GLOBAL EXPONENTIAL STABILITY OF COHEN-GROSSBERG NEURAL NETWORKS WITH TIME-VARYING DELAYS

Xiaohui Wang
College of Science
Hunan Agricultural University
Changsha, Hunan 410128
People's Republic of China

Xumeng Li*
College of Science
Hunan Agricultural University
Changsha, Hunan 410128
People's Republic of China
xm.li@hunau.edu.cn

Wenxin Zhou
College of Science
Hunan Agricultural University
Changsha, Hunan 410128
People's Republic of China

Abstract. In this paper, without the assumptions for boundedness, monotonicity, and differentiability on activation functions and symmetry of interconnections, a class of Cohen-Grossberg neural networks with time-varying delays is studied. A new useful criteria on the uniqueness of equilibrium is obtained by utilizing the nonlinear measure. Combining with Dini derivatives and Young inequality, new sufficient condition for the global exponential stability is established by directly estimating the upper bound of solutions of the system. All results are presented in M-matrix form, which extended and generalized the corresponding results in previous literature.
Keywords: Cohen-Grossberg neural networks, time-varying delays, unique equilibrium, global exponential stability, nonlinear measure, M-matrix.

1. Introduction

The classic Cohen-Grossberg neural network was initially proposed and studied by Cohen and Grossberg in 1983([1]), which can be described by the following ordinary differential equations:

$$(1) \qquad \dot{x}_i(t) = a_i(x_i(t))\{-b_i(x_i(t)) + \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) + I_i\}, i = 1, 2, \ldots, n.$$

$*$. Corresponding author

Here $n \geq 2$ is the number of neurons in the network, $x_i(t)$ is the state of neuron i at time t, $a_i(\cdot)$ represents an amplification function, $b_i(\cdot)$ is an appropriately behaved function to keep the solutions of system (1) bounded, the activation function $f_j(\cdot)$ shows how neuron i reacts to the input, $W = (w_{ij})_{n\times n}$ is a real constant matrix and denotes the normal weights of the neuron interconnections. System(1) is a very general neural network model. Models such as the Hopfield neural networks, cellular neural networks, and bidirectional associative memory neural networks are its special cases(see for instance [4],[7],[8]). All these neural neural networks have attracted much attention for they successful or promising potential applications in the pattern recognition, associative memory,signal processing,and optimization([5],[9],[10]).

In reality, however, time delays universally exist in biological and artificial neural networks due to the finite switching speed of neurons and amplifiers. It is well known that, with symmetric connection matrix and the so called sigmoid activation functions, system (1) has the property of absolute stability([1]), i.e. given any initial conditions, the solution of system (1) converges to some equilibrium of the system. On the other hand, the existence of time delays is frequently a source of oscillation and instability([6]). Marcus and Westervelt ([2]) first introduced a single delay into the model and observed sustained oscillations even with symmetric connections. Moreover, the delays in artificial neural networks are usually time-varying([3]). Therefore, it is natural and important to incorporate time-varying delays into the model. A general Cohen-Grossberg neural network with time-varying delays can be described by the following retarded differential difference equations:

$$\begin{aligned} \dot{x}_i(t) &= a_i(x_i(t))\{-b_i(x_i(t)) + \sum_{j=1}^{n} w_{ij} f_j(x_j(t)) \\ &\quad + \sum_{j=1}^{n} w_{ij}^d f_j(x_j(t-d_{ij}(t))) + I_i\}, i = 1, 2, \dots, n \end{aligned} \tag{2}$$

where n, $x_i(t)$, $a_i(\cdot)$, $b_i(\cdot)$ and $W = (w_{ij})_{n\times n}$ are the same as these in system (1), $W^d = (w_{ij}^d)_{n\times n}$ is a real constant matrix and denotes the delayed weights of the neuron interconnections, $d_{ij}(t)$ is the time delay required in processing and transmitting signals from neuron j to neuron i at time t.

The stability of neural networks is a prerequisite for almost all applications. In applications of neural networks to parallel computation, signal processing and other problems involving the solutions of optimization problems, it is frequently required that the network have a unique global attractive equilibrium. Meanwhile, in designing and implementing a network, it is preferable and desirable that the neural network not only converges to an equilibrium, but also converges as fast as possible. It is well known that the exponential stability gives a fast convergence rate to the equilibrium. Thus, the global exponential stability of system(1), system (2) and their special cases are of great importance, and has been widely investigated. Many useful results have been obtained by some authors in the previous literature([14][16], [17],[18],[19],[20],[21],[22],[23],[24],[25],[26]).

In applications, the activation functions maybe just continuous. On the other hand, the assumption of symmetry connections also lays a restriction on the connection topology of the networks. In this paper, without any assumption on the boundedness, monotonicity, and differentiability of activation functions and symmetry of interconnections, we will study the existence and global exponential stability of an equilibrium for System (2). A new useful criteria on the uniqueness of equilibrium is obtained by utilizing the nonlinear measure, which was initially introduced in [11]. Combining with Dini derivatives and Young inequality, new sufficient conditions for the global exponential stability are established by inroducing many real paremeters and directly estimating the upper bound of solutions of the system. This mathod has been utilized by some authors (see for instance,[14],[22],[17]), but we do introduce a different type parameters. Our results extend and generalize the corresponding results in previous literature. By the way, for the common use of M-matrix in the study of qualitative properties of various neural networks, we represent our results in M-matrix form.

The remainder of this paper is organized as follows. In section 2, the basic notations, definitions, and some useful lemmas are introduced. Some assumptions used in the main results are listed there, too. In section 3, the main results of this paper are proposed and proved. Some remarks and corollaries are given. In section 4, some examples are given to demonstrate the main results.

2. Preliminaries

In this section, we state some notations, definitions and lemmas.

Let $N = \{1, 2, \ldots, n\}$, R denote the set of real numbers, R^n denote the n-dimensional real vector space, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_r$ $(r \geq 1)$ denote the usual inner product and l^r norm of vectors in R^n respectively. For $x \in R^n$, x_i denotes the ith coordinate of x, x^T denotes the transpose of x, $sign(x) = (sign(x_1), \ldots, sign(x_n))^T$ denotes the sign vector of x, where $sign(\cdot)$ is the sign function of real numbers. $R^n_+ = \{x | x \in R^n,\, x_i > 0,\, for\ i \in N\}$. $R^{n\times n}$ denotes the set of all $n \times n$ real matrices, $diag\{a_1, \ldots, a_n\}$ denotes the usual diagonal matrix. For $A = (a_{ij}) \in R^{n\times n}$, $|A| = (|a_{ij}|)$, A^T denotes the transpose of A, A^{-1} denotes the inverse of A (if it has). For two matrices $A = (a_{ij}), B = (b_{ij}) \in R^{n\times n}$, we say $A \leq B$ if $a_{ij} \leq b_{ij},\ for\ i, j \in N$.

Throughout this paper, we always assume that for $i, j \in N, a_i, b_i, f_i, d_{ij} : R \to R$ are continuous functions, and there exist a real number d such that $0 \leq d_{ij}(t) \leq d$, $t \in R$. Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ denote the solution of system (2). System (2) is supplemented with initial values of the type

$$x(t) = \varphi(t), \varphi(t) \in C([-d, 0], R^n),\ t \in [-d, 0],$$

where $C([-d,0],R^n)$ denotes the space of continuous functions $\varphi:[-d,0]\to R^n$.

Definition 1. *Suppose that $x^* \in R^n$ is an equilibrium of system (2), x^* is said to be globally exponentially stable, if there exist $\lambda>0$ and $C>0$ such that for any solution $x(t)$ of system (2), we have*

$$|x_i(t)-x_i^*| \le C\sum_{i=1}^{n}\sup_{t\in[-d,0]}|\varphi_i(t)-x_i^*|e^{-\lambda t}, for\ t\ge 0,\ i\in N.$$

In order to obtain the uniqueness of equilibrium of system (2), we introduce some results about nonlinear measure from J.Peng,H.Qiao and Z.Xu([11]).

Definition 2 ([11])**.** *Suppose that Ω is an open subset of R^n, F is an operator from Ω to R^n, the constant*

$$m_\Omega(F)=\sup_{x,y\in\Omega,x\ne y}\frac{\langle F(x)-F(y),sign(x-y)\rangle}{\|x-y\|_1}$$

is called the nonlinear measure of F on Ω.

Lemma 2.1 ([11])**.** *If $m_\Omega(F)<0$, then F is injective on Ω. If in addition $\Omega=R^n$, then F is a homeomorphism of R^n.*

For the study of the global exponential stability of an equilibrium by our methods, we introduce the Dini derivatives and the Young inequality now.

Definition 3. *Suppose that $V(t)$ is a real function, the left upper Dini derivative and the left lower Dini derivative of $V(t)$, denoted by $D^-V(t)$ and $D_-V(t)$ respectively , are defined by*

$$D^-V(t)=\limsup_{h\to 0^-}\frac{V(t+h)-V(t)}{h},\ D_-V(t)=\liminf_{h\to 0^-}\frac{V(t+h)-V(t)}{h}.$$

Lemma 2.2 ([12],Young inequality)**.** *Assume that $a\ge 0, b\ge 0, p>1, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then we have inequality*

$$ab\le \frac{1}{p}a^p+\frac{1}{q}b^q.$$

For the common use of M-matrix in the study of qualitative properties of various neural networks, we represent all of our results in the M-matrix form. The definition of M-matrix and some useful results about it are given as follows.

Definition 4 ([14])**.** *Let $A=(a_{ij})\in R^{n\times n}$, and $a_{ij}\le 0$ for $i\ne j$, $i,j\in N$. A is called M-matrix if there exist $P\in R_+^n$, such that $AP>0$ or $P^TA>0$.*

Remark 2.3. From the definition, it is obvious that A is a M-matrix if and only if for every diagonal matrix D with positive diagonal elements, DA and AD are M-matrices.

Lemma 2.4 ([13]). *Let $A = (a_{ij}) \in R^{n\times n}$, and $a_{ij} \le 0$ for $i \ne j$, $i, j \in N$. Then the following conditions are equivalent:*

(1) *A is a M-matrix;*

(2) *All the leading principal minors of A are positive;*

(3) *$A^{-1} \ge 0$.*

Remark 2.5. From lemma 2.4 (2), we can obtain that A is a M-matrix if and only if A^T is. Thus in the definition, The statement " there exist $P \in R^n_+$, such that $AP > 0$ or $P^T A > 0$ " is equivalent to the statement " there exist P_1, $P_2 \in R^n_+$, such that $AP_1 > 0$ and $P_2^T A > 0$ ".

Lemma 2.6 ([25]). *Let $B = (b_{ij}) \in R^{n\times n}$, and $b_{ij} \le 0$ for $i \ne j$, $i, j \in N$. If $B \ge A$ and A is a M-matrix, then B is a M-matrix.*

At the end of this section, we list some assumptions which will be used in the main results of system (2).

(H_1) For each $i \in N$, $a_i(s) > 0, for\ s \in R$.

(H_1') For each $i \in N$, there exist positive real numbers $\underline{\alpha}_i$ and $\overline{\alpha}_i$ such that

$$\underline{\alpha}_i \le a_i(s) \le \overline{\alpha}_i, for\ s \in R.$$

(H_2) For each $i \in N$, b_i is global left Lipschitz continuous, i.e. there exists a positive constant $\beta_i > 0$ such that

$$\frac{b_i(s) - b_i(t)}{s - t} \ge \beta_i, for\ s, t \in R\ and\ s \ne t.$$

(H_3) For each $i \in N$, f_i is global Lipschitz continuous, i.e. there exists a positive constant L_i such that

$$|f_i(s) - f_i(t)| \le L_i|s - t|, for\ s, t \in R.$$

3. Main results

In this section, a new criteria on uniqueness of equilibrium of system (2) is firstly proposed and proved. The remainder is mainly concerned with the global exponential stability.

On the existence of equilibrium of system (2), we refer to the powerful result initially proposed recently in K. Lu, D. Xu and Z. Yang ([14]).

Theorem 3.1 ([14]). *Assume that (H_1) holds, and there are nonnegative constants such that*

$$b_i(s) \ge \beta_i^0|s| - \xi_i, |f_i(s)| \le L_i^0|s| + \mu_i, for\, s \in R,\ i \in N.$$

If

$$M_0 \triangleq B_0 - |W + W^d|L_0 \tag{3}$$

is a M-matrix, where $B_0 = diag\{\beta_1^0, \dots, \beta_n^0\}$ and $L_0 = diag\{L_1^0, \dots, L_n^0\}$, then System (2) has at least a equilibrium x^*. Furthermore, $|x^*| \le M_0^{-1}(|W+W^d|\xi+\mu + |I|)$, where $\xi = (\xi_1, \dots, \xi_n)^T$, and $\mu = (\mu_1, \dots, \mu_n)^T$.

Now, we propose and prove our result on uniqueness of equilibrium of system (2).

Theorem 3.2. *Assume that* (H_1), (H_2), (H_3) *hold. If*

$$M \triangleq B - |W + W^d|L \tag{4}$$

is a M-matrix, where $B = diag\{\beta_1, \dots, \beta_n\}$ *and* $L = diag\{L_1, \dots, L_n\}$, *then System* (2) *has a unique equilibrium* x^*. *Furthermore, for any* $x \in R^n$, *let* $y = x - x^*$, *we have* $|y| \le M^{-1}J(x)$, *where* $J : R^n \to R^n$ *is defined by*

$$J_i(x) = \sum_{j=1}^{n} |w_{ij} + w_{ij}^d||f_j(x_j)| + |I_i - b_i(x_i)|,\ for\ i \in N.$$

In particular, $|x^*| \le M^{-1}J(0)$.

Proof. By (H_1), x^* is an equilibrium of System (2) if and only if $x = x^*$ is a solution of equations

$$F_i(x) \triangleq -b_i(x_i) + \sum_{j=1}^{n}(w_{ij} + w_{ij}^d)f_j(x_j) + I_i = 0,\ for\ i \in N. \tag{5}$$

Using (4), there exists $P \in R_+^n$ such that $P^T M > 0$, i.e.

$$P_i\beta_i - \sum_{j=1}^{n} P_j|w_{ji} + w_{ji}^d|L_i > 0,\ for\ i \in N. \tag{6}$$

Define $F : R^n \to R^n$ by $F(x) = (P_1F_1(x), \dots, P_nF_n(x))^T, \forall x \in R^n$. Then x^* is an equilibrium of System (2) if and only if $F(x^*) = 0$.

We now prove that $m_{R^n}(F) < 0$. For all $x, y \in R^n$, Using (H_2) and (H_3),

$$\begin{aligned}
&\langle F(x) - F(y), sign(x-y)\rangle \\
&= \sum_{i=1}^{n} P_i\{-(b_i(x_i) - b_i(y_i)) + \sum_{j=1}^{n} |w_{ij} + w_{ij}^d|(f_j(x_j) - f_j(y_j))\}sign(x_i - y_i) \\
&\le \sum_{i=1}^{n} P_i(-\beta_i|x_i - y_i| + \sum_{j=1}^{n} |w_{ij} + w_{ij}^d|L_j|x_j - y_j|) \\
&= -\sum_{i=1}^{n}(P_i\beta_i - \sum_{j=1}^{n} P_j|w_{ji} + w_{ji}^d|L_i)|x_i - y_i| \\
&\le -\min_{i\in N}\{P_i\beta_i - \sum_{j=1}^{n} P_j|w_{ji} + w_{ji}^d|L_i\} \sum_{i=1}^{n} |x_i - y_i|.
\end{aligned}$$

Therefore, by means of (6),

$$m_{R^n}(F) \le -\min_{i\in N}\{P_i\beta_i - \sum_{j=1}^{n} P_j|w_{ji}+w_{ji}^d|L_i\} < 0.$$

According to lemma 2.1, F is a homeomorphism of R^n, which indicates that $F(x)=0$ has a unique solution x^*, and thus system (2) has a unique equilibrium x^*. To estimate the existence region of x^*, for $\forall x \in R^n$ and $i \in N$, using (H_2) , (H_3) and (5), we can obtain

$$\begin{aligned}
&\beta_i|x_i^*-x_i| \le (b_i(x_i^*)-b_i(x_i))sign(x_i^*-x_i)\\
&=(\sum_{j=1}^{n}(w_{ij}+w_{ij}^d)f_j(x_j^*)+I_i-b_i(x_i))sign(x_i^*-x_i)\\
&\le |\sum_{j=1}^{n}(w_{ij}+w_{ij}^d)f_j(x_j^*)+I_i-b_i(x_i)|\\
&\le |\sum_{j=1}^{n}|w_{ij}+w_{ij}^d|L_j|x_j^*-x_i|+\sum_{j=1}^{n}|w_{ij}+w_{ij}^d||f_j(x_j)|+|I_i-b_i(x_i)|,
\end{aligned}$$

namely, $M|y| \le J(x)$. Utilizing lemma 2.4, $M^{-1} \ge 0$, then we have $|y| \le M^{-1}J(x)$. In particular, Let $x=0$, $|x^*| \le M^{-1}J(0)$. The proof is completed.□

Remark 3.3. As far as existence of equlibrium concerned, theorem 3.2 is just a direct corollary of theorem 3.1. But theorem 3.1 can not ensure the uniqueness of equilibrium. To the best of our knowledge, there are few results about the uniqueness of equilibrium ([21]), most of the existing results in the previous literature just deal with the existence ([18], [16]). Moreover, our estimate of the existence region of equilibrium can be used many times by setting different x to obtain more exact estimation.

For the global exponential stability of equilibrium of system (2), we have the following main result.

Theorem 3.4. *Assume that all the conditions in theorem 3.1 hold. Moreover,* $(H_1^{'})$,(H_2) *and* (H_3) *hold. If there exist* $r \ge 1$, $P=(P_1,\dots,P_n)^T \in R_+^n$, *and real numbers* h_{ij}, l_{ij}, h_{ij}^*, l_{ij}^*, *for* $i,j \in N$, *such that*

$$M_{r,P,h,l,h^*,l^*} \triangleq (m_{ij}) \tag{7}$$

is a M-matrix, where for $i,j\in N$,

$$\begin{aligned}
m_{ii} &= \{r\frac{\underline{\alpha}_i}{\overline{\alpha}_i}\beta_iP_i-(r-1)\sum_{k=1}^{n}(|w_{ik}|^{\frac{r-h_{ik}}{r-1}}L_k^{\frac{r-l_{ik}}{r-1}}+|w_{ik}^d|^{\frac{r-h_{ik}^*}{r-1}}L_k^{\frac{r-l_{ik}^*}{r-1}})P_k\\
&\quad -(|w_{ii}|^{h_{ii}}L_i^{l_{ii}}+|w_{ii}^d|^{h_{ii}^*}L_i^{l_{ii}^*})P_i\},\\
m_{ij} &= -(|w_{ij}|^{h_{ij}}L_j^{l_{ij}}+|w_{ij}^d|^{h_{ij}^*}L_j^{l_{ij}^*})P_j, i\ne j,
\end{aligned}$$

then System (2) *has a unique equilibrium, which is globally exponentially stable.*

Proof. From theorem 3.1, system (2) has at least one equilibrium, say x^*. Uniqueness of equilibrium can be induced directly from the global exponential stability of x^*. So we just need to prove the global exponential stability of x^*. Let $y(t) = x(t) - x^*$, substitute $x(t) = y(t) + x^*$ into system (2), for each $i \in N$,

$$(8)\quad \dot{y}_i(t) = A_i(y_i(t))\{-B_i(y_i(t)) + \sum_{j=1}^{n} w_{ij} g_j(y_j(t)) + \sum_{j=1}^{n} w_{ij}^d g_j(y_j(t - d_{ij}(t)))\}.$$

Here, for each $i, j \in N$, $A_i(y_i(t)) = a_i(y_i(t) + x_i^*)$, $B_i(y_i(t)) = b_i(y_i(t)) - b_i(x_i^*)$, $g_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$.

We will show that $y(t) = 0$ is globally exponentially stable. Firstly, using $(H_1^{'})$, (H_2) and (H_3),

$$(9)\qquad \underline{\alpha}_i \leq A_i(s) \leq \overline{\alpha}_i,\ \frac{B_i(s)}{s} \geq \beta_i,\ |g_i(s)| \leq L_i|s|,\ for\, s \in R,\ i \in N.$$

Using (7), let $\overline{A}$=$diag\{\overline{\alpha}_1, \ldots, \overline{\alpha}_n\}$, there exists $Q \in R_+^n$ such that $M_{r,P,h,l,h^*,l^*}\overline{A}Q > 0$, i.e.

$$r\underline{\alpha}_i\beta_i P_i Q_i - (r-1)\overline{\alpha}_i \sum_{j=1}^{n}(|w_{ij}|^{\frac{r-h_{ij}}{r-1}} L_j^{\frac{r-l_{ij}}{r-1}} + |w_{ij}^d|^{\frac{r-h_{ij}^*}{r-1}} L_j^{\frac{r-l_{ij}^*}{r-1}})P_j Q_i$$
$$(10)\qquad - \sum_{j=1}^{n}|w_{ij}|^{h_{ij}} L_j^{l_{ij}}\overline{\alpha}_j P_j Q_j - \sum_{j=1}^{n}|w_{ij}^d|^{h_{ij}^*} L_j^{l_{ij}^*}\overline{\alpha}_j P_j Q_j > 0,\ for\ i \in N.$$

So, we can choose a real number $0 < \lambda \ll 1$ such that

$$r\underline{\alpha}_i\beta_i P_{i_0} Q_i - (r-1)\overline{\alpha}_i \sum_{j=1}^{n}(|w_{ij}|^{\frac{r-h_{ij}}{r-1}} L_j^{\frac{r-l_{ij}}{r-1}} + |w_{ij}^d|^{\frac{r-h_{ij}^*}{r-1}} L_j^{\frac{r-l_{ij}^*}{r-1}})P_j Q_i$$
$$(11)\qquad - \sum_{j=1}^{n}|w_{ij}|^{h_{ij}} L_j^{l_{ij}}\overline{\alpha}_j P_j Q_j - \sum_{j=1}^{n}|w_{ij}^d|^{h_{ij}^*} L_j^{l_{ij}^*}\overline{\alpha}_j P_j Q_j e^{\lambda d} - P_i Q_i \lambda > 0,$$

for $i \in N$.

Let $y_i(t) = P_i z_i(t)$, $U_i(t) = \left|\int_0^{z_i(t)} \frac{|s|^{r-1}}{A_i(P_i s)} ds\right|$. It follows From (9) that

$$(12)\qquad \frac{|z_i(t)|^r}{r\overline{\alpha}_i} \leq U_i(t) \leq \frac{|z_i(t)|^r}{r\underline{\alpha}_i}$$

For any $\varepsilon > 0$, let $V(t) = m\sum_{j=1}^{n}(\sup_{s\in[-d,0]}|z_j(s)|^r + \varepsilon)e^{-\lambda t}$, $V_i(t) = Q_i V(t)$, where $m \gg 1$ is a constant such that $mQ_i e^{-\lambda t} > \frac{1}{r\underline{\alpha}_i},\ for\ t \in [-d, 0]\, i \in N$. Then $U_i(t) < V_i(t),\ for\ t \in [-d, 0], i \in N$. We claim that

$$(13)\qquad U_i(t) < V_i(t), for\ i \in N, t \in [-d, \infty).$$

Contrarily, there must exist $i_0 \in N$ and $t_0 > 0$ such that

$$U_{i_0}(t_0) = V_{i_0}(t_0),\ and\ U_j(t) < V_j(t),\ for j \in N,\ t \in [-d, t_0), \tag{14}$$

and we have

$$D_{-}U_{i_0}(t_0) \geq \dot{V}_{i_0}(t_0) = -\lambda V_{i_0}(t_0). \tag{15}$$

Now, By (8), (9), and (12), for each $i \in N$, we estimate $D^{-}U_i(t)$. Noting that $\frac{|s|^{r-1}}{A_i(P_i s)} > 0$, for each $i \in N$, we have

$$\begin{aligned}
D^{-}U_i(t) &= D^{-}\{sign(z_i(t)) \int_0^{z_i(t)} \frac{|s|^{r-1}}{A_i(P_i s)} dx\} \\
&\leq \frac{1}{P_i}|z_i(t)|^{r-1}\{-B_i(P_i z_i(t)) + \sum_{j=1}^{n} w_{ij} g_j(P_j z_j(t)) \\
&+ \sum_{j=1}^{n} w_{ij}^{d} g_j(P_j z_j(t - d_{ij}(t)))\} sign(z_j(t)) \\
&\leq \frac{1}{P_i}\{-\beta_i P_i |z_i(t)|^r + \sum_{j=1}^{n} |w_{ij}| P_j L_j |z_j(t)||z_i(t)|^{r-1} \\
&+ \sum_{j=1}^{n} |w_{ij}^{d}| P_j L_j |z_j(t - d_{ij}(t))||z_i(t)|^{r-1}\}
\end{aligned}$$

According to the Young inequality (lemma 2.2), for $i, j \in N$ and $r > 1$,

$$\begin{aligned}
&|w_{ij}|L_j|z_j(t)||z_i(t)|^{r-1} \\
&= \{|w_{ij}|^{\frac{r-h_{ij}}{r-1}} L_j^{\frac{r-l_{ij}}{r-1}} |z_i(t)|^r\}^{\frac{r-1}{r}} \{|w_{ij}|^{h_{ij}} L_j^{l_{ij}} |z_j(t)|^r\}^{\frac{1}{r}} \\
&\leq \frac{r-1}{r}|w_{ij}|^{\frac{r-h_{ij}}{r-1}} L_j^{\frac{r-l_{ij}}{r-1}} |z_i(t)|^r + \frac{1}{r}|w_{ij}|^{h_{ij}} L_j^{l_{ij}} |z_j(t)|^r,
\end{aligned}$$

similarly,

$$\begin{aligned}
&|w_{ij}^{d}|L_j|z_j(t - d_{ij}(t))||z_i(t)|^{r-1} \\
&\leq \frac{r-1}{r}|w_{ij}^{d}|^{\frac{r-h_{ij}^{*}}{r-1}} L_j^{\frac{r-l_{ij}^{*}}{r-1}} |z_i(t)|^r + \frac{1}{r}|w_{ij}^{d}|^{h_{ij}^{*}} L_j^{l_{ij}^{*}} |z_j(t - d_{ij}(t))|^r.
\end{aligned}$$

We regulate it that $0 \cdot \infty = 0$,then the preceding two inequality hold trivially for $r = 1$. Thus, for each $i \in N$,

$$\begin{aligned}
D^{-}U_i(t) \leq &\frac{1}{P_i}\{-\beta_i P_i |z_i(t)|^r \\
&+ \frac{r-1}{r} \sum_{j=1}^{n} (|w_{ij}|^{\frac{r-h_{ij}}{r-1}} L_j^{\frac{r-l_{ij}}{r-1}} + |w_{ij}^{d}|^{\frac{r-h_{ij}^{*}}{r-1}} L_j^{\frac{r-l_{ij}^{*}}{r-1}}) P_j |z_i(t)|^r \\
&+ \frac{1}{r} \sum_{j=1}^{n} |w_{ij}|^{h_{ij}} L_j^{l_{ij}} P_j |z_j(t)|^r + \frac{1}{r} \sum_{j=1}^{n} |w_{ij}^{d}|^{h_{ij}^{*}} L_j^{l_{ij}^{*}} P_j |z_j(t - d_{ij}(t))|^r\}.
\end{aligned} \tag{16}$$

In light of (14) and (16), we estimate $D_-U_{i_0}(t_0) - \dot{V}_i(t_0)$ as follows.

$$
\begin{aligned}
&D_-U_{i_0}(t_0) - \dot{V}_i(t_0) \leq D^-U_{i_0}(t_0) - \dot{V}_i(t_0) \\
&\leq \frac{1}{P_{i_0}}\{-\beta_{i_0}P_{i_0}|z_{i_0}(t)|^r + \frac{r-1}{r}\sum_{j=1}^{n}(|w_{i_0j}|^{\frac{r-h_{i_0j}}{r-1}}L_j^{\frac{r-l_{i_0j}}{r-1}} \\
&+ |w^d_{i_0j}|^{\frac{r-h^*_{i_0j}}{r-1}}L_j^{\frac{r-l^*_{i_0j}}{r-1}})P_j|z_{i_0}(t)|^r + \frac{1}{r}\sum_{j=1}^{n}|w_{i_0j}|^{h_{i_0j}}L_j^{l_{i_0j}}P_j|z_j(t)|^r \\
&+ \frac{1}{r}\sum_{j=1}^{n}|w^d_{i_0j}|^{h^*_{i_0j}}L_j^{l^*_{i_0j}}P_j|z_j(t-d_{i_0j}(t))|^r\} + \lambda V_{i_0}(t_0) \\
&\leq \frac{1}{P_{i_0}}\{-\beta_{i_0}P_{i_0}r\underline{\alpha}_{i_0}U_{i_0}(t_0) + (r-1)\sum_{j=1}^{n}(|w_{i_0j}|^{\frac{r-h_{i_0j}}{r-1}}L_j^{\frac{r-l_{i_0j}}{r-1}} \\
&+ |w^d_{i_0j}|^{\frac{r-h^*_{i_0j}}{r-1}}L_j^{\frac{r-l^*_{i_0j}}{r-1}})P_j\overline{\alpha}_{i_0}U_{i_0}(t_0) + \sum_{j=1}^{n}|w_{i_0j}|^{h_{i_0j}}L_j^{l_{i_0j}}P_j\overline{\alpha}_jU_j(t_0) \\
&+ \sum_{j=1}^{n}|w^d_{i_0j}|^{h^*_{i_0j}}L_j^{l^*_{i_0j}}P_j\overline{\alpha}_jU_j(t_0-d_{i_0j}(t_0)) + P_{i_0}\lambda V_{i_0}(t_0)\} \\
&\leq \frac{1}{P_{i_0}}\{-\beta_{i_0}P_{i_0}r\underline{\alpha}_{i_0}V_{i_0}(t_0) + (r-1)\sum_{j=1}^{n}(|w_{i_0j}|^{\frac{r-h_{i_0j}}{r-1}}L_j^{\frac{r-l_{i_0j}}{r-1}} \\
&+ |w^d_{i_0j}|^{\frac{r-h^*_{i_0j}}{r-1}}L_j^{\frac{r-l^*_{i_0j}}{r-1}})P_j\overline{\alpha}_{i_0}V_{i_0}(t_0) + \sum_{j=1}^{n}|w_{i_0j}|^{h_{i_0j}}L_j^{l_{i_0j}}P_j\overline{\alpha}_jV_j(t_0) \\
&+ \sum_{j=1}^{n}|w^d_{i_0j}|^{h^*_{i_0j}}L_j^{l^*_{i_0j}}P_j\overline{\alpha}_jV_j(t_0-d_{i_0j}(t_0)) + P_{i_0}\lambda V_{i_0}(t_0)\} \\
&\leq \frac{1}{P_{i_0}}\{-\beta_{i_0}P_{i_0}r\underline{\alpha}_{i_0}V_{i_0}(t_0) + (r-1)\sum_{j=1}^{n}(|w_{i_0j}|^{\frac{r-h_{i_0j}}{r-1}}L_j^{\frac{r-l_{i_0j}}{r-1}} \\
&+ |w^d_{i_0j}|^{\frac{r-h^*_{i_0j}}{r-1}}L_j^{\frac{r-l^*_{i_0j}}{r-1}})P_j\overline{\alpha}_{i_0}V_{i_0}(t_0) + \sum_{j=1}^{n}|w_{i_0j}|^{h_{i_0j}}L_j^{l_{i_0j}}P_j\overline{\alpha}_jV_j(t_0) \\
&+ \sum_{j=1}^{n}|w^d_{i_0j}|^{h^*_{i_0j}}L_j^{l^*_{i_0j}}P_j\overline{\alpha}_jV_j(t_0-d) + P_{i_0}\lambda V_{i_0}(t_0)\} \\
&= -\frac{1}{P_{i_0}}\{r\underline{\alpha}_{i_0}\beta_{i_0}P_{i_0}Q_{i_0} - (r-1)\overline{\alpha}_{i_0}\sum_{j=1}^{n}(|w_{i_0j}|^{\frac{r-h_{i_0j}}{r-1}}L_j^{\frac{r-l_{i_0j}}{r-1}} \\
&+ |w^d_{i_0j}|^{\frac{r-h^*_{i_0j}}{r-1}}L_j^{\frac{r-l^*_{i_0j}}{r-1}})P_jQ_{i_0} - \sum_{j=1}^{n}|w_{i_0j}|^{h_{i_0j}}L_j^{l_{i_0j}}\overline{\alpha}_jP_jQ_j \\
&- \sum_{j=1}^{n}|w^d_{i_0j}|^{h^*_{i_0j}}L_j^{l^*_{i_0j}}\overline{\alpha}_jP_jQ_je^{\lambda d} - P_{i_0}Q_{i_0}\lambda\}V(t_0).
\end{aligned}
$$

Using (11), we obtain that $D_{-}U_{i_0}(t_0) < \dot{V}_i(t_0)$, which contradicts (15). Hence, (13) holds. With the help of (12),

$$\begin{aligned}|y_i(t)| &= P_i|z_i(t)| < (r\overline{\alpha}_i V_i(t))^{\frac{1}{r}}\\ &= (mr\overline{\alpha}_i Q_i)^{\frac{1}{r}}(\sum_{j=1}^{n}(\sup_{s\in[-d,0]}|y_j(s)|^r+\varepsilon))^{\frac{1}{r}}e^{-\frac{\lambda}{r}t}, \text{ for } i\in N.\end{aligned}$$

For the arbitrariness of ε, it follows that

$$|y_i(t)| \le P_i(mr\overline{\alpha}_i Q_i)^{\frac{1}{r}}(\sum_{j=1}^{n}\sup_{s\in[-d,0]}|y_j(s)|^r)^{\frac{1}{r}}e^{-\frac{\lambda}{r}t}. \tag{17}$$

Noting that all norms in R^n are equivalent, then there exist $C_r > 0$ such that,

$$\|x\|_r = (\sum_{j=1}^{n}|x_j|^r)^{\frac{1}{r}} \le C_r\|x\|_1 = C_r\sum_{j=1}^{n}|x_j|, \text{ for } x\in R^n \tag{18}$$

By virtue of (17) and (18), we can obtain that

$$|y_i(t)| \le C\sum_{j=1}^{n}\sup_{s\in[-d,0]}|y_j(s))|e^{-\frac{\lambda}{r}t}, \text{ for } i\in N,$$

where $C = \max_{i\in N}\{P_i(m\overline{\alpha}_i Q_i)^{\frac{1}{r}}C_r\}$. Namely, system(8) and thus system(2) is globally exponentially stable.

Remark 3.5. From the proof of theorem 3.4, (H_2) and (H_3) are only use to ensure that " $\frac{B_i(s)}{s} \ge \beta_i,\ |g_i(s)| \le L_i|s|,\ for\, s\in R,\ i\in N$" in (9). If we have known the equlibrium x^* of system (2), the global left Lipschitz continuity of $b_i(t)$ and the global Lipschitz continuity of $f_i(t)$ can be relaxed to the so called global left quasi-Lipschtiz continuity and global quasi-Lipschtiz continuity at x_i^*, respectively. Namely,

$$\tfrac{b_i(s)-b_i(x_i^*)}{s-x_i^*} \ge \beta_i,\ |f_i(s)-f_i(x_i^*)| \le L_i|s-x_i^*| for\ s\ne x_i^*\in R,\ i\in N.$$

Remark 3.6. When theorem 3.4 ensures global exponential stability of system (2), it really ensures the robust global exponential stability in the following meaning: changing the involved system parameters (namely, all the parameters in theorem 3.4 except r, P, and h_{ij}, l_{ij}, h_{ij}^*, l_{ij}^*, $for\ i,j\in N$) small enough has no harm on the global exponential stability. This can be infered by lemma 2.4 (2) and the well-known fact that the determinant of every matrix in $R^{n\times n}$ is a continuous function of its elements.

By letting $P = (1,\dots,1)^T$ in theorem 3.4, we can obtain the following result.

Corollary 3.7. *Assume that all the conditions in theorem 3.1 hold. Moreover,* $(H_1^{'})$,(H_2) *and* (H_3) *hold. If there exist* $r \geq 1$, *real numbers* h_{ij}, l_{ij}, h_{ij}^*, l_{ij}^*, *for* $i, j \in N$, *such that*

$$M_{r,h,l} \triangleq (m_{ij}) \tag{19}$$

is a M-matrix, where for $i, j \in N$,

$$\begin{aligned} m_{ii} = \{ & r\frac{\alpha_i}{\overline{\alpha}_i}\beta_i - (r-1)\sum_{k=1}^{n}(|w_{ik}|^{\frac{r-h_{ik}}{r-1}}L_k^{\frac{r-l_{ik}}{r-1}} + |w_{ik}^d|^{\frac{r-h_{ik}^*}{r-1}}L_k^{\frac{r-l_{ik}^*}{r-1}}) \\ & - (|w_{ii}|^{h_{ii}}L_i^{l_{ii}} + |w_{ii}^d|^{h_{ii}^*}L_i^{l_{ii}^*})\ \}, \\ m_{ij} = & -(|w_{ij}|^{h_{ij}}L_j^{l_{ij}} + |w_{ij}^d|^{h_{ij}^*}L_j^{l_{ij}^*}), i \neq j, \end{aligned}$$

then System (2) *has a unique equilibrium, which is globally exponentially stable.*

Remark 3.8. Because coditions in theorem 3.1 are always satisfied by letting $L_0 = diag\{0, \dots, 0\}$ and $\mu = (\sup_{s\in R} f_1(s) \dots, \sup_{s\in R} f_n(s))^T$ for bounded activation functions. Corollary 3.7 contains the corresponding results in [22]. In [22] the activation functions are assumed to be bounded, and (H_2) is replaced by the assumption $(H_2^{'})$: "For each $i \in N$, $b_i \in C^1(R, R)$, $\dot{b}_i(\cdot) > 0$, $b_i(\cdot)$ and $b_i^{-1}(\cdot)$ is global Lipschtiz continuous." Obviously, we have relaxed this condition. Moreover, Corollary 3.7 can be used when the activation functions are unbounded.

Corollary 3.9. *Assume that all the conditions in theorem 3.1 hold. Moreover,* $(H_1^{'})$,(H_2) *and* (H_3) *hold. If there exist* $r \geq 1$ *and* $P = (P_1, \dots, P_n)^T \in R_+^n$ *such that*

$$M_{r,P} \triangleq (m_{ij}) \tag{20}$$

is a M-matrix, where for $i, j \in N$,

$$\begin{aligned} m_{ii} &= r\{\frac{\alpha_i\beta_i}{\overline{\alpha}_i L_i} - (|w_{ii}| + |w_{ii}^d|)\}P_i - (r-1)\sum_{k=1,k\neq i}^{n}(|w_{ik}| + |w_{ik}^d|)P_k, \\ m_{ij} &= -(|w_{ij}| + |w_{ij}^d|)P_j, i \neq j, \end{aligned}$$

then System (2) has a unique equilibrium, which is globally exponentially stable.

Proof. Let $P = LP^{'}$ with $L = diag\{L_1, \dots, L_n\}$, then we can see that $M_{r,LP^{'}}$ is the same as $M_{r,P^{'},h,l,h^*,l^*}$ in theorem 3.4 with $h_{ij} = l_{ij} = h_{ij}^* = l_{ij}^* = 1,\ for\ i, j \in N$. Using (20), $M_{r,P^{'},h,l,h^*,l^*}$ is also a M-matrix. Thus, according to theorem 3.4, corollary 3.9 is proved. □

Corollary 3.10. *Assume that* $(H_1^{'})$, (H_2) *and* (H_3) *hold. If*

$$M_1 \triangleq A - (|W| + |W^d|) \tag{21}$$

is a M-matrix, where $A = diag\{\frac{\alpha_1\beta_1}{\overline{\alpha}_1 L_1}, \dots, \frac{\alpha_n\beta_n}{\overline{\alpha}_n L_n}\}$, *then System (2) has a unique equilibrium, which is globally exponentially stable.*

Proof. From (21), M_1L is a M-matrix. Noting that $M \geq M_1L$ with $L = diag\{L_1, \dots, L_n\}$, by lemma 2.6, M is a M-matrix, too. Then theorem 3.2 ensures a unique equilibrium. In the corollary 3.9, let $r = 1$ and $P = (1, \dots, 1)^T$, we immediately get the result. □

Remark 3.11. Corollary 3.10 contains the corresponding results in [17]. But in [17], (H_2) is replaced by $(H_2^{'})$.

4. Acknowledgements

This work was supported by the State's Key R & D Program of China (Grant No.2017YFD0301507), the Natural Science Foundation of Hunan Province, China (Grant No.2018JJ3227) and Key R & D Program in Hunan Province, China (No.2017NK2382).

References

[1] M. Cohen, S. Grossberg, *Absolute stability and global pattern formation and parallel memory storage by competitive neural networks*, IEEE Transactions on Systems, Man and Cybernetics, SMC, 13 (1983), 815-826.

[2] C. Marcus, R. Westervelt, *Stability of analog neural networks with delay*, Physical Review A, 39 (1989), 347-359.

[3] Z. Liu, L. Liao, *Existence and global exponential stability of periodic solution of cellular neural networks with time-varying delays*, J. Math. Anal. Appl., 290 (2004), 247-262.

[4] J. Hopfield, *Neural networks and physical systems with emergent collective computational ablities*, Proceedings of the National Academy of Sciences, 71 (1982), 2252-2258.

[5] J. Hopfield, D. Tank, *Computing with neural circuits: a model, Science*, 233 (1986), 625-633.

[6] P. Baldi, A. Atiya, *How delays affect neural dynamics and leaning*, IEEE Transactions on Neural Networks, 5 (1994), 612-621.

[7] K. Gopalsamy, X. He, *Delay-independent stability in bidirectional associative memory networks*, IEEE Transactions on Neural Networks, 5 (1994), 998-1002.

[8] L. Chua, L. Yang, *Cellular neural networks: theory*, IEEE Transactions on Circuits and Systems, 35 (1988), 1257-1272.

[9] L. Chua, L. Yang, *Cellular neural networks: applications*, IEEE Transactions on Circuits and Systems, 35 (1988), 1273-1290.

[10] M. Kennedy, L. Chua, *Neural networks for nonlinear programming*, IEEE Transactions on Circuits and Systems, 35 (1988), 554-562.

[11] J. Peng, H. Qiao, Z. Xu, *A new approach to stability of neural networks with time-varying delays*, Neural Networks, 15 (2002), 95-103.

[12] G. Hardy, H. Littlewood, G. Polya, *Inequality*, Cambridge University Press, Cambridge, 1934.

[13] A. Berman, R. Plemmons, *Nonnegative matrices in the mathematical sciences*, New York: Academic, 1979.

[14] K. Lu, D. Xu, Z. Yang, *Global attraction and stability for Cohen-Grossberg neural networks with delays*, Neural Networks, 19 (2006), 1538-1549.

[15] D. Xu, Z. Xu, *Stability analysis of linear delay-differential systems*, Control Theory and Advanced Technology, 7 (1991), 629-642.

[16] X. Liao, C. Li, K. Wong, *Criteria for exponential stability of Cohen-Grossberg neural networks*, Neural Networks, 17 (2004), 1401-1404.

[17] C. Huang and L. Huang, *Dynamics of a class of Cohen-Grossberg neural networks with time-varying delays*, Nonlinear Analysis: Real World Applications, 8 (2007), 40-52.

[18] L. Wang, X. Zou, *Exponential stability of Cohen-Grossberg neural networks,* Neural Networks, 15 (2002), 415-422.

[19] L. Wang, X. Zou, *Harmless delays in Cohen-Grossberg neural networks*, Physica D, 170 (2002), 162-173.

[20] T. Chen, L. Rong, *Robust global exponential stability of Cohen-Grossberg neural networks with time delays*, IEEE Transactions on Neural Networks, 15 (2004), 203-206.

[21] A. Wan, M. Wang, J. Peng, H. Qiao, *Exponential stability of Cohen-Grossberg neural networks with a general class of activation functions*, Physics Letters A, 350 (2006), 96-102.

[22] H. Jiang, J. Cao, Z. Teng, *Dynamics of Cohen-Grossberg neural networks with time-varying delays*, Physics Letters A, 354 (2006), 414-422.

[23] S. Mandal, N. C. Majee, A. B. Roy, *Global exponential stability of periodic solution to Cohen-Grossberg type neural network with both ordinary and neutral type discrete time varying delays: an LMI approach,* Applied Mathematical Sciences, 7 (2013), 2979 - 2991.

[24] Zhengqiu Zhang, Jinde Cao, Dongming Zhou, *Novel LMI-based condition on global asymptotic stability for a class of Cohen-Grossberg BAM networks with extended activation functions*, IEEE Trans on Neural Networks and Learning System, 256 (2014), 1161-1172.

[25] C. Xu, Q. Zhang, *Existence and global exponential stability of anti-periodic solutions for BAM neural networks with inertial term and delay*, Neurocomputing, 153 (2015), 207-215.

[26] Z. Zhang, S. Yu, *Global asymptotic stability for a class of complex-valued Cohen-Grossberg neural network with time delays*, Neurocomputing, 171 (2016), 1158-1166.

Accepted: 25.02.2017