = a'u = a'u'$ and $us = u's')$.

Also we recall that a monoid S is called *left(right) collapsible* if for every $s, t \in S$ there exists $u \in S$ such that $us = ut(su = tu)$. Let S be a monoid and $x, y \in S$ then $l(x, y) = \{z \in S \mid zx = zy\}$. Evidently, $l(x, y) = \emptyset$ or $l(x, y)$ is a left ideal. If S is a left collapsible monoid, then for every $x, y \in S$, $l(x, y) \neq \emptyset$, and so $l(x, y)$ is a left ideal. Similarly, for every $x, y \in S$, we define $r(x, y) = \{t \in S \mid xt = yt\}$. Clearly $r(x, y) = \emptyset$ or $r(x, y)$ is a right ideal of S. If S is a right collapsible monoid, then for every $x, y \in S$, $r(x, y) \neq \emptyset$ and so $r(x, y)$ is a right ideal.

We use the following abbreviations

weak pullback flatness = WPF

weak kernel flatness = WKF

principal weak kernel flatness = $PWKF$

translation kernel flatness = TKF

principal weak homoflatness = (PWP)

torsion freeness = TF

Theorem 1.1 (12, Theorem 1.3)**.** *Let S be a monoid and α be an act property which is preserved under retraction. Then the following statements are equivalent:*

1. *all generators satisfy property α;*
2. *$S_S \times A_S$ satisfies property α for right S-act A_S;*

3. *$S_S \times A_S$ satisfies property α for every generator A_S;*

4. *a right S-act A_S satisfies property α if $Hom(A_S, S_S) \neq \emptyset$.*

It is obvious that all properties under discussion here are preserved under retraction.

2. Monoids over which all generators are GP-flat, strongly (P)-cyclic, (P)-regular

A monoid S is called *regular*, if for every $s \in S$, there exists $x \in S$ such that $s = sxs$. We recall from 18 that a monoid S is called a *generally regular*, if for every $s \in S$, there exist a natural number n and $x \in S$ such that $s^n = sxs^n$. It is clear that the class of generally regular monoids contains all regular monoids. Also we recall from [18] that an act A_S is called GP-flat, if for every $s \in S$ and $a, a' \in A_S, a \otimes s = a' \otimes s$ in $A_S \otimes {}_SS$ implies the existence of a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_SSs^n$. It is obvious that every principally weakly flat act is GP-flat, but not the converse, also every GP-flat act is torsion free.

First see the following result.

Lemma 2.1 (18, Lemma 2.2)**.** *Let S be a monoid and A_S be a right $S-$act. Then The following statements are equivalent:*

1. *A_S is GP-flat;*

2. *for every $s \in S$, and $a, a' \in A$, $a \otimes s = a' \otimes s$ in $A \otimes S$ implies that there exist $m, n \in \mathbb{N}$ and elements $a_1, \cdots a_m \in A_S, s_1, t_1, \cdots, s_m, t_m \in S$ such that*

$$\begin{array}{rl} a = a_1 s_1 & \\ a_1 t_1 = a_2 s_2 & s_1 s^n = t_1 s^n \\ a_2 t_2 = a_3 s_3 & s_2 s^n = t_2 s^n \\ \vdots & \vdots \\ a_m t_m = a' & s_m s^n = t_m s^n. \end{array}$$

Now we begin our investigations for monoids over which all (finitely generated) generators are GP-flat.

Theorem 2.2. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts are GP-flat;*

2. *all finitely generated generators right S-acts are GP-flat;*

3. *all generators right S-acts generated by at most three elements are GP-flat;*

4. *$S \times A_S$ is GP-flat for every generator right S-act A_S;*

5. *$S \times A_S$ is GP-flat for every finitely generated generator right S-act A_S;*

6. *$S \times A_S$ is GP-flat for every generator right S-act A_S generated by at most three elements;*

7. *$S \times A_S$ is GP-flat for every right S-act A_S;*

8. *$S \times A_S$ is GP-flat for every finitely generated right S-act A_S;*

9. *$S \times A_S$ is GP-flat for every right S-act A_S generated by at most two elements;*

10. *a right S-act A_S is GP-flat if $Hom(A_S, S_S) \neq \emptyset$;*

11. *a finitely generated right S-act A_S is GP-flat if $Hom(A_S, S_S) \neq \emptyset$;*

12. *a right S-act A_S generated by at most two elements is GP-flat if $Hom(A_S, S_S) \neq \emptyset$;*

13. *all right S-acts are GP-flat;*

14. *all finitely generated right S-acts are GP-flat;*

15. *S is generally regular.*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) and (13) $\Leftrightarrow$ (14) $\Leftrightarrow$ (15) are clear from Theorem 1.1 and [18, Theorem 3.4].

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9), (10) $\Rightarrow$ (11) $\Rightarrow$ (12) and (13) $\Rightarrow$ (1) are obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator and suppose $\pi : A_S \longrightarrow S_S$ is an epimorphism. Let $a \otimes s = a' \otimes s$ in $A \otimes S$, for $a, a' \in A_S, s \in S$, then $as = a's$ in A_S. Since π is an epimorphism, there exists $a'' \in A_S$ such that $\pi(a'') = 1$. If $A' = aS \cup a'S \cup a''S$, Then $as = a's$ in A_S implies $as = a's$ in A', and so $a \otimes s = a' \otimes s$ in $A' \otimes S$. But A' is a generator and so A' is GP-flat by assumption. Thus there exists a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A'_S \otimes {}_S Ss^n \subseteq A_S \otimes {}_S Ss^n$. Hence A_S is GP-flat.

(6) $\Rightarrow$ (4). Let A_S be a generator and suppose $(l_1, a) \otimes s = (l_2, a') \otimes s$ in $(S \times A)_S \otimes {}_S S$, for $l_1, l_2, s \in S$ and $a, a' \in A_S$. Let $A' = aS \cup a'S \cup a''S$ is as in the proof of (3) $\Rightarrow$ (1), then $S \times A'$ is GP-flat by assumption. Thus there exists a natural number n such that $(l_1, a) \otimes s^n = (l_2, a') \otimes s^n$ in $(S \times A')_S \otimes {}_S Ss^n \subseteq (S \times A)_S \otimes {}_S Ss^n$, and so the result follows.

(7) $\Rightarrow$ (13) Let A_S be a right S-act and suppose $a \otimes s = a' \otimes s$ in $A \otimes S$, for $a, a' \in A_S$ and $s \in S$. Then $as = a's$ in A_S and so $(1, a)s = (1, a')s$ in $S \times A_S$. Thus $(1, a) \otimes s = (1, a') \otimes s$ in $(S \times A) \otimes S$. Since $S \times A_S$ is GP-flat, by lemma 2.1 there exist $m, n \in \mathbb{N}, s_1, t_1, s_2, t_2, \cdots, s_m, t_m \in S$ and

$(l_1, a_1), (l_2, a_2), \cdots, (l_m, t_m) \in S \times A_S$ such that:

$$\begin{array}{rcll} (1,a) &=& (l_1,a_1)s_1 & \\ (l_1,a_1)t_1 &=& (l_2,a_2)s_2 & s_1s^n = t_1s^n \\ (l_2,a_2)t_2 &=& (l_3,a_3)s_3 & s_2s^n = t_2s^n \\ &\vdots& & \quad\vdots \\ (l_m,a_m)t_m &=& (1,a') & s_ms^n = t_ms^n. \end{array}$$

The above tossing implies that

$$\begin{array}{rcll} a &=& a_1s_1 & \\ a_1t_1 &=& a_2s_2 & s_1s^n = t_1s^n \\ a_2t_2 &=& a_3s_3 & s_2s^n = t_2s^n \\ &\vdots& & \quad\vdots \\ a_mt_m &=& a' & s_ms^n = t_ms^n\ , \end{array}$$

and so A_S is GP-flat by Lemma 2.1.

(9) $\Rightarrow$ (7). Let A_S be a right S-act and suppose $(l_1, a) \otimes s = (l_2, a') \otimes s$, for $a, a' \in A_S$ and $l_1, l_2, s \in S$. Let $A' = aS \cup a'S$. Then $S \times A'_S$ is GP-flat by assumption and so there exists a natural number n such that $(l_1, a) \otimes s^n = (l_2, a') \otimes s^n$ in $(S \times A')_S \otimes {}_SSs^n \subseteq (S \times A)_S \otimes {}_SSs^n$ and so the result follows.

(12) $\Rightarrow$ (10). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$ and suppose $a \otimes s = a' \otimes s$, for $a, a' \in A_S$ and $s \in S$. Since $Hom(A_S, S_S) \neq \emptyset$, there exists a homomorphism $f : A_S \longrightarrow S_S$. If $A' = aS \cup a'S$ and $f' = f\mid_{A'}$, then A' is GP-flat by assumption and so $a \otimes s = a' \otimes s$ in $A' \otimes S$ implies that there exists a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A' \otimes Ss^n \subseteq A \otimes Ss^n$, and so the result follows. □

We recall from [19] that a right congruence ρ on S_S is called *right subannihilator congruence* if $\rho \leq ker\lambda_s$ for some $s \in S$. Also we recall from [10] that S is a *right PCP monoid*, if all principal right ideals of S satisfy Condition (P) and a right S-act A_S is called *strongly (P)-cyclic*, if for every $a \in A_S$ there exists $z \in S$ such that $ker\lambda_a = ker\lambda_z$ and zS satisfies Condition (P).

Now similar to Theorem 2.2, we give equivalents of when all (finitely generated) generators right S-acts are strongly (P)-cyclic.

Theorem 2.3. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts are strongly (P)-cyclic;*
2. *all finitely generated generators right S-acts are strongly (P)-cyclic;*
3. *all generators right S-acts generated by at most two elements are strongly (P)-cyclic;*
4. *$S \times A_S$ is strongly (P)-cyclic for every generator right S-act A_S;*

5. *$S \times A_S$ is strongly (P)-cyclic for every finitely generated generator right S-act A_S;*

6. *$S \times A_S$ is strongly (P)-cyclic for every generator right S-acts generated by at most two elements;*

7. *$S \times A_S$ is strongly (P)-cyclic for every right S-act A_S;*

8. *$S \times A_S$ is strongly (P)-cyclic for every finitely generated right S-act A_S;*

9. *$S \times A_S$ is strongly (P)-cyclic for every cyclic right S-act A_S;*

10. *a right S-act A_S is strongly (P)-cyclic if $Hom(A_S, S_S) \neq \emptyset$;*

11. *a finitely generated right S-act A_S is strongly (P)-cyclic if $Hom(A_S, S_S) \neq \emptyset$;*

12. *a cyclic right S-act A_S is strongly (P)-cyclic if $Hom(A_S, S_S) \neq \emptyset$;*

13. *for every right subannihilator congruence ρ, S/ρ is strongly (P)-cyclic.*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) and (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator and let $\pi : A_S \longrightarrow S_S$ be an epimorphism. Let $a \in A_S$. Since π is an epimorphism, there exists $a' \in A_S$ such that $\pi(a') = 1$. If $A' = aS \cup a'S$ then A' is a generator and so it is strongly (P)-cyclic by assumption. Thus there exists $z \in S$ such that $ker\lambda_a = ker\lambda_z$ and zS satisfies Condition (P), and so the result follows.

(6) $\Rightarrow$ (4). Let A_S be a generator and let $(l, a) \in S \times A$, for $a \in A_S$ and $l \in S$. Let $A' = aS \cup a'S$ be as in the proof of (3) $\Rightarrow$ (1), then A' is a generator and so $S \times A'$ is strongly (P)-cyclic by assumption. Thus there exists $z \in S$ such that $ker\lambda_{(l,a)} = ker\lambda_z$ and zS satisfies Condition (P), and so the result follows.

(9) $\Rightarrow$ (10). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$. Let $a \in A_S$ and let $f : A_S \longrightarrow S_S$ be an S-homomorphism. Consider $(f(a), a) \in S \times aS$. Since $S \times aS$ is strongly (P)-cyclic by assumption, there exists $z \in S$ such that zS satisfy Condition (P) and $ker\lambda_{(f(a),a)} = ker\lambda_z$. So we have:

$$ker\lambda_z = ker\lambda_{(f(a),a)} = ker\lambda_{f(a)} \cap ker\lambda_a \subseteq ker\lambda_a.$$

Hence $ker\lambda_z \subseteq ker\lambda_a$. Now we show that $ker\lambda_a \subseteq ker\lambda_z$. Let $(l_1, l_2) \in ker\lambda_a$ for $l_1, l_2 \in S$, then $(l_1, l_2) \in ker\lambda_{f(a)}$. Thus $(l_1, l_2) \in ker\lambda_a \cap ker\lambda_{f(a)} = ker\lambda_z$ and so $ker\lambda_a \subseteq ker\lambda_z$. Therefore, $ker\lambda_a = ker\lambda_z$ and zS satisfy Condition (P), as required.

(12) $\Rightarrow$ (13). Let ρ be a right subannihilator congruence. Thus there exists $s \in S$ such that $\rho \leq ker\lambda_s$. Define $f : S/\rho \longrightarrow S$ by $f([t]_\rho) = st$. Clearly f is

an S-homomorphism. So $Hom(S/\rho, S) \neq \emptyset$. Thus S/ρ is strongly (P)-cyclic by assumption.

(13) $\Rightarrow$ (1). Let A_S be a generator and $a \in A_S$. Thus, there exists an epimorphism $\pi : A_S \longrightarrow S$. Let $\pi(a) = t$ and let $(l_1, l_2) \in ker\lambda_a$, for $l_1, l_2 \in S$. Then $(l_1, l_2) \in ker\lambda_t$. So $ker\lambda_a \leq ker\lambda_t$. Thus $ker\lambda_a$ is a right subannihilator congruence and so $aS \cong S/ker\lambda_a$ is strongly (P)-cyclic by assumption. Thus there exists $z \in S$ such that $ker\lambda_a = ker\lambda_z$ and zS satisfies Condition (P), and so the result follows. □

Corollary 2.4. *Let S be a monoid over which all generators right S-acts are strongly (P)-cyclic. Then for any nonempty family $\{A_i \mid i \in I\}$ of strongly (P)-cyclic right S-acts, $\prod_I A_i$ is strongly (P)-cyclic.*

Proof. Since S_S is a generator, it is strongly (P)-cyclic by assumption and so S is right PCP by [10, Theorem 2.2]. Thus by Theorem 2.3, $(S_S)^I$ is strongly (P)-cyclic for any nonempty set I and so the result follows by [16, Corollary 5.5]. □

Definition 2.5 ([3])**.** *Let S be a monoid. A right S-act A_S is called (P)-regular if all cyclic subacts of A_S satisfy Condition (P).*

It is obvious that every regular right act is (P)-regular, but not the converse.

Theorem 2.6 ([3], Theorem 2.2)**.** *Let S be a monoid and A_S be a right S-act. Then A_S is (P)-regular if and only if for every $a \in A$ and $x, y \in S, ax = ay$ implies that there exist $u, v \in S$ such that $a = au = av$ and $ux = vy$.*

Similar to Theorem 2.3 we have the following result for (P)-regularity.

Theorem 2.7. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts are (P)-regular;*
2. *all finitely generated generators right S-acts are (P)-regular;*
3. *all generators right S-acts generated by at most two elements are (P)-regular;*
4. *$S \times A_S$ is (P)-regular for every generator right S-act A_S;*
5. *$S \times A_S$ is (P)-regular for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ is (P)-regular for every generator right S-act A_S generated by at most two elements;*
7. *$S \times A_S$ is (P)-regular for every right S-act A_S;*
8. *$S \times A_S$ is (P)-regular for every finitely generated right S-act A_S;*
9. *$S \times A_S$, is (P)-regular for every cyclic right S-act A_S;*

10. *a right S-act A_S is (P)-regular if $Hom(A_S, S_S) \neq \emptyset$;*

11. *a finitely generated right S-act A_S is (P)-regular if $Hom(A_S, S_S) \neq \emptyset$;*

12. *a cyclic right S-act A_S is (P)-regular if $Hom(A_S, S_S) \neq \emptyset$;*

13. *for every right subannihilator congruence ρ, S/ρ is (P)-regular.*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) and (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator and $\pi : A_S \longrightarrow S_S$ be an epimorphism. Let $a \in A_S$. Since π is an epimorphism, there exists $a' \in A_S$ such that $\pi(a') = 1$. If $A' = aS \cup a'S$ then A' is a generator and so it is (P)-regular by assumption. Thus aS satisfies Condition (P) and the result follows.

(6) $\Rightarrow$ (4). Let A_S be a generator and let $(l, a) \in S \times A$, for $a \in A_S$ and $l \in S$. Let $A' = aS \cup a'S$ be as in the proof of (3) $\Rightarrow$ (1), then A' is a generator and so $S \times A'$ is (P)-regular by assumption. Thus $(l, a)S$ satisfies Condition (P) and the result follows.

(9) $\Rightarrow$ (10). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$. Let $a \in A_S$ and let $f : A_S \longrightarrow S_S$ be an S-homomorphism such that $ax = ay$, for $x, y \in S$. Thus $(f(a), a)x = (f(a), a)y$. Since $S \times aS$ is (P)-regular, the last equality implies that there exist $u, v \in S$ such that $(f(a), a) = (f(a), a)u = (f(a), a)v$ and $ux = vy$. Thus $a = au = av$ and $ux = vy$, and so A_S is (P)-regular by Theorem 2.6.

(12) $\Rightarrow$ (13). Let ρ be a right subannihilator congruence. Thus $Hom(S/\rho, S) \neq \emptyset$ and so S/ρ is (P)-regular by assumption.

(13) $\Rightarrow$ (1). Let A_S be a generator and let $a \in A_S$. Similar to (13) $\Rightarrow$ (1) in Theorem 2.3, $aS \cong S/ker\lambda_a$ is (P)-regular and so aS satisfy Condition (P), thus the result follows. $\square$

3. Monoids over which all generators satisfy Condition (EP), $(E'P)$, (E'), (P_E), (PWP_E), (PWP), (P), (WP), WPF, WKF, $PWKF$, TKF or (P')

In this section by using Theorem 1.1, we give a characterization of monoids over which all generators satisfy each of the above conditions.

Lemma 3.1 ([12], Lemma 3.3)**.** *Suppose for every $x, y \in S$,($l(x, y) = \emptyset \vee l(x, y) = S \vee xS \cup yS = S$). If $x, x' \in S$ such that $xx' = 1$, then $xx' = 1 = x'x$.*

Now we investigate monoids over which all generators satisfy Condition (EP).

Theorem 3.2. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (EP);*

2. *all finitely generated generators right S-acts satisfy Condition (EP);*
3. *all generators right S-acts generated by at most two elements satisfy Condition (EP);*
4. $S \times A_S$ *satisfies Condition (EP) for every generator right S-act A_S;*
5. $S\times A_S$ *satisfies Condition (EP) for every finitely generated generator right S-act A_S;*
6. $S \times A_S$ *satisfies Condition (EP) for every generator right S-act A_S generated by at most two elements*
7. $S \times A_S$ *satisfies Condition (EP) for every right S-act A_S;*
8. $S \times A_S$ *satisfies Condition (EP) for every finitely generated right S-act A_S;*
9. $S \times A_S$ *satisfies Condition (EP) for every cyclic right S-act A_S;*
10. $S \times A_S$ *satisfies Condition (EP) for every monocyclic right S-act A_S;*
11. *a right S-act A_S satisfies Condition (EP) if $Hom(A_S, S_S) \neq \emptyset$;*
12. *a finitely generated right S-act A_S satisfies Condition (EP) if $Hom(A_S, S_S) \neq \emptyset$;*
13. *a cyclic right S-act A_S satisfies Condition (EP) if $Hom(A_S, S_S) \neq \emptyset$;*
14. *a monocyclic right S-act A_S satisfies Condition (EP) if $Hom(A_S, S_S) \neq \emptyset$;*
15. $(\forall x, y \in S)\ (l(x, y) = \emptyset\ \vee\ l(x, y) = S\ \vee\ xS \cup yS = S)$;
16. *for every right subannihilator congruence ρ, S/ρ satisfies Condition (EP);*
17. $(\forall x, y \in S)(\ l(x, y) = \emptyset \vee S/\rho(x, y)$ *satisfies Condition* $(EP))$;
18. $(\forall x, y \in S)(\ l(x, y) = \emptyset \vee (\exists u, v \in S,\ ux = vy\ \wedge\ 1\ \rho(x, y)\ u\ \rho(x, y)\ v))$.
19. $(\forall x, y, t \in S)(\ l(tx, ty) = \emptyset \vee S/\rho(tx, ty)$ *satisfies Condition* $(EP))$;
20. $(\forall x, y, t \in S)(\ l(tx, ty) = \emptyset \vee (\exists u, v \in S,\ t\ \rho(tx, ty)\ u\ \rho(tx, ty)\ v\ \wedge\ ux = vy))$.

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (11) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10), (11) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (14) and (17) $\Rightarrow$ (19) are obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator and $\pi : A_S \longrightarrow S_S$ be an epimorphism. Let $as = at$, for $a \in A_S$ and $s, t \in S$. Since π is an epimorphism, there exists $a' \in A_S$ such that $\pi(a') = 1$. If $A' = aS \cup a'S$, then A' is a generator and

so it satisfies Condition (EP) by assumption. Thus $as = at$ in A' implies the existence of $a'' \in A' \subseteq A$ and $u, v \in S$ such that $a = a''u = a''v$ and $us = vt$. That is, A_S satisfies Condition (EP), as required.

(6) $\Rightarrow$ (4). Let A_S be a generator and let $(l, a)s = (l, a)t$, for $a \in A_S$ and $l, s, t \in S$. If $A' = aS \cup a'S$, then A' is a generator and so $S \times A'$ satisfies Condition (EP) by assumption. Thus there exist $(l', a'') \in S \times A' \subseteq S \times A$ and $u, v \in S$ such that $(l, a) = (l', a'')u = (l', a'')v$ and $us = vt$. Thus $S \times A$ satisfies Condition (EP).

(11) $\Rightarrow$ (16). Let ρ be a right subannihilator congruence. Thus $Hom(S/\rho, S) \neq \emptyset$, and so S/ρ satisfies Condition (EP) by assumption.

(16) $\Rightarrow$ (17). Let $x, y \in S$ and suppose $l(x, y) \neq \emptyset$. Then there exists $s \in S$ such that $sx = sy$, and so $(x, y) \in ker\lambda_s$, that is, $\rho(x, y) \leq ker\lambda_s$. Thus $\rho(x, y)$ is a right subannihilator congruence and so $S/\rho(x, y)$ satisfies Condition (EP) as required.

(17) $\Rightarrow$ (18). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then $S/\rho(x, y)$ satisfies Condition (EP) and so $[1]_\rho x = [1]_\rho y$ implies that there exist $\alpha, u_1, u_2 \in S$, such that $[1]_\rho = [\alpha]_\rho u_1 = [\alpha]_\rho u_2$ and $u_1 x = u_2 y$. If $\alpha u_1 = u$ and $\alpha u_2 = v$, then $[1]_\rho = [u]_\rho = [v]_\rho$ and $ux = vy$. Hence $1\ \rho(x, y)\ u\ \rho(x, y)\ v$ and $ux = vy$.

(18) $\Rightarrow$ (20). Let $x, y \in S$ such that $l(tx, ty) \neq \emptyset$. By assumption there exist $u_1, u_2 \in S$ such that $u_1 tx = u_2 ty, 1\ \rho(tx, ty)\ u_1\ \rho(tx, ty)\ u_2$. Therefore, $t\ \rho(tx, ty)\ u_1 t\ \rho(tx, ty)\ u_2 t$ and $u_1 tx = u_2 ty$. If $u_1 t = u$ and $u_2 t = v$, then $t\ \rho(tx, ty)\ u\ \rho(tx, ty)\ v$ and $ux = vy$ as required.

(19) $\Rightarrow$ (20). Let $x, y, t \in S$ such that $l(tx, ty) \neq \emptyset$, and let $\rho(tx, ty) = \rho$. Since $[t]_\rho x = [t]_\rho y$ and S/ρ satisfy Condition (EP), there exist $\alpha, u_1, u_2 \in S$, such that $[t]_\rho = [\alpha]_\rho u_1 = [\alpha]_\rho u_2$ and $u_1 x = u_2 y$. If $\alpha u_1 = u$ and $\alpha u_2 = v$, then $t\ \rho\ u\ \rho\ v$ and $ux = vy$.

(20) $\Rightarrow$ (15). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$ and suppose $l(x, y) \neq S$. Since $l(x, y) \neq S$, thus $x \neq y$. Let $K_S = xS \cup yS$. Clearly, K_S is a right ideal and $|K_S| \geq 2$. Since $\rho_{K_S} = (K_S \times K_S) \cup 1_S$, we have $(x, y) \in \rho_{K_S}$ and so $\rho(x, y) \leq \rho_{K_S}$. By assumption there exists $u, v \in S$ such that $1\ \rho(x, y)\ u\ \rho(x, y)\ v$ and $ux = vy$. Now $\rho(x, y) \leq \rho_{K_S}$ implies that, $1\ \rho_{K_S}\ u\ \rho_{K_S} v$ and $ux = vy$. If $K_S \neq S$ then $1\ \rho_{K_S}\ u\ \rho_{K_S}\ v$ implies that $u = v = 1$ and so $x = y$, a contradiction. Hence $K_S = xS \cup yS = S$.

(14) $\Rightarrow$ (17). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then there exists $z \in S$ such that $zx = zy$ and so $\rho(x, y) \leq ker\lambda_z$. Define the mapping $f : S/\rho(x, y) \longrightarrow S_S$ by $f([t]_{\rho(x,y)}) = zt$, for $t \in S$. Clearly f is a well defined S-homomorphism. Therefore, $Hom(S/\rho(x, y), S_S) \neq \emptyset$ and so by assumption $S/\rho(x, y)$ satisfies Condition (EP), as required.

(14) $\Rightarrow$ (15). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$ and $l(x, y) \neq S$. Thus $x \neq y$ and there exists $z \in S$ such that $zx = zy$, which implies $(x, y) \in ker\lambda_z$ and so $\rho(x, y) \leq ker\lambda_z$. Let $\rho = \rho(x, y)$. Thus similarly as (14) $\Rightarrow$ (17), $Hom(S/\rho, S_S) \neq \emptyset$ and so by assumption S/ρ satisfies Condition (EP). Since $x\ \rho\ y$ and S/ρ satisfies Condition (EP), $[1]_\rho x = [1]_\rho y$ implies that there exist $w, u, v \in S$ such that $[1]_\rho = [w]_\rho u = [w]_\rho v$ and $ux = vy$. If $1 = wu = wv$ then

$x = y$, which is a contradiction. Thus without loss of generality we suppose $1 \neq wu$. Using [13, Lemma I, 4.37], we get the following sequence of equalities:

$$\begin{array}{cccc} 1 = p_1w_1 & q_2w_2 = p_3w_3 & \cdots & q_nw_n = wu \\ q_1w_1 = p_2w_2 & & \cdots & \end{array}$$

where $p_i, q_i \in \{x, y\}, w_i \in S$ for each $1 \leq i \leq n$. Thus $1 = p_1w_1$ implies that $xS \cup yS = S$.

(15) $\Rightarrow$ (1). Let A_S be a generator and let $\pi : A_S \longrightarrow S_S$ be an epimorphism. Suppose $as = as'$, for $a \in A_S$ and $s, s' \in S$. Thus $\pi(a)s = \pi(a)s'$ implies that $l(s, s') \neq \emptyset$ and so $l(s, s') = S$ or $sS \cup s'S = S$ by assumption. If $l(s, s') = S$ then $s = s'$ and so the result follows. Otherwise, $sS \cup s'S = S$. Without loss of generality we suppose $1 \in sS$, thus there exists $x \in S$ such that $sx = 1$. Then $xs = 1$ by Lemma 3.1 and so $xss' = s'1 = s'xs$. Let $xs = u$ and $s'x = v$. Thus $a = au = av$ and $us' = vs$, as required.

(10) $\Rightarrow$ (17). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then there exists $z \in S$ such that $zx = zy$ and so $\rho(x, y) \leq ker\lambda_z$. Suppose $\rho(x, y) = \rho$ and let $l_1 \ \rho \ l_2$ for $l_1, l_2 \in S$. Then $zl_1 = zl_2$. Thus $(z, [1]_\rho)l_1 = (z, [1]_\rho)l_2$ in $S \times S/\rho$. The last equality implies that there exist $(w, [a]_\rho) \in S \times S/\rho$ and $u, v \in S$ such that $(z, [1]_\rho) = (w, [a]_\rho)u = (w, [a]_\rho)v, ul_1 = vl_2$. If $au = u'$ and $av = v'$. Thus $S/\rho(x, y)$ satisfies Condition (EP) by [7, Theorem 3.2]. □

Since $(E) \Longrightarrow (EP)$, the following can be concluded immediately.

Corollary 3.3 ([19], Corollary 2.6)**.** *Let S be a monoid over which all generators satisfy Condition (E). Then for each pair $(x, y) \in S \times S$, $l(x, y) = \emptyset$ or $l(x, y) = S$ or $xS \cup yS = S$.*

Now using an argument similar to that of the proof of Theorem 3.2, we have the following.

Theorem 3.4. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition $(E'P)$;*
2. *all finitely generated generators right S-acts satisfy Condition $(E'P)$;*
3. *all generators right S-acts generated by at most two elements satisfy Condition $(E'P)$;*
4. *$S \times A_S$ satisfies Condition $(E'P)$ for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition $(E'P)$ for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition $(E'P)$ for every generator right S-act A_S generated by at most two elements*
7. *$S \times A_S$ satisfies Condition $(E'P)$ for every right S-act A_S;*

8. *$S \times A_S$ satisfies Condition $(E'P)$ for every finitely generated right S-act A_S;*

9. *$S \times A_S$ satisfies Condition $(E'P)$ for every cyclic right S-act A_S;*

10. *$S \times A_S$ satisfies Condition $(E'P)$ for every monocyclic right S-act A_S;*

11. *a right S-act A_S satisfies Condition $(E'P)$ if $Hom(A_S, S_S) \neq \emptyset$;*

12. *a finitely generated right S-act A_S satisfies Condition $(E'P)$ if $Hom(A_S, S_S) \neq \emptyset$;*

13. *a cyclic right S-act A_S satisfies Condition $(E'P)$ if $Hom(A_S, S_S) \neq \emptyset$;*

14. *a monocyclic right S-act A_S satisfies Condition $(E'P)$ if $Hom(A_S, S_S) \neq \emptyset$;*

15. $(\forall x, y \in S)$ $(l(x,y) = \emptyset \vee r(x,y) = \emptyset \vee l(x,y) = r(x,y) = S \vee xS \cup yS = S)$;

16. *for every right subannihilator congruence ρ, S/ρ satisfies Condition $(E'P)$;*

17. $(\forall x, y \in S)(\ l(x,y) = \emptyset \vee S/\rho(x,y)$ *satisfies Condition* $(E'P))$;

18. $(\forall x, y \in S)(\ l(x,y) = \emptyset \vee r(x,y) = \emptyset \vee (\exists u, v \in S,\ ux = vy\ \wedge\ 1\ \rho(x,y)\ u\ \rho(x,y)\ v))$;

19. $(\forall x, y, t \in S)(\ l(tx,ty) = \emptyset \vee S/\rho(tx,ty)$ *satisfies Condition* $(E'P))$;

20. $(\forall x, y, t \in S)(\ l(tx,ty) = \emptyset \vee\ \ r(tx,ty) = \emptyset \vee\ \ (\exists u, v \in S,\ t\ \rho(tx,ty)\ u\ \rho(tx,ty)\ v \wedge\ ux = vy))$.

Corollary 3.5 ([8], Corollary 2.12)**.** *If S is a right collapsible monoid, then for cyclic acts Conditions (P) and $(E'P)$ coincide.*

Theorem 3.6. *If S is a collapsible monoid, then the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition $(E'P)$;*

2. *all right S-acts satisfy Condition $(E'P)$;*

3. *all finitely generated right S-acts satisfy Condition $(E'P)$;*

4. *all cyclic right S-acts satisfy Condition $(E'P)$;*

5. $S = \{1\}$ *or* $S = \{1, 0\}$.

Proof. Implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4), (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (2) are obvious.

(1) $\Rightarrow$ (2). Let A_S be a right S-act and $as = at, sz = tz$, for $a \in A_S, s, t, z \in S$. By Theorem 3.4, $S \times A_S$ satisfies Condition $(E'P)$. Since S is left collapsible, there exists $u \in S$ such that $us = ut$. Thus $(u, a)s = (u, a)t, sz = tz$ implies that there exist $(w, a') \in S \times A_S, u, v \in S$ such that $(u, a) = (w, a')u = (w, a')v, us = vt$. Hence $a = a'u = a'v, us = vt$ and so A_S satisfies Condition $(E'P)$.

(4) $\Rightarrow$ (5). By Corollary 3.5, all cyclic right S-acts satisfy Condition (P) and so by [13, IV, 9.9] S is a group or a group with a zero adjoined. Since S is right collapsible, $S = \{1\}$ or $S = \{1, 0\}$. □

Now for a monoid S we answer the question of when all generators right S-acts satisfy Condition (E'). Recall from [15] that A_S satisfies Condition (E'), if for all $a \in A_S, s, s', z \in S$

$as = as', sz = s'z \Rightarrow (\exists a' \in A_S)(\exists u \in S)(a = a'u \text{ and } us = us')$.

Theorem 3.7. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (E');*
2. *all finitely generated generators right S-acts satisfy Condition (E');*
3. *all generators right S-acts generated by at most two elements satisfy Condition (E');*
4. *$S \times A_S$ satisfies Condition (E') for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition (E') for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition (E') for every generator right S-act A_S generated by at most two elements;*
7. *$S \times A_S$ satisfies Condition (E') for every right S-act A_S;*
8. *$S \times A_S$ satisfies Condition (E') for every finitely generated right S-act A_S;*
9. *$S \times A_S$ satisfies Condition (E') for every cyclic right S-act A_S;*
10. *$S \times A_S$ satisfies Condition (E') for every monocyclic right S-act A_S;*
11. *a right S-act A_S satisfies Condition (E') if $Hom(A_S, S_S) \neq \emptyset$;*
12. *a finitely generated right S-act A_S satisfies Condition (E') if $Hom(A_S, S_S) \neq \emptyset$;*
13. *a cyclic right S-act A_S satisfies Condition (E') if $Hom(A_S, S_S) \neq \emptyset$;*
14. *a monocyclic right S-act A_S satisfies Condition (E') if $Hom(A_S, S_S) \neq \emptyset$;*
15. *$(\forall x, y \in S)(\ l(x, y) = \emptyset \vee\ r(x, y) = \emptyset \vee (\exists e \in E(S),\ \rho(x, y) = ker\lambda_e))$;*

16. *for every right subannihilator congruence* ρ*,* S/ρ *satisfies Condition* (E')*;*

17. $(\forall x, y \in S)(l(x,y) = \emptyset \vee S/\rho(x,y)$ *satisfies Condition* $(E'))$*;*

18. $(\forall x, y \in S)(l(x,y) = \emptyset \vee r(x,y) = \emptyset \vee (\exists u \in S, ux = uy \ \wedge \ 1 \ \rho(x,y) \ u))$*;*

19. $(\forall x, y, t \in S)(l(tx,ty) = \emptyset \vee S/\rho(tx,ty)$ *satisfies Condition* $(E'))$*;*

20. $(\forall x, y, t \in S)(l(tx,ty) = \emptyset \vee \ r(tx,ty) = \emptyset \vee (\exists u \in S, \ t \ \rho(tx,ty) \ u \ \wedge \ ux = uy))$*;*

21. $(\forall x, y \in S)(l(x,y) = \emptyset \vee \ r(x,y) = \emptyset \vee (\exists e \in E(S), \ ex = ey \wedge 1 \ \rho(x,y) \ e))$*;*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (11) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10), (11) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (14) and (17) $\Rightarrow$ (19) are obvious.

(20) $\Rightarrow$ (21). Let $x, y \in S$ such that $l(x,y) \neq \emptyset$ and $r(x,y) \neq \emptyset$. If $t = 1$, then there exists $u \in S$ such that $ux = uy$ and $1 \ \rho(x,y) \ u$. If $\rho = \rho(x,y)$, then $(x,y) \in ker\lambda_u$ implies that $\rho \subseteq ker\lambda_u$. Since $1 \ \rho \ u$ we have $(1,u) \in ker\lambda_u$, that is, $u = u^2$ and so u is an idempotent. If $u = e$, then we are done.

(21) $\Rightarrow$ (15). Let $x, y \in S$ such that $l(x,y) \neq \emptyset$ and $r(x,y) \neq \emptyset$. If $\rho = \rho(x,y)$, then by assumption there exists $e \in E(S)$ such that $ex = ey$ and $1 \ \rho \ e$. Thus $ex = ey$ implies $\rho \subseteq ker\lambda_e$. Let $l_1, l_2 \in S$, such that $(l_1, l_2) \in ker\lambda_e$. Then $el_1 = el_2$, and that $1 \ \rho \ e$ we have $l_1 \ \rho \ el_1$, $l_2 \ \rho \ el_2$ and so $l_1 \ \rho \ l_2$. Thus, $ker\lambda_e \subseteq \rho$, and so $ker\lambda_e = \rho$ as required.

(15) $\Rightarrow$ (1). Let A_S be a generator and let $\pi : A_S \longrightarrow S_S$ be an epimorphism. Let $a \in A_S, x, y, z \in S$ and suppose $ax = ay$ and $xz = yz$. Then $\pi(a)x = \pi(a)y$ and so $l(x,y) \neq \emptyset$. Also $xz = yz$ implies that $r(x,y) \neq \emptyset$. Thus by assumption there exists $e \in E(S)$ such that $\rho(x,y) = ker\lambda_e$. On the other hand $ax = ay$ implies that $(x,y) \in ker\lambda_a$ and so $ker\lambda_e = \rho(x,y) \subseteq ker\lambda_a$. Since $(1,e) \in ker\lambda_e \subseteq ker\lambda_a$ so $a = ae$. Also $(x,y) \in \rho(x,y) = ker\lambda_e$ implies that $ex = ey$, and so A_S satisfies Condition (E') as required.

(3) $\Rightarrow$ (1). Let A_S be a generator and $\pi : A_S \longrightarrow S_S$ be an epimorphism. Suppose $as = at, sz = tz$, for $a \in A_S$ and $s, t, z \in S$. Since π is an epimorphism, there exists $a' \in A_S$ such that $\pi(a') = 1$. If $A' = aS \cup a'S$, then A' is a generator and so by assumption satisfies Condition (E'). Thus $as = at, sz = tz$ implies the existence of $a'' \in A' \subseteq A$ and $u \in S$ such that $a = a''u$ and $us = ut$. That is A_S satisfies Condition (E'), as required.

(6) $\Rightarrow$ (4). Let A_S be a generator and let $(l,a)s = (l,a)t, sz = tz$, for $a \in A_S$ and $l, s, t, z \in S$. If $A' = aS \cup a'S$ is as in the proof of (3) $\Rightarrow$ (1), then A' is a generator and so $S \times A'$ satisfies Condition (E') by assumption. Thus there exist $(l', a'') \in S \times A' \subseteq S \times A$ and $u \in S$ such that $(l,a) = (l', a'')u$ and $us = ut$. Therefore $S \times A$ satisfies Condition (E').

(17) $\Rightarrow$ (18). Let $x, y \in S$ such that $l(x,y) \neq \emptyset$ and $r(x,y) \neq \emptyset$. Since $r(x,y) \neq \emptyset$, there exists $z \in S$ such that $xz = yz$. Then $S/\rho(x,y)$ satisfies Condition (E') and so $[1]_\rho x = [1]_\rho y, \ xz = yz$ implies that there exist $\alpha, u_1 \in S$

such that $[1]_\rho = [\alpha]_\rho u_1$ and $u_1x = u_1y$. If $\alpha u_1 = u$, then $[1]_\rho = [u]_\rho$ and $ux = uy$. Hence $1\ \rho(x, y)\ u$ and $ux = uy$.

(18) $\Rightarrow$ (20). Let $x, y, t \in S$ such that $l(tx, ty) \neq \emptyset$ and $r(tx, ty) \neq \emptyset$. Then by assumption there exists $u_1 \in S$ such that $u_1tx = u_1ty$ and $1\ \rho(tx, ty)\ u_1$. Thus $t\ \rho(tx, ty)\ u_1t$ and $u_1tx = u_1ty$. If $u_1t = u$, then $t\ \rho(tx, ty)\ u$ and $ux = uy$ as required.

(19) $\Rightarrow$ (20). Let $x, y, t \in S$ such that $l(tx, ty) \neq \emptyset$ and $r(tx, ty) \neq \emptyset$. Since $r(tx, ty) \neq \emptyset$, there exists $z \in S$ such that $txz = tyz$. Suppose $\rho(tx, ty) = \rho$. Since $[1]_\rho tx = [1]_\rho ty, txz = tyz$, there exist $\alpha, u \in S$, such that $[1]_\rho = [\alpha]_\rho u_1$ and $u_1tx = u_1ty$. If $\alpha u_1t = u$ then $t\ \rho(tx, ty)\ u$ and $ux = uy$ and so we are done.

(14) $\Rightarrow$ (17). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then there exists $z \in S$ such that $zx = zy$ and so $\rho(x, y) \leq ker\lambda_z$. Therefore, $Hom(S/\rho(x, y), S_S) \neq \emptyset$ and so by assumption $S/\rho(x, y)$ satisfies Condition (E'), as required.

(10) $\Rightarrow$ (17). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then there exists $l \in S$ such that $lx = ly$ and so $\rho(x, y) \leq ker\lambda_l$. Suppose $\rho(x, y) = \rho$ and let $l_1, l_2, z \in S$ such that $l_1\rho\ l_2$ and $l_1z = l_2z$, then $ll_1 = ll_2$. Thus $(l, [1]_\rho)l_1 = (l, [1]_\rho)l_2$ in $S \times S/\rho$ and $l_1z = l_2z$. By assumption there exist $(w, [a]_\rho) \in S \times S/\rho$ and $v \in S$ such that $(l, [1]_\rho) = (w, [a]_\rho)v$ and $vl_1 = vl_2$. If $av = u$, then $1\rho(x, y)u$ and $ul_1 = ul_2$. Thus $S/\rho(x, y)$ satisfies Condition (E') by [17, Lemma 2.2]. □

Recall from [1] and [4] that an act A_S satisfies *Condition* (PWP), if for all $a, a' \in A_S, s \in S$, $as = a's \Rightarrow (\exists a'' \in A_S)(\exists u, v \in S)(a = a''u,\ a' = a''v$ and $us = vs)$. Also we say that A_S satisfies *Condition* (PWP_E), if for all $a, a' \in A_S, s \in S$, $as = a's \Rightarrow (\exists a'' \in A_S)(\exists u, v \in S)(\exists e, f \in E(S))(ae = a''ue,\ a'f = a''vf, es = s = fs$ and $us = vs)$.

Theorem 3.8. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (PWP);*
2. *all finitely generated generators right S-acts satisfy Condition (PWP);*
3. *all generators right S-acts generated by at most three elements satisfy Condition (PWP);*
4. *$S \times A_S$ satisfies Condition (PWP) for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition (PWP) for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition (PWP) for every generator right S-act A_S generated by at most three elements;*
7. *$S \times A_S$ satisfies Condition (PWP) for every right S-act A_S;*
8. *$S \times A_S$ satisfies Condition (PWP) for every finitely generated right S-act A_S;*

9. *$S \times A_S$ satisfies Condition (PWP) for every right S-act A_S generated by at most two elements;*

10. *a right S-act A_S satisfies Condition (PWP) if $Hom(A_S, S_S) \neq \emptyset$;*

11. *a finitely generated right S act A_S satisfies Condition (PWP) if $Hom(A_S, S_S) \neq \emptyset$;*

12. *a right S-act A_S generated by at most two elements satisfies Condition (PWP) if $Hom(A_S, S_S) \neq \emptyset$;*

13. *all right S-acts satisfy Condition (PWP).*

14. *S is group.*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) and (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious.

Implications (13) $\Rightarrow$ (14) and (14) $\Rightarrow$ (1) follows from [1, Proposition 9].

(9) $\Rightarrow$ (7). Let A_S be a right S-act and $(s, x)c = (t, y)c$, for $s, t, c \in S$ and $x, y \in A_S$. If $A' = xS \cup yS$, then $S \times A'$ satisfies Condition (PWP) by assumption. Thus there exist, $(w, z) \in S \times A' \subseteq S \times A$ for $w \in S, z \in A'$ and $u, v \in S$ such that $(s, x) = (w, z)u, (t, y) = (w, z)v$ and $us = vs$. That is, $S \times A_S$ satisfies Condition (PWP) as required.

(12) $\Rightarrow$ (10). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$ and suppose $as = a's$, for $a, a' \in A_S$ and $s \in S$. Since $Hom(A_S, S_S) \neq \emptyset$, there exists homomorphism $f : A_S \longrightarrow S_S$. If $A' = aS \cup a'S$ and $f' = f\mid_{A'}$, the result follows by assumption.

(3) $\Rightarrow$ (1). Let A_S be a generator. Then there exists an epimorphism $\pi : A_S \longrightarrow S_S$. Suppose now that $as = a's$, for $a, a' \in A_S$ and $s \in S$. Since π is an epimorphism, there exists $c \in A_S$ such that $\pi(c) = 1$. Let $A' = aS \cup a'S \cup cS$ and so $\pi\mid_{A'}: A' \longrightarrow S$ is an epimorphism. Thus A' is a generator and so A' satisfies Condition (PWP) by assumption. Hence $as = a's$ implies that there exist $a'' \in A' \subseteq A, u, v \in S$ such that $a = a''u, a' = a''v$ and $us = vs$, thus the result follows.

(6) $\Rightarrow$ (1). Let A_S be a generator and let $as = a's$, for $a, a' \in A_S$ and $s \in S$. If $A' = aS \cup a'S \cup cS$ is as in the proof of (3) $\Rightarrow$ (1), then $(1, a)s = (1, a')s$. Clearly, A' is a generator and so $S \times A'$ satisfies Condition (PWP) by assumption. Thus $(1, a)s = (1, a')s$ implies that there exist $(w, z) \in S \times A' \subseteq S \times A_S$, $w \in S, a \in A' \subseteq A$ and $u, v \in S$ such that $(1, a) = (w, z)u, (1, a') = (w, z)v$ and $us = vs$. Thus $a = zu, a' = zv$ and $us = vs$ and so the result follows.

(7) $\Rightarrow$ (13). Let A_S be a right S-act and $as = a's$, for $a, a' \in A_S, s \in S$. Thus $(1, a)s = (1, a')s$. Since $S \times A$ satisfies Condition (PWP), there exist $(w, a'') \in S \times A_S, u, v \in S$ such that $(1, a) = (w, a'')u, (1, a') = (w, a'')v$ and $us = vs$. Thus $a = a''u, a' = a''v$ and $us = vs$, as required. □

From Theorems 3.2, 3.4, 3.7, 3.8 and [16, Theorems 3.21, 3.22, 3.23], and [16, Proposition 3.16], we have the following corollary.

Corollary 3.9. *Let S be a monoid over which all generators satisfy Condition (EP)(Condition $(E'P)$, Condition (E'), Condition (PWP)). Then for any nonempty family $\{A_i \mid i \in I\}$ of right S-acts satisfying Condition (EP)(Condition $(E'P)$, Condition (E'), Condition (PWP)), $\prod_I A_i$ satisfies Condition (EP) (Condition $(E'P)$, Condition (E'), Condition (PWP)).*

By a similar argument as in the proof of Theorem 3.8 and using [4, Theorem 3.1] we can show the following theorem.

Theorem 3.10. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (PWP_E);*
2. *all finitely generated generators right S-acts satisfy Condition (PWP_E);*
3. *all generators right S-acts generated by at most three elements satisfy Condition (PWP_E);*
4. *$S \times A_S$ satisfies Condition (PWP_E) for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition (PWP_E) for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition (PWP_E) for every generator right S-act A_S generated by at most three elements;*
7. *$S \times A_S$ satisfies Condition (PWP_E) for every right S-act A_S;*
8. *$S \times A_S$ satisfies Condition (PWP_E) for every finitely generated right S-act A_S;*
9. *$S \times A_S$ satisfies Condition (PWP_E) for every right S-act A_S generated by at most two elements;*
10. *a right S-act A_S satisfies Condition (PWP_E) if $Hom(A_S, S_S) \neq \emptyset$;*
11. *a finitely generated right S act A_S satisfies Condition (PWP_E) if $Hom(A_S, S_S) \neq \emptyset$;*
12. *a right S act A_S generated by at most two elements satisfies Condition (PWP_E) if $Hom(A_S, S_S) \neq \emptyset$;*
13. *all right S-acts satisfy Condition (PWP_E);*
14. *S is regular.*

We recall from [5] and [13] that a right $S-$act A_S satisfies *Condition* (P_E), if for all $a, a' \in A, s, s' \in S$, $as = a's' \Rightarrow (\exists a'' \in A)(\exists u, u' \in S)(\exists e, f \in E(S))(ae = a''ue,\ a'f = a''u'f, es = s, fs' = s'$ and $us = u's')$ also S is called *right reversible*, if for any $s, t \in S$, there exist $u, v \in S$ such that $us = vt$.

We recall from [19], a right S-act A_S is called *almost weakly flat* if A_S is principally weakly flat and satisfies Condition

(W') If $as = a't$, and $Ss \cap St \neq \emptyset$, for $a, a' \in A_S, s, t \in S$, then there exists $a'' \in A_S$, $u \in Ss \cap St$ such that $as = a't = a''u$. It is proved in [19, Theorem 3.4] that all generators are weakly flat if and only if all right S-acts are almost weakly flat.

Theorem 3.11. *For any monoid S the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (P_E);*
2. *all finitely generated generators right S-acts satisfy Condition (P_E);*
3. *all generators right S-acts generated by at most three elements satisfy Condition (P_E);*
4. *$S \times A_S$ satisfies Condition (P_E) for every generator right S-act A_S;*
5. *$S \times A_S$ satisfies Condition (P_E) for every finitely generated generator right S-act A_S;*
6. *$S \times A_S$ satisfies Condition (P_E) for every generator right S-act A_S generated by at most three elements;*
7. *$S \times A_S$ satisfies Condition (P_E) for every right S-act A_S;*
8. *$S \times A_S$ satisfies Condition (P_E) for every finitely generated right S-act A_S;*
9. *$S \times A_S$ satisfies Condition (P_E) for every right S-act A_S generated by at most two elements;*
10. *a right S-act A_S satisfies Condition (P_E) if $Hom(A_S, S_S) \neq \emptyset$;*
11. *a finitely generated right S act A_S satisfies Condition (P_E) if $Hom(A_S, S_S) \neq \emptyset$;*
12. *a right S act A_S generated by at most two elements satisfies Condition (P_E) if $Hom(A_S, S_S) \neq \emptyset$;*
13. *all right S-acts are almost weakly flat.*
14. *S is regular and for every $s, t \in S$ with $Ss \cap St \neq \emptyset$, there exists $w \in Ss \cap St$ such that $1\ (ker\lambda_s \vee ker\lambda_t)\ w$.*

Proof. Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) are clear from Theorem 1.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9), (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious.

(1) $\Rightarrow$ (13). Since Condition (P_E) implies weak flatness by [9, Theorem 2.3], the result follows by [19, Theorem 3.4].

(13) $\Rightarrow$ (14). It follows from [19, Theorems 3.4, 3.8].

(3) $\Rightarrow$ (1). It is similar to (3) $\Rightarrow$ (1) of Theorem 3.8.

(14) $\Rightarrow$ (1). Since S is regular, it is left PP and so by [9, Theorem 2.5] Condition (P_E) and weak flatness are the same and so the result follows by [19, Theorem 3.8] .

(9) $\Rightarrow$ (7). Let A_S be a right S-act and $(w_1, a)s = (w_2, a')t$, for $w_1, w_2 \in S, a, a' \in A$. If $A' = aS \cup a'S$, then $S \times A'$ satisfies Condition (P_E) by assumption. Hence there exists $(w, a'') \in S \times A' \subseteq S \times A_S$ and $u, v \in S, e, f \in E(S)$, such that $(w_1, a)e = (w, a'')ue, (w_2, a')f = (w, a'')vf, es = s, ft = t$ and $us = vt$, as required.

(12) $\Rightarrow$ (10). Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \emptyset$ and $as = a't$, for $a, a' \in A_S, s, t \in S$. Since $Hom(A_S, S_S) \neq \emptyset$, there exists an S-homomorphism $f : A_S \longrightarrow S_S$. Let $A' = aS \cup a'S$ and $f' = f\mid_{A'}$. Thus $as = a't$ in A' implies that there exists $a'' \in A' \subseteq A_S, u, v \in S, e, f \in E(S)$ such that $ae = a''ue, a'f = a''vf, es = s, ft = t, us = vt$, and so A_S satisfies Condition (P_E) as required.

(6) $\Rightarrow$ (1). Let A_S be a generator and let $as = a's'$ for $a, a' \in A_S, s, s' \in S$. If $A' = aS \cup a'S \cup cS$ is as in the proof of (3) $\Rightarrow$ (1) of Theorem 3.8, then $(\pi(a), a)s = (\pi(a'), a')s'$. Clearly, A' is a generator and so $S \times A'$ satisfies Condition (P_E) by assumption. Thus the last equality implies that there exist $(w, a'') \in S \times A' \subseteq S \times A, u, v \in S$ and $e, f \in E(S)$ such that $(\pi(a), a)e = (w, a'')ue, (\pi(a'), a')f = (w, a'')vf, es = s, fs' = s'$ and $us = u's'$ and so the result follows. □

Corollary 3.12. *Let S be a right reversible monoid. Then the following statements are equivalent:*

1. *all generators right S-acts satisfy Condition (P_E);*
2. *all right S-acts satisfy Condition (P_E);*
3. *S is regular and satisfies Condition (R).*

$$(R) : (\forall s, t \in S)(\exists w \in Ss \cap St),\ w\ \rho(s, t)\ s$$

Proof. (2) $\Leftrightarrow$ (3) By [5,Theorem 2.1].

(2) $\Rightarrow$ (1). It is obvious.

(1) $\Rightarrow$ (3). Since S is right reversible, weak flatness and almost weak flatness are equivalent. By Theorem 3.11, all S-acts are weakly flat, and so the result follows from [13, IV,7.5]. □

From Theorems 3.10, 3.11 and [16, Theorems 3.18, 3.15] we have the following corollary.

Corollary 3.13. *Let S be a commutative monoid over which all generators satisfy Condition $(PWP_E)((P_E))$. If $A_1, \cdots A_n, n \in \mathbb{N}$ satisfy Condition (PWP_E) $((P_E))$, then $\prod_{i=1}^{n} A_i$ satisfies Condition $(PWP_E)((P_E))$.*

Definition 3.14 ([20]). *An act A_S is called strongly torsion free (STF) if for any $a, b \in A$ and any $s \in S$ the equality $as = bs$ implies $a = b$.*

It is obvious that $STF \Rightarrow Condition(PWP) \Rightarrow PWF \Rightarrow TF$.

Corollary 3.15. *Let S be a monoid and (U) be any of the conditions or properties $WPF, WKF, PWKF, TKF, (P), (WP), (P'), STF$ of right S-acts, then the following statements are equivalent:*

1. *all generators right S-acts satisfy (U);*
2. *$S \times A_S$ satisfies (U) for every generator right S-act A_S;*
3. *$S \times A_S$ satisfies (U) for every right S-act A_S;*
4. *a right S-act A_S satisfies (U) if $Hom(A_S, S_S) \neq \emptyset$;*
5. *all right S-acts satisfy (U).*
6. *S is group.*

Proof. Implications $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$ are clear from Theorem 1.1 and $(5) \Leftrightarrow (6)$ follows from [1, Proposition 9], [2, Theorem 2.5] and [20, Theorem 3.2].

$(6) \Rightarrow (1)$. It is obvious.

$(1) \Rightarrow (6)$. Since $(U) \Longrightarrow (PWP)$ the result follows from Theorem 3.8. □

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