SANDWICH SETS AND CONGRUENCES IN COMPLETELY INVERSE AG^{**}-GROUPOIDS

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Abstract. In this paper, we investigate sandwich set in a completely inverse AG**-groupoid and make use of it for congruence pairs of a completely inverse AG**-groupoid. We discuss completely inverse AG**-groupoids in terms of partial order and show that the natural partial order is an equality relation in an AG-group. Further, we study idempotent-separating and idempotent-pure congruences on completely inverse AG**-groupoids and show that the quotient structure for a maximum idempotent-separating congruence on an AG**-groupoid is fundamental. Further, we characterize E-unitary completely inverse AG**-groupoids and provide a condition for a compatibility relation to be transitive.

Keywords: Completely inverse AG**-groupoid, AG-group, partial order relation, trace of congruence, kernel of congruence, idempotent-separating, idempotent-pure, compatibility relation, E-unitary inverse AG**-groupoid.

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1. Introduction

An *Abel-Grassmann's* groupoid (abbreviated as AG-groupoid) or Left Almost Semigroup (briefly LA-semigroup) is a groupoid S satisfying the left invertive law $(ab)c = (cb)a$ for all $a, b, c \in S$. An AG-groupoid satisfying the identities $a(bc) = b(ac)$ and $(ab)(cd) = (dc)(ba)$ for all $a,\ b,\ c,\ d \in S$ is called an AG**-groupoid (cf. [5, 6, 8]).

The inverse AG**-groupoids and completely inverse AG**-groupoids constitute the most important class of AG-groupoids. The structural properties, characterizations and congruences on inverse AG**-groupoids via kernel-normal system and kernel-trace approaches can be found in literature (see for example [1, 4, 9, 11, 12]). In [3], study on congruences and lattices of congruences in completely inverse AG**-groupoids has been carried out.

In this paper, our purpose is to study some congruences in completely inverse AG**-groupoids, the natural partial order relation on completely inverse AG**-groupoids and E-unitary completely inverse AG**-groupoids. In the second section, some preliminaries and basic results on completely inverse AG**-groupoids are recalled. We also introduce sandwich set for a completely inverse AG**-groupoid. In section 3, we introduce completely left inverse AG**-groupoids and investigate some basic congruences using the congruence pairs. We show that if ρ is a congruence on a completely left inverse AG**-groupoid, then $(\ker\rho, \mathrm{tr}\rho)$ is a congruence. In section 4, the natural order relation and its relation with the AG-group is investigated. We further show that the set of all permissible subsets of a completely inverse AG**-groupoid is an inverse AG**-groupoid. In section 5, we investigate minimum and maximum congruences on completely inverse AG**-groupoids. The relation between a smallest congruence on an AG-group is established in this section. Idempotent-separating and idempotent-pure congruences are also studied in this section. We show that if μ is the maximum idempotent-separating congruence, then S/μ is fundamental. Finally, in section 6, E-unitary completely inverse AG**-groupoids and their different characterizations are provided. We show that if the compatibility relation is transitive, then S is E-unitary and conversely.

In the following section, we recall some necessary basic notions and then we present a few auxiliary results that will be used throughout the paper.

2. Preliminaries

An AG-groupoid S is regular if $a \in (aS)a$ for all $a \in S$. If for $a \in S$, there exists an element a' such that $a = (aa')a$ and $a' = (a'a)a'$, then we say that a' is inverse of a. In addition, if inverses commute, that is $a'a = aa'$, then S is called completely regular. If $a \in S$, then

$$V(a) = \{a' \in S : a = aa' \cdot a \text{ and } a' = a'a \cdot a'\}$$

is called the set of all inverses of $a \in S$. Note that if $a' \in V(a)$ and $b' \in V(b)$, then $a \in V(a')$ and $a'b' \in V(ab)$. AG-groupoid S in which every element has an unique inverse is called inverse AG-groupoid. If a^{-1} is the unique inverse of $a \in S$, then a groupoid satisfying the following identities is called a completely inverse AG**-groupoid, that is for all $a, b, c \in S$

$$(ab)c = (cb)a,\ a(bc) = b(ac)$$
$$a = (aa^{-1})a,\quad a^{-1} = (a^{-1}a)a^{-1} \text{ and } aa^{-1} = a^{-1}a.$$

If S is a completely inverse AG**-groupoid, then $a^{-1}a \in E_S$, where E_S is the set of idempotents of S. If S is a completely inverse AG**-groupoid, then E_S is either empty or a semilattice. If S is an AG-groupoid such that for all $x \in S, x^2 = x$ and for all $a, b \in S, a = (ab)a$, then we say that S is a rectangular AG-band. For any idempotent $e \in E_S$, $e^{-1} = e$. Moreover, the set E_S of an AG-groupoid S is a rectangular AG-band, that is for all $e, f \in E_S$ and $e = (ef)e$. For further concepts and results, the reader is referred to [3].

The sandwich set $S(e, f)$ of $e, f \in E_S$ is defined as below:

$$S(e, f) = \{g \in E_S : eg \cdot f = ef\}.$$

Proposition 2.1 ([3, Proposition 4.1])**.** *Let S be a completely inverse* AG**-*groupoid and let $a, b \in S$ such that $ab \in E_S$. Then $ab = ba$.*

Theorem 2.2. *Let S be a completely inverse* AG***-groupoid and $a, b \in S$. If $g \in S(aa^{-1}, bb^{-1})$, then $a^{-1} \cdot gb^{-1} \in V(ab)$.*

Proof. Let $a, b \in S$, $a^{-1} \in V(a)$, $b^{-1} \in V(b)$ and $g \in S(aa^{-1}, bb^{-1})$. Then

$$\begin{aligned}
\{(ab)(a^{-1} \cdot gb^{-1})\}(ab) &= \{(aa^{-1})(b \cdot gb^{-1})\}(ab) \\
&= \{(aa^{-1})(g \cdot bb^{-1})\}(ab) \\
&= \{(aa^{-1} \cdot bb^{-1})\}(ab) \qquad (\text{since } g \in S(aa^{-1}, bb^{-1})) \\
&= (aa^{-1} \cdot a)(bb^{-1} \cdot b) \\
&= ab
\end{aligned}$$

and

$$\begin{aligned}
\{(a^{-1} \cdot gb^{-1})(ab)\}(a^{-1} \cdot gb^{-1}) &= \{(a^{-1}a)(gb^{-1} \cdot b)\}(a^{-1} \cdot gb^{-1}) \\
&= \{(a^{-1}a)(g \cdot b^{-1}b)\}(a^{-1} \cdot gb^{-1}) \\
&= (a^{-1}a \cdot b^{-1}b)(a^{-1} \cdot gb^{-1}) \\
&\qquad (\text{since } g \in S(aa^{-1}, bb^{-1})) \\
&= (a^{-1}a \cdot a^{-1})(g(b^{-1}b \cdot b^{-1})) \\
&= a^{-1} \cdot gb^{-1}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(a^{-1} \cdot gb^{-1})(ab) &= (a^{-1}a)(gb^{-1} \cdot b) = (a^{-1}a)(bb^{-1} \cdot g) = (g \cdot bb^{-1})(aa^{-1}) \\
&= (aa^{-1} \cdot bb^{-1})g = g(aa^{-1} \cdot bb^{-1}) = (aa^{-1})(g \cdot bb^{-1}) \\
&= (ab)(a^{-1} \cdot gb^{-1}).
\end{aligned}$$

This completes the proof. □

3. Congruences in completely inverse AG**-groupoids

A congruence pair for a completely inverse AG**-groupoid which is based on a pair of a normal AG**-groupoid N and a congruence ζ has been established in this section. We will make use of the sandwich set to prove the main theorem.

Let ρ be a congruence on a completely inverse AG**-groupoid S and E_S be the set of idempotents of S. The restriction of ρ on E_S, that is, $\rho|_{E_S}$ is the trace of ρ denoted by trρ. The subset ker$\rho = \{a \in S : (\exists e \in E_S)\ (a, e) \in \rho\}$ is the kernel of ρ.

Definition 3.1 ([10, Definition 4.1])**.** A nonempty subset N of a completely inverse AG**-groupoid S is said to be normal ($N \lhd S$) if

(1) $E_S \subseteq N$,
(2) for every $a \in S$, $a \cdot Na^{-1} \subseteq N$,
(3) for every $a \in N$, $a^{-1} \in N$.

Definition 3.2 ([10, Definition 4.2])**.** Let N be normal subgroupoid of a completely inverse AG**-groupoid S and ζ be a congruence on E_S such that for every $a \in S$ and $e \in E_S$, $ea \in N$ and $(e,\ a^{-1}a) \in \zeta$ implies $a \in N$. Then the pair (N, ζ) is a congruence pair for S.

If (N, ζ) is a congruence pair where $N \lhd S$ and ζ is a congruence, then $\rho_{(N,\zeta)}$ defined as $(a, b) \in \rho_{(N,\zeta)}$ if and only if $a^{-1}b \in N,\ (aa^{-1}, bb^{-1}) \in \zeta$ is a relation on S.

Notice that if $a^{-1}b \in N \lhd S$, then $ab^{-1} \in V(a^{-1}b) \in N$. In the light of Proposition 2.1, the condition $ab^{-1} \in N$ is equivalent to $b^{-1}a \in N$.

Lemma 3.3. *If (N, ζ) is a congruence pair on a completely inverse* AG**-*groupoid S and $(a, b) \in \rho_{(N,\zeta)}$, then*

$$(aa^{-1}, aa^{-1} \cdot bb^{-1}) \in \zeta \text{ and } (bb^{-1}, bb^{-1} \cdot aa^{-1}) \in \zeta.$$

Proof. Let $(a, b) \in \rho_{(N,\zeta)}$. Then $aa^{-1} = (aa^{-1})(aa^{-1}) \equiv_\zeta (aa^{-1})(bb^{-1})$ (since $(aa^{-1}, bb^{-1}) \in \zeta$).

In similar way, replacing a and b, we get $bb^{-1} \equiv_\zeta bb^{-1} \cdot aa^{-1}$. □

Lemma 3.4. *Let (N, ζ) be a congruence pair on a completely inverse* AG**-*groupoid S. Then for $a, b \in S$ and $e \in E_S$, if $ab \in N$ and $(e, aa^{-1}) \in \zeta$, then $a \cdot eb \in N$.*

Proof. Proof is straight forward. □

Lemma 3.5. *Let (N, ζ) be a congruence pair on a completely inverse* AG**-*groupoid S and $g \in S(aa^{-1}, cc^{-1})$ and $h \in S(bb^{-1}, cc^{-1})$. If $(aa^{-1}, bb^{-1}) \in \zeta$ and $a^{-1}b \in N$, then $(a \cdot ga^{-1}, b \cdot hb^{-1}) \in \zeta$.*

Proof. Let (N, ζ) be a congruence pair. Let $a, b \in S$ such that $(aa^{-1}, bb^{-1}) \in \zeta$ and $a^{-1}b \in N$. Suppose $x \in S(aa^{-1}, bb^{-1})$. Then $a \cdot xb^{-1} \in V(a^{-1}b)$. Also if $t \in S((a^{-1}b)(a \cdot xb^{-1}), g)$, then $(a^{-1}b)(tg) \in V((a \cdot xb^{-1})g)$. Since $(aa^{-1}, bb^{-1}) \in \zeta$, then

$$(a^{-1}b)(t(a \cdot xb^{-1})) = t((a^{-1}a)(b \cdot xb^{-1})) = t((a^{-1}a \cdot x)(bb^{-1})) = t(b \cdot xb^{-1}). \tag{1}$$

Again $(aa^{-1}, bb^{-1}) \in \zeta$ and $x \in S(aa^{-1}, bb^{-1})$, then

$$(bb^{-1})(x \cdot bb^{-1}) \equiv_\zeta (aa^{-1})(x \cdot bb^{-1}) \equiv_\zeta (aa^{-1})(bb^{-1}) \equiv_\zeta (bb^{-1})(bb^{-1}) \equiv_\zeta bb^{-1}.$$

On the other hand $(bb^{-1})(x \cdot bb^{-1}) = x(bb^{-1} \cdot bb^{-1}) = b \cdot xb^{-1}$ and because $b \cdot xb^{-1}$ is an idempotent,

$$(b \cdot xb^{-1}, bb^{-1}) \in \zeta. \tag{2}$$

Since ζ is compatible, it follows from (2) that

$$(t(b \cdot xb^{-1}), t \cdot bb^{-1}) \in \zeta. \tag{3}$$

Using (1) and (3) with transitivity of ζ, we have

$$((a^{-1}b)(t(a \cdot xb^{-1})), t \cdot bb^{-1}) \in \zeta. \tag{4}$$

Similarly, since $x \in S(aa^{-1}, bb^{-1})$,

$$(aa^{-1})(x \cdot aa^{-1}) \equiv_\zeta (aa^{-1})(x \cdot bb^{-1}) \equiv_\zeta (aa^{-1})(bb^{-1}) \equiv_\zeta (aa^{-1})(aa^{-1}) \equiv_\zeta aa^{-1}.$$

Thus, we have

$$(a \cdot xa^{-1}, aa^{-1}) \in \zeta. \tag{5}$$

From the compatibility of ζ, we get

$$\begin{aligned} b(((a \cdot xb^{-1})(t \cdot a^{-1}b))b^{-1}) &= ((a \cdot xb^{-1})(t \cdot a^{-1}b))(bb^{-1}) \\ &= (t((aa^{-1})(x \cdot bb^{-1})))(b^{-1}b) \\ &= (t((aa^{-1})(bb^{-1})))(b^{-1}b) \quad (\text{since } \ x \in S(aa^{-1}, bb^{-1})) \\ &= t \cdot aa^{-1} \qquad\qquad\qquad\quad (\text{since } (aa^{-1}, bb^{-1}) \in \zeta) \end{aligned}$$

and since $t \cdot aa^{-1} \in E_S$, it follows

$$(b(((a \cdot xb^{-1})(t \cdot a^{-1}b))b^{-1}), t \cdot aa^{-1}) \in \zeta. \tag{6}$$

Further, we show that $(a \cdot ta^{-1}, a \cdot ga^{-1}) \in \zeta$.

Since $x \in S(aa^{-1}, bb^{-1})$ and $(aa^{-1}, bb^{-1}) \in \zeta$, then $(aa^{-1} \cdot x)(tg) = (aa^{-1}(x \cdot bb^{-1}))(tg) = (aa^{-1} \cdot bb^{-1}))(tg) \equiv_\zeta (aa^{-1} \cdot aa^{-1}))(tg) = aa^{-1} \cdot tg$. Because of the compatibility of ζ, (5) gives $((a \cdot xa^{-1})(tg), (aa^{-1})(tg)) \in \zeta \Longrightarrow ((aa^{-1} \cdot tg), a \cdot ga^{-1}) \in \zeta$ which further shows that

$$(a \cdot ta^{-1}, a \cdot ga^{-1}) \in \zeta. \tag{7}$$

Finally, we prove that $(b \cdot tb^{-1}, b \cdot hb^{-1}) \in \zeta$. Since $b \cdot tb^{-1} \in E_S$ and $g \in S(aa^{-1}, cc^{-1})$. Thus

$$\begin{aligned}
b \cdot tb^{-1} &\equiv_\zeta (t \cdot aa^{-1})(bb^{-1}) \\
&\equiv_\zeta (g \cdot aa^{-1})(bb^{-1}) && \text{(it follows from (7))} \\
&\equiv_\zeta g \cdot bb^{-1} && \text{(since } (aa^{-1}, bb^{-1}) \in \zeta) \\
&\equiv_\zeta g(bb^{-1} \cdot cc^{-1}) && \text{(using Lemma 3.3)} \\
&\equiv_\zeta aa^{-1} \cdot cc^{-1} && \text{(since } (aa^{-1}, bb^{-1}) \in \zeta \text{ and } g \in S(aa^{-1}, cc^{-1})) \\
&\equiv_\zeta (bb^{-1} \cdot h)(cc^{-1}) && \text{(since } h \in S(bb^{-1}, cc^{-1})) \\
&\equiv_\zeta b \cdot hb^{-1} && \text{(using Lemma 3.3)}
\end{aligned}$$

thus, we have

$$(b \cdot tb^{-1}, b \cdot hb^{-1}) \in \zeta. \tag{8}$$

Combining (4) and (8) with the compatibility of ζ and since both $b \cdot hb^{-1}$ and $b(((a \cdot xb^{-1})(t \cdot a^{-1}b))b^{-1})$ are in E_S, we have

$$(b(((a \cdot xb^{-1})(t \cdot a^{-1}b))b^{-1}), b \cdot hb^{-1}) \in \zeta. \tag{9}$$

Using (6) and (7) and the transitivity of ζ, we get

$$(b(((a \cdot xb^{-1})(t \cdot a^{-1}b))b^{-1}), a \cdot ga^{-1}) \in \zeta. \tag{10}$$

Finally, (9) and (10) with the transitivity of ζ, completes the proof as $(a \cdot ga^{-1}, b \cdot hb^{-1}) \in \zeta$. □

Theorem 3.6. *If (N, ζ) is a congruence pair of a completely inverse* AG**-*groupoid S, then $\rho_{(N,\zeta)}$ defined as*

$$(a, b) \in \rho_{(N,\zeta)} \text{ if and only if } a^{-1}b \in N,\ (aa^{-1}, bb^{-1}) \in \zeta$$

is congruence on S with kernel N and trace ζ. Conversely, if ρ is a congruence on S, then $(\ker\rho, \mathrm{tr}\rho)$ is a congruence pair and $\rho = \rho_{(\ker\rho, \mathrm{tr}\rho)}$.

Proof. Let $\rho_{(N,\zeta)}$ be a relation with congruence pair (N, ζ). First, we show that $\rho_{(N,\zeta)}$ is a congruence. $\rho_{(N,\zeta)}$ is reflexive. Symmetry is evident because ζ is symmetric and $(bb^{-1}, aa^{-1}) \in \rho_{(N,\zeta)}$. Since $a^{-1}b \in N$, then by Proposition 2.1, $b^{-1}a \in N$. For transitivity, let $(a, b),\ (b, c) \in \rho_{(N,\zeta)}$. Then $(aa^{-1}, bb^{-1}) \in \zeta,\ a^{-1}b \in N$ and $(bb^{-1}, cc^{-1}) \in \zeta,\ b^{-1}c \in N$. But ζ is transitive, thus $(aa^{-1}, cc^{-1}) \in \rho_{(N,\zeta)}$. Since $b^{-1}a,\ bc^{-1} \in N$ and N is subgroupoid of S, $b^{-1}a \cdot$

$bc^{-1} \in N$. We need to show $ac^{-1} \in N$. Let us take g, h such that $g \in S(bb^{-1}, cc^{-1})$ and $h \in (aa^{-1}, cc^{-1})$. Then by Lemma 3.3, we have

$$N \ni b^{-1}a \cdot bc^{-1} = a(bb^{-1} \cdot c^{-1}) = a((bb^{-1} \cdot c^{-1}c)c^{-1}) = a(((b^{-1}c)(b \cdot gc^{-1}))c^{-1})$$

because $b^{-1}c \in V(b \cdot gc^{-1})$. Therefore $(b^{-1}c)(b \cdot gc^{-1}) \in E_S$. We have

$$cc^{-1} \equiv_\zeta cc^{-1} \cdot cc^{-1} \equiv_\zeta bb^{-1} \cdot cc^{-1} \equiv_\zeta bb^{-1}(g \cdot cc^{-1}) \equiv_\zeta cc^{-1}(g \cdot cc^{-1}) = g \cdot cc^{-1}$$

that is, $(cc^{-1}, c \cdot gc^{-1}) \in \zeta$. In the same manner, as $h \in S(aa^{-1}, cc^{-1})$ and $(aa^{-1}, cc^{-1}) \in \zeta$, we have $(cc^{-1}, c \cdot hc^{-1}) \in \zeta$. So

$$(b^{-1}c)(b \cdot gc^{-1}) = c((bb^{-1} \cdot g)c^{-1}) \equiv_\zeta c \cdot gc^{-1} = cc^{-1} \equiv_\zeta c \cdot hc^{-1} = (a^{-1}c)(a \cdot hc^{-1}).$$

Thus $((b^{-1}c)(b \cdot gc^{-1}), (a^{-1}c)(a \cdot hc^{-1})) \in \zeta$. From (10)-(11), $ac^{-1} \in N$ and by Definition 3.1(3), $a^{-1}c \in N$. Thus $\rho_{(N,\zeta)}$ is an equivalence relation. To show that $\rho_{(N,\zeta)}$ is compatible, let $g \in S(aa^{-1}, cc^{-1})$ and $h \in S(cc^{-1}, bb^{-1})$. Then

$$a^{-1} \cdot gc^{-1} \in V(ac) \text{ and } b^{-1} \cdot hc^{-1} \in V(bc)$$

$(ac)(a^{-1} \cdot gc^{-1}) = (aa^{-1} \cdot g)(cc^{-1}) = c \cdot gc^{-1}$ and $(bc)(b^{-1} \cdot hc^{-1}) = (bb^{-1} \cdot h)(cc^{-1}) = c \cdot hc^{-1}$. Since $g \in S(aa^{-1}, cc^{-1})$ and $h \in S(cc^{-1}, bb^{-1})$, it follows

$$\begin{aligned} c \cdot gc^{-1} &= (aa^{-1} \cdot g)(cc^{-1}) = (aa^{-1})(cc^{-1}) \equiv_\zeta (bb^{-1})(cc^{-1}) \\ &= (bb^{-1})(h \cdot cc^{-1}) \equiv_\zeta (cc^{-1})(h \cdot cc^{-1}) = c \cdot hc^{-1}. \end{aligned}$$

Thus $((ac)(a^{-1} \cdot gc^{-1}), (bc)(b^{-1} \cdot hc^{-1})) = (c \cdot gc^{-1}, c \cdot hc^{-1}) \in \zeta$. Also $(a^{-1} \cdot gc^{-1})(bc) = (a^{-1}b)(cc^{-1} \cdot g) \in N$ because $a^{-1}b \in N$ and $cc^{-1} \cdot g \in E_S$. Therefore $(ac, bc) \in \rho_{(N,\zeta)}$.

For $(ca, cb) \in \rho_{(N,\zeta)}$, let us suppose again that $g \in S(aa^{-1}, cc^{-1})$ and $h \in S(cc^{-1}, bb^{-1})$. Then $(c^{-1} \cdot ga^{-1}) \in V(ca)$, $(c^{-1} \cdot hb^{-1}) \in V(cb)$ and

$$\begin{aligned} (ca)(c^{-1} \cdot ga^{-1}) &= (aa^{-1} \cdot g)(cc^{-1}) \equiv_\zeta (c \cdot gc^{-1}), \\ (cb)(c^{-1} \cdot hb^{-1}) &= (cc^{-1})(h \cdot bb^{-1}) \equiv_\zeta (c \cdot hc^{-1}). \end{aligned}$$

Since $c \cdot gc^{-1}$ and $c \cdot hc^{-1} \in E_S$, then $((ca)(c^{-1} \cdot ga^{-1}), (cb)(c^{-1} \cdot hb^{-1})) = (c \cdot gc^{-1}, c \cdot hc^{-1}) \in \zeta$ and since $a^{-1}b \in N$, we have $(c^{-1} \cdot ga^{-1})(cb) = (c \cdot gc^{-1})(a^{-1}b) \in N$. Hence $(ca, cb) \in \rho_{(N,\zeta)}$.

We further show that $N = \ker \rho_{(N,\zeta)}$ and $\zeta = \mathrm{tr}\rho_{(N,\zeta)}$. Suppose that $a \in \ker\rho_{(N,\zeta)}$. Then there exists $e \in E_S$ such that $(e, a)\rho_{(N,\zeta)}$. Using the definition of $\rho_{(N,\zeta)}$, it is clear that $(ee, aa^{-1}) \in \zeta$, $ea \in N$ i.e. $a \in N$. Hence $\ker \rho_{(N,\zeta)} \subseteq N$.

Conversely suppose that $a \in N$. Because N is normal subgroupoid, then $(a^{-1}a, aa^{-1} \cdot a^{-1}a) \in \zeta$ and $a^{-1} \cdot aa^{-1} \in N$, that is, $(a, a^{-1}a) \in \rho_{(N,\zeta)} \Longrightarrow a \in \ker\rho_{(N,\zeta)}$. Thus $N \subseteq \ker\rho_{(N,\zeta)}$. Hence $\ker\rho_{(N,\zeta)} = N$.

Suppose $a, b \in E_S$ such that $(a, b) \in \rho_{(N,\zeta)}$. Then $(aa^{-1}, bb^{-1}) \in \zeta$ and $a^{-1}b \in N$. Since ζ is a congruence on E_S, it follows that $(a)_\zeta = (a)_\zeta(a^{-1})_\zeta = (b)_\zeta(b^{-1})_\zeta = (b)_\zeta$. Thus $(a, b) \in \zeta$. Hence $\mathrm{tr}\rho_{(N,\zeta)} \subseteq \zeta$.

Conversely, $(a,b) \in \zeta$ for $a,b \in E_S$. Then $(a^{-1})_\zeta, (b^{-1})_\zeta \in E_{S/\zeta}$. Because $(a,b) \in \zeta$, then $(a)_\zeta(a^{-1})_\zeta = (b)_\zeta(b^{-1})_\zeta$. That is, $(aa^{-1}, bb^{-1}) \in \zeta$, $a^{-1}b \in N$. Thus $(a,b) \in \rho_{(N,\zeta)}$. Hence $\mathrm{tr}\rho_{(N,\zeta)} = \zeta$.

To prove the converse of the theorem, first we show that $\ker\rho \lhd S$. Let $a,b \in \ker\rho$, then $(a)_\rho, (b)_\rho \in E_{S/\rho}$ and $(a)_\rho(b)_\rho = (ab)_\rho \in E_{S/\rho}$. We show that $\ker\rho$ is self-conjugate. Let $a \in S$ and $x \in \ker\rho$, then for $(x)_\rho \in E_{S/\rho}$, we have $(a \cdot xa^{-1})_\rho = (a)_\rho \cdot (x)_\rho(a^{-1})_\rho \in E_{S/\rho}$. Also, if $a \in \ker\rho$, then $(a)_\rho \in E_{S/\rho}$. Thus $(a^{-1})_\rho \in E_{S/\rho} \subseteq \ker\rho$ showing $\ker\rho$ is normal. Hence by Definition 3.2, $(\ker\rho, \mathrm{tr}\rho)$ is a congruence pair.

Finally, for any $a,b \in S$, let $(a,b) \in \rho$. Then $(a^{-1}, b^{-1}) \in \rho$, $(aa^{-1}, bb^{-1}) \in \rho$ and $(ab^{-1}, bb^{-1}) \in \rho$. Since $aa^{-1}, bb^{-1} \in E_S$, $(aa^{-1}, bb^{-1}) \in \mathrm{tr}\rho$ and $a^{-1}b \in \ker\rho$, it follows that $\rho \subseteq \rho_{(\ker\rho, \mathrm{tr}\rho)}$.

Conversely, to show $\rho_{(\ker\rho,\mathrm{tr}\rho)} \subseteq \rho$. Let$(a,b) \in \rho_{(\ker\rho,\mathrm{tr}\rho)}$ such that $(aa^{-1}, bb^{-1}) \in \mathrm{tr}\rho$ and $a^{-1}b \in \ker\rho$. Since $ab^{-1} \in V(a^{-1}b) \in \ker\rho$, it follows that for some $(e)_\rho \in E_{S/\rho}$, $(ab^{-1})_\rho = (e)_\rho = (e)_\rho(e^{-1})_\rho = (ab^{-1})_\rho((ab^{-1})^{-1})_\rho$. Then

$$\begin{aligned}(a)_\rho &\equiv_\rho (aa^{-1} \cdot a)_\rho \equiv_\rho (ab^{-1})_\rho \cdot (b)_\rho \equiv_\rho ((ab^{-1})(ab^{-1})^{-1})_\rho (b)_\rho \\ &\equiv_\rho (aa^{-1} \cdot bb^{-1})_\rho (b)_\rho \equiv_\rho (b^{-1}b)_\rho \cdot (b)_\rho \equiv_\rho (b)_\rho.\end{aligned}$$

Hence, $\rho_{(\ker\rho,\ \mathrm{tr}\rho)} = \rho$. □

4. Natural partial order relation

In this section, we discuss the natural partial order relation on completely inverse AG**-groupoids which indeed constitute AG-groups containing exactly one idempotent. We show that the natural partial order is the equality relation if and only if S is an AG-group.

Definition 4.1. Let S be a completely inverse AG**-groupoid. Then the relation

$$a \leq b \text{ if and only if } a \in E_S b.$$

on S is a natural partial order relation.

Lemma 4.2. *Let S be a completely inverse* AG**-*groupoid. Then the following are equivalent:*

(1) $a \leq b$.
(2) $a = fb$ *for some* $f \in E_S$.
(3) $a^{-1} \leq b^{-1}$.
(4) $a = aa^{-1} \cdot b$.

Proof. Let $a,b \in S$, $a^{-1} \in V(a)$. Then

(1)⇒(2): It is clear from Definition 4.1.

(2)⇒(3): Since $a = fb$, it follows that $a^{-1} = (fb)^{-1} = fb^{-1} \leq b^{-1}$.

(3)⇒(4): By assumption, we have $a^{-1} = ea^{-1}$ for some $e \in E_S$. Then $a = eb$ and $ea = eb = a$ and thus $aa^{-1} \cdot b = (ea \cdot a^{-1})b = (be)(a^{-1}a) = aa^{-1} \cdot a = a$.

(4)⇒(1): This is obvious because $a^{-1}a$ is an idempotent. □

Proposition 4.3. *Let S be a completely inverse* AG***-groupoid. Then the following statements hold:*

(1) *The relation $\leq$ is a partial order on S.*
(2) *For $e, f \in E_S$, $e \leq f$ if and only if $e = ef = fe$.*
(3) *If $a \leq b$ and $c \leq d$, then $ab \leq cd$.*
(4) *If $a \leq b$, then $a^{-1}a \leq b^{-1}b$.*

Proof. (1) The relation $\leq$ is reflexive, since $a = (aa^{-1})a \leq a$. Let $a \leq b$ and $b \leq a$, then $a = (aa^{-1})b$ and $b = (bb^{-1})a$. We have $a = (aa^{-1})b = (aa^{-1})(bb^{-1} \cdot a) = (bb^{-1})(aa^{-1} \cdot a) = (bb^{-1})a = b$. Further, let $a \leq b$ and $b \leq c$. Then there exist idempotents e, f such that $a = eb$ and $b = fc$. It follows that $a = eb = e(fc) = (ef)c \leq c$.

(2) If for $e, f \in E_S$, $e \leq f$, then $e = if$ for some $i \in E_S$. Now $fe = f(if) = if = e$. Similarly, $ef = (if)f = if = e$. The converse is trivial.

(3) Suppose that $a \leq b$ and $c \leq d$ then for idempotents e, f we have $a = eb$ and $c = fd$. Hence $ac = (eb)(fd) = (ef)(bd) \leq bd$.

(4) This follows from Lemma 4.2(3) and (3) above. □

Note that a completely inverse AG**-groupoid including only one idempotent is an AG-group, that is if $E_S = \{e\}$, then $ea = (a^{-1}a)a = (aa^{-1})a = a$ for $e = a^{-1}a = aa^{-1}$. A connection between an AG-group which simply contains an idempotent and the natural partial order relation is established in the following proposition.

Proposition 4.4. *The natural partial order on a completely inverse* AG***-groupoid S is an equality relation if and only if S is an* AG*-group.*

Proof. Suppose the natural partial order relation is the equality relation. Then for idempotents $e, f \in S$, $ef \leq e,\ f$. Thus $e = ef = f$. Hence S contains exactly one idempotent. The converse is trivial. □

Definition 4.5. Let S be a completely inverse AG**-groupoid. For all $a, b \in S$ the compatibility relation is defined as

$$a \sim b \text{ if and only if } ab^{-1}, a^{-1}b \in E_S.$$

The above mentioned relation is definitely reflexive. It is symmetric because

$$b^{-1}a = b^{-1}(aa^{-1} \cdot a) = (aa^{-1} \cdot b^{-1}a) = (b^{-1}a \cdot aa^{-1}) = (aa^{-1} \cdot a)b^{-1} = ab^{-1}$$
$$ba^{-1} = (bb^{-1} \cdot b)a^{-1} = (a^{-1}b \cdot bb^{-1}) = (bb^{-1} \cdot a^{-1}b) = a^{-1}(bb^{-1} \cdot b) = a^{-1}b.$$

For A, B nonempty subsets of a completely inverse AG**-groupoid S, we define the set $A^{-1} = \{a^{-1} : a \in A\}$ and $AB = \{ab : a \in A,\ b \in B\}$. A nonempty subset A of a completely inverse AG**-groupoid S is compatible if the elements of A are compatible, that is $AA^{-1} \subset E_S$ and $A^{-1}A \subset E_S$.

Definition 4.6. A nonempty subset A of a completely inverse AG**-groupoid S is an order ideal if it satisfies the following property:

$$a \in A,\ x \leq a \Longrightarrow x \in A.$$

Definition 4.7. A subset A of a completely inverse AG**-groupoid S is said to be permissible if it is a compatible order ideal. The set of all permissible subsets of S is denoted by $C(S)$.

The set $A^{-1} = \{a^{-1} : a \in A\}$ for a permissible subset A of S is also permissible and is the inverse of A.

Proposition 4.8. *The set $C(S)$ of a completely inverse* AG***-groupoid S is a completely inverse* AG***-groupoid under the multiplication of subsets of S. The mapping $\imath : S \longrightarrow C(S)$, defined by $x \mapsto \lfloor x \rfloor$, where $\lfloor x \rfloor$ represents the permissible subset of $x \in S$, is an injective morphism.*

Proof. To show that $AB = \{ab : a \in A,\ b \in B\}$ is a compatible order ideal, let $A, B \in C(S)$. Also for $a \in A, b \in B$, let $x \leq ab$. Then by Lemma 4.2(4)

$$x = (xx^{-1})(ab) = a(xx^{-1} \cdot b) = ab^{*}.$$

Since, B is an order ideal, $b^{*} = xx^{-1} \cdot b \in B$. Hence $ab^{*} \in AB$ and AB is order ideal. Further, to show that AB is compatible subset of S, let $ab, cd \in AB$. Then

$$(ab)^{-1}(cd) = (a^{-1}b^{-1} \cdot cd) = (a^{-1}c \cdot b^{-1}d) \in E_S.$$

Since $a \sim c$, $b \sim d$, and $a^{-1}c$, $b^{-1}d \in E_S$, similarly, $(ab)(cd)^{-1} \in E_S$. Thus $ab \sim cd$. Hence AB is a compatible subset of S and $AB \in C(S)$. To show that any $A \subseteq C(S)$ is completely inverse, let we consider $ab^{-1} \cdot c$ for $a, b, c \in A$. Also consider that $s = ab^{-1} \cdot b$. Then $s = bb^{-1} \cdot a$ which shows that $s \leq a$ and since A is an order ideal, then $s \in A$. Moreover, since $a \sim b$, then $ab^{-1} \in E_S$. Thus we have

$$\begin{aligned} ss^{-1} &= (ab^{-1} \cdot b)(ab^{-1} \cdot b)^{-1} = (ab^{-1} \cdot b)(a^{-1}b \cdot b^{-1}) = (ab^{-1} \cdot a^{-1}b)(bb^{-1}) \\ &= (ab^{-1} \cdot (ab^{-1})^{-1})(bb^{-1}) = (b^{-1}b)(ab^{-1}) = a(b^{-1}b \cdot b^{-1}) = ab^{-1}. \end{aligned}$$

So we can write $ab^{-1} \cdot c = ss^{-1} \cdot c \leq c$. But since A is an order ideal, then $ab^{-1} \cdot c \in A$. Hence $AA^{-1} \cdot A \subseteq A$. For the inverse inclusion it is clear that $A \subseteq AA^{-1} \cdot A$. Thus $A = AA^{-1} \cdot A$. The condition $AA^{-1} = A^{-1}A$ is clear.

Suppose that $\lfloor x \rfloor$ represents the permissible subset of $x \in S$. Then it is an order ideal. For compatibility, let $a, b \leq x$. Then $a = ex,\ b = fx$ for some $e, f \in E_S$. Now $a^{-1}b = (ex^{-1})(fx) = (fx \cdot x^{-1})e = (ef)(x^{-1}x) \in E_S$ and $ab^{-1} = (ex)(fx^{-1}) = (ef)(xx^{-1}) \in E_S$. Finally, let $x, t \in S$. Then $\lfloor x \rfloor, \lfloor t \rfloor \in C(S)$ and $\lfloor x \rfloor \cdot \lfloor t \rfloor = \lfloor xt \rfloor$. Thus $\imath$ is a morphism. If $\lfloor x \rfloor = \lfloor t \rfloor$, then $x = t$. Hence $\imath$ is one-one. □

5. Minimum and maximum congruences

In this section, we study minimum congruence and maximum congruence. Minimum congruence is important in the sense that it develops a connection between an AG-group and a completely inverse AG**-groupoid.

Definition 5.1. Let S be a completely inverse AG**-groupoid and let $a, b \in S$. We define a relation σ by

$$a\sigma b \iff \exists x \in S : x \leq a, b.$$

From the fact that the natural partial order is compatible with the multiplication, we conclude that σ is compatible, that is, if $c \in S$, then

$$a\sigma b \Longrightarrow \exists x : x \leq a, b \Longrightarrow xc \leq ac, bc \text{ and } cx \leq ca, cb.$$

In the following main result of this section, we show that σ is the smallest congruence containing the compatibility relation $\sim$ and itself is contained in any congruence ρ such that S/ρ is an AG-group.

Theorem 5.2. *Let S be a completely inverse* AG***-groupoid. Then*
(1) *σ is the smallest congruence on S containing the compatibility relation $\sim$.*
(2) *S/σ is an* AG*-group.*
(3) *If ρ is any congruence such that S/ρ is an* AG*-group, then $\sigma \subseteq \rho$.*

Proof. (1) Reflexivity and symmetry are clear. For transitivity, suppose that $a\sigma b$ and $b\sigma c$, where $a, b, c \in S$. Then there exist r, s such that $s \leq a, b$ and $r \leq b, c$, which further give $s, r \leq b$. Consequently, $s, r \in \lfloor b \rfloor$. But $\lfloor b \rfloor$ is compatible subset of S, thus $s \sim r$. Consider the quantity $ss^{-1} \cdot r$ which gives $ss^{-1} \cdot r \leq r$ and $ss^{-1} \cdot r = rs^{-1} \cdot s \leq s$, because $s \sim r$ and $rs^{-1} \in E_S$. This further implies $ss^{-1} \cdot r \leq a$ and $ss^{-1} \cdot r \leq r \leq c$. Thus $a \sim c$. Since σ is compatible with multiplication, then σ is a congruence.

Let $a \sim b$. Then $a^{-1}b \cdot a \leq a$ and $a^{-1}b \cdot a = (a^{-1}b)(aa^{-1} \cdot a) = (a^{-1}b \cdot aa^{-1})a = (a^{-1}a \cdot ba^{-1})a = (ba^{-1})a = aa^{-1} \cdot b$. Thus $a^{-1}b \cdot a \leq a, b$ and so $a \equiv_\sigma b$. Hence $\sim \subseteq \sigma$.

Let ρ be any congruence such that $\sim \subseteq \rho$. For $a \equiv_\sigma b$ there is $s \in S$ such that $s \leq a, b$ and $s = ea, s = fb$ for some $e, f \in E_S$. Then we have

$$s^{-1}a = (ea)^{-1}a = (aa^{-1})e \in E_S \Longleftrightarrow s \sim a$$
$$sa^{-1} = (ea)a^{-1} = a^{-1}a \cdot e \in E_S \Longleftrightarrow s \sim a.$$

In the same manner, $s^{-1}b, sb^{-1} \Longleftrightarrow s \sim b$. Thus $s \sim a, b$ and by assumption $a \equiv_\rho s$ and $s \equiv_\rho b$. But ρ is a congruence, so $a \equiv_\rho b$. Hence $\sigma \subseteq \rho$.

(2) Since σ is a congruence on S, then S/σ is a completely inverse AG**-groupoid. Suppose that $(a)_\sigma \in E_{S/\sigma}$, then

$$(a)_\sigma = ((a)_\sigma)((a^{-1})_\sigma) = (aa^{-1})_\sigma = (a^{-1}a)_\sigma = (e)_\sigma \text{ for some } a^{-1}a = e \in E_S.$$

Hence for every $(a)_\sigma \in E_{S/\sigma}$, there is $(e)_\sigma$ such that $a_\sigma = (e)_\sigma$, that is, $(a, e) \in \sigma$. Also $(ea)_\sigma = (aa^{-1} \cdot a)_\sigma = (a)_\sigma$. For $e, f \in E_S$, it is clear that $ef \leq e, f$, that is $ef \sim e$ and $ef \sim f$ and thus $e \sim f$. Since σ is a congruence, then $e \sim f$ implies $e \equiv_\sigma f$, that is, all idempotents are contained in a single σ-class. Thus S/σ containing a single σ-class of idempotents is an AG-group.

(3) Suppose that ρ is a congruence such that S/ρ is an AG-group. Suppose $a \equiv_\sigma b$. Then for some $s \in S$, $s \leq a, b$. Therefore there exists $e, f \in E_S$ such that $ea = s = fb$. Hence

$$(s)_\rho = (ea)_\rho = (e)_\rho(a)_\rho, \qquad (s)_\rho = (fb)_\rho = (f)_\rho(b)_\rho,$$

where $(e)_\rho, (f)_\rho \in E_{S/\rho}$. Since S/ρ is an AG-group, $(e)_\rho = (f)_\rho$, that is $(e, f) \in \rho$. Thus $(a)_\rho = (s)_\rho = (b)_\rho$ and $a \equiv_\rho b$. Therefore $\sigma \subseteq \rho$. □

Lemma 5.3. *Let S be a completely left inverse* AG***-groupoid. Then $a \equiv_\sigma b$ if and only if there exists an idempotent i such that $ia = ib$.*

Proof. Suppose $a\sigma b$. Then for some $x \in S$ and $e, f \in E_S$, we have $ea = x = fb$. Thus $(ef)a = e(fa) = (ef)(ea) = (ef)b$ where ef is the required element. The converse is trivial. □

An idempotent-separating congruence μ is a congruence on a completely inverse AG**-groupoid in which distinct idempotents lie in distinct congruence classes, that is $(e)_\rho = (f)_\rho \Longrightarrow e = f$ for $e, f \in E_S$. In the following proposition, we investigate a maximum idempotent-separating congruence of a completely inverse AG**-groupoid S.

Proposition 5.4. *Let S be a completely inverse* AG***-groupoid. Then the relation*

$$(a, b) \in \mu \text{ if and only if } a^{-1} \cdot ea = b^{-1} \cdot eb \text{ for all } e \in E_S$$

is the maximum idempotent-separating congruence.

Proof. The relation μ indeed is an equivalence relation. To show that μ is a congruence relation let $(a, b), (c, d) \in \mu$ and $e \in E_S$. Then

$$\begin{aligned}(ac)^{-1}(e(ac)) &= (a^{-1}c^{-1})(e(ac)) = (a^{-1}e)(c^{-1}(ac)) \\ &= c^{-1}((a \cdot ea^{-1})c) = d^{-1}((b \cdot eb^{-1})d) = (bd)^{-1}(e(bd))\end{aligned}$$

which shows that μ is a congruence. For idempotent-separating, choose idempotents e, f such that $(e, f) \in \mu$. Then

$$e = e^{-1} \cdot ee = f^{-1} \cdot ef = ef = e^{-1} \cdot ef = e^{-1} \cdot fe = f^{-1} \cdot ff = f.$$

Hence μ is an idempotent-separating congruence. Further suppose ρ be another idempotent-separating congruence. Then for $(a, b) \in \rho$ and any idempotent e, both (a^{-1}, b^{-1}) and $(a^{-1} \cdot ea, b^{-1} \cdot eb)$ are in ρ. But since ρ is idempotent-separating, we have $a^{-1} \cdot ea = b^{-1} \cdot eb$. Hence $(a, b) \in \mu$.

□

Green's relations are the most important tools for understanding semigroups. Green's relations for AG-groupoids are defined the same way as in semigroup theory [2]. There are five Green's relations denoted by $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$. It is important to be noted that every $\mathcal{H}$-class contains exactly one idempotent. In the following proposition, the largest congruence μ that is contained in $\mathcal{H}$ is considered.

Proposition 5.5. *Let S be a completely inverse* AG***-groupoid. Then μ is the largest congruence contained in $\mathcal{H}$.*

Proof. For $\mu \subseteq \mathcal{H}$, it is enough to show that $a \mathcal{L} b$. Let $(a, b) \in \mu$, then for some $bb^{-1} \in E_S$, $a^{-1}(bb^{-1} \cdot a) = b^{-1}(bb^{-1} \cdot b)$. But $b^{-1}(bb^{-1} \cdot b) = bb^{-1} \cdot b^{-1}b = b^{-1}(bb^{-1} \cdot b) = b^{-1}b$ and then $b = bb^{-1} \cdot b = (a^{-1}(bb^{-1} \cdot a))b = (b(bb^{-1} \cdot a^{-1}))a \in Sa$. Similarly, $a \in Sb$ and $a \mathcal{L} b$. Thus $a \mathcal{H} b$. Suppose ν be another congruence such that $(a, b) \in \nu \subseteq \mathcal{H}$. Then $(a^{-1}, b^{-1}) \in \nu$ and for some $e \in E_S$, $(a^{-1} \cdot ea, b^{-1} \cdot eb) \in \nu$. Then $a^{-1} \cdot ea = b^{-1} \cdot eb$, since every H-class contains only one idempotent and thus $(a^{-1} \cdot ea, b^{-1} \cdot eb) \in \mu$. Hence μ contains an arbitrary congruence $\nu \subseteq H$. □

A completely inverse AG**-groupoid is fundamental if μ is the equality relation. Every completely inverse AG**-groupoid can be made *fundamental* if the factor AG**-groupoid is kept with respect to μ. The following proposition is a direct connection between the fundamental completely inverse AG**-groupoid and maximum idempotent-separating congruence.

Proposition 5.6. *Let μ be the maximum idempotent-separating congruence on a completely inverse* AG***-groupoid S. Then S/μ is fundamental and it has the semilattice of idempotents isomorphic to E_S.*

Proof. Consider the morphism $\natural : S \to S/\mu$. Then for any $e \in E_S$ idempotent in S/μ it can be written as $(e)_\mu$. Suppose that $((a)_\mu, (b)_\mu) \in \mu_{S/\mu}$. Then $(a^{-1} \cdot ea)_\mu = (a^{-1})_\mu \cdot (e)_\mu (a)_\mu = ((a)_\mu)^{-1} \cdot (e)_\mu (a)_\mu \ \mu_{S/\mu} ((b)_\mu)^{-1} \cdot (e)_\mu (b)_\mu = (b^{-1})_\mu \cdot (e)_\mu (b)_\mu = (b^{-1} \cdot eb)_\mu$. Thus $((a^{-1} \cdot ea)_\mu, (b^{-1} \cdot eb)_\mu) \in \mu_{S/\mu}$. But $\mu_{S/\mu}$ is idempotent-separating, so we have $(a^{-1} \cdot ea)_\mu = (b^{-1} \cdot eb)_\mu$. Since μ is idempotent-separating, it follows that $a^{-1} \cdot ea = b^{-1} \cdot eb$. Hence $(a)_\mu = (b)_\mu$.

For each $(e)_\mu \in E_{S/\mu}$, there exists $e \in E_S$ such that $e \to (e)_\mu$. Hence $E_S \cong E_{S/\mu}$. □

Theorem 5.7. *Let S be a completely inverse* AG***-groupoid on S. Then*

$$\varrho = \{(a, b) \in S \times S : ea^2 = b^2 f \text{ for some } e, f \in E_S\}$$

is a congruence relation on S.

Proof. Reflexivity and symmetry are immediate. We show that ϱ is transitive and compatible. Let $(a, b) \in \varrho$ and $(b, c) \in \varrho$, then for some idempotents e, f, g, h, we have $ea^2 = b^2 f$ and $gb^2 = c^2 h$. Since $ge \in E_S$,

$$(ge)a^2 = g(ea^2) = g(b^2 f) = (c^2 h)f = c^2(fh)$$

where $fh \in E_S$. Further, suppose $(a, b) \in \varrho$. Then for some $c \in S$,

$$e(ca)^2 = c^2(ea^2) = c^2(b^2 f) = (fb^2)c^2 = (cb)^2 f.$$

Similarly, if $(a, b) \in \varrho$, then for $c \in S$,

$$e(ac)^2 = (ea^2)c^2 = (b^2 f)c^2 = c^2(fb^2) = f(bc)^2.$$

Hence ϱ is a congruence on S. □

A congruence ρ on a completely inverse AG**-groupoid S is idempotent-pure if for $e \in E_S,\ a \in S$ and $(a, e) \in \rho \implies a \in E_S$.

Proposition 5.8. *Let S be a completely inverse AG^{**}-groupoid. Then a congruence ρ is idempotent-pure if and only if $\rho \subseteq \sim$.*

Proof. Suppose that $(a, b) \in \rho$. Then $(ab^{-1}, bb^{-1}) \in \rho$. Clearly ab^{-1} is an idempotent because bb^{-1} is idempotent and ρ is idempotent-pure. Similarly, $a^{-1}b$ is an idempotent and hence $a \sim b$.

Conversely, suppose that the congruence ρ is contained in $\sim$. Let for an idempotent $e,\ (a, e) \in \rho$. Then $(e^{-1}, a^{-1}) \in \rho$. Thus $e = e^{-1}e = a^{-1}a$. Since ρ is a congruence, it is clear that $a \equiv_\rho a^{-1}a$. By assumption $\rho \subseteq \sim$, thus we have $a \sim a^{-1}a$. Then $a = aa^{-1} \cdot a = a^{-1}a \cdot a = (aa^{-1})^{-1}a \in E_S$. Hence ρ is idempotent-pure. □

6. E-unitary completely inverse AG**-groupoid

In this section, we characterize E-unitary completely inverse AG**-groupoid and investigate idempotent-pure congruence with a connection in compatibility relation containing the idempotent-pure congruence.

A subgroupoid A of an AG-groupoid S is called left (right) unitary if for $a \in A,\ s \in S$ and $as \in A$ ($sa \in A$), imply that $s \in A$. A is unitary if it is both left and right unitary. An AG-groupoid S is E-unitary if E_S is unitary.

Proposition 6.1 ([3, Proposition 4.4])**.** *Let S be an* AG***-groupoid. Then the following are equivalent:*

(1) *E_S is right unitary.*
(2) *E_S is left unitary.*
(3) *If $e \in E_S$ and $e \leq a$, then $a \in E_S$.*

Proof. (1) $\Rightarrow$ (2) : Suppose $e,\ ea \in E_S$. Then we have $E_S \ni af = ea$ for some idempotent f. Thus E_S is left unitary.

(2) $\Rightarrow$ (3) : Suppose E_S is left unitary and $e \leq a$ for $e \in E_S$. Then $e = ea$ and hence $a \in E_S$.

(3) $\Rightarrow$ (1) : Suppose $ea = f \in E_S$. Then $f \leq a$ and thus E_S is right unitary. □

Proposition 6.2. *The compatibility relation on a completely inverse* AG**-*groupoid S is transitive if and only if S is E-unitary.*

Proof. Let the compatibility relation $\sim$ be transitive and for $e \in E_S,\ e \leq a$. Clearly, $e^{-1}a = ea = e$ and $ea^{-1} \leq aa^{-1}$ implies that ea^{-1} is an idempotent because $aa^{-1} \in E_S$. Hence $a \sim e$. Since e and aa^{-1} both are idempotents, therefore $e \sim aa^{-1}$. But $\sim$ is transitive and hence $a \sim aa^{-1}$. It is clear that a is an idempotent, since $a = aa^{-1} \cdot a = a^{-1}a \cdot a = (aa^{-1})^{-1}a \in E_S$ and thus E_S is E-unitary.

Conversely, suppose that E_S is E-unitary and $a \sim b$ and $b \sim c$. Then by assumption $a^{-1}b,\ ab^{-1}$ and $b^{-1}c,\ bc^{-1}$ are idempotents. Thus we have

$$\begin{aligned} a^{-1}b \cdot b^{-1}c &= (a^{-1}b)((b^{-1}b \cdot b^{-1})c) = (a^{-1}b)(cb^{-1} \cdot b^{-1}b) \\ &= (a^{-1}b \cdot cb^{-1})(b^{-1}b) \\ &= (a^{-1}c \cdot bb^{-1})(b^{-1}b) = (bb^{-1} \cdot b^{-1}b)(a^{-1}c) \leq a^{-1}c. \end{aligned}$$

In the same manner,

$$\begin{aligned} ab^{-1} \cdot bc^{-1} &= (ab^{-1})((bb^{-1} \cdot b)c^{-1}) = (ab^{-1})(c^{-1}b \cdot b^{-1}b) \\ &= (ab^{-1} \cdot c^{-1}b)(b^{-1}b) = (ac^{-1} \cdot bb^{-1})(b^{-1}b) = (bb^{-1} \cdot b^{-1}b)(ac^{-1}) \\ &\leq ac^{-1}. \end{aligned}$$

Since S is E-unitary, $a^{-1}c$ and ac^{-1} are idempotents. Hence $a \sim c$. □

Corollary 6.3. Let S be E-unitary completely inverse AG**-groupoid. Then the compatibility relation $\sim$ is a congruence.

Lemma 6.4. *Let ρ be an idempotent-pure congruence on a completely inverse* AG**-*groupoid S. Then S is E-unitary if and only if S/ρ is E-unitary.*

Proof. Suppose S is E-unitary. Let $a, b \in S$ such that $(a)_\rho, (b)_\rho \in S/\rho$ and $(b)_\rho(a)_\rho \in E_{S/\rho}$. Then there is $e \in E_S$ such that $(b)_\rho = (e)_\rho$, that is $(b, e) \in \rho$. Since ρ is idempotent-pure, $ea \in E_S$. But S is E-unitary, we have $a \in E_S$. Thus $(a)_\rho \in E_{S/\rho}$. The converse is simple. □

Theorem 6.5. *Let S be a completely inverse* AG**-*groupoid. Then the following are equivalent:*

(1) *S is E-unitary.*
(2) $\sim = \sigma$.
(3) *σ is idempotent-pure.*
(4) *for all $e \in E_S,\ \sigma(e) = E_S$.*
(5) *AA^{-1} and $A^{-1}A$ are ideals of E_S for every σ-class.*
(6) *$A^{-1} \cdot eA$ is an ideal of E_S for every $e \in E_S$ and every σ-class A.*

Proof. (1) $\Rightarrow$ (2) : Suppose S is E-unitary. It is clear from Theorem 5.2 that $\sim\subseteq\sigma$. Let $(a,b)\in\sigma$ then $s\leq a,b$ for some $s\in S$. Therefore for idempotents e,f, we have $ea=s=fb$. It follows that

$$ss^{-1}=(ea)(fb)^{-1}=(ef)(ab^{-1})\leq ab^{-1}$$

and similarly $s^{-1}s\leq a^{-1}b$. But S is E-unitary so ab^{-1} and $a^{-1}b$ are idempotents. Thus a and b are compatible.

(2) $\Rightarrow$ (3) : From Proposition 5.8, since $\sigma\subseteq\sim$, it follows immediately that σ is idempotent-pure.

(3) $\Rightarrow$ (4) : It follows from the definition of idempotent-pure congruence.

(4) $\Rightarrow$ (5) : We only show AA^{-1} is an ideal. If $(a,b)\in\sigma$, then $ab^{-1}\in E_S$. For $e\in E_S$,

$$(ab^{-1})e=e(ab^{-1})=a(eb)^{-1}\in AA^{-1}.$$

Hence AA^{-1} is an ideal of E_S.

(5) $\Rightarrow$ (6) : Let $a,b\in A$. Then for any idempotent e, we have $a^{-1}\cdot eb=e\cdot a^{-1}b\in E_S$, because by hypothesis $a^{-1}b\in E_S$. Thus $A^{-1}\cdot eA$ is contained in E_S. For every $f\in E_S$,

$$f(a^{-1}\cdot eb)=(a^{-1}\cdot eb)f=(fe\cdot a^{-1}b)=(fa^{-1}\cdot eb)=(fa)^{-1}(eb)\in A^{-1}\cdot eA.$$

Hence $A^{-1}\cdot eA$ is an ideal of E_S.

(6) $\Rightarrow$ (1) : Suppose $ea,e\in E_S$. Then $(a,ea)\in\sigma$. Also

$$ea=ee\cdot a=e\cdot ea=ea\cdot e\leq e$$

which implies that $(ea,e)\in\sigma$. Thus $(a,e)\in\sigma$. Moreover, $(aa^{-1},ee^{-1})\in\sigma$. Letting $A=(a)_\sigma$ we have

$$a=(aa^{-1})a=(aa^{-1})(aa^{-1}\cdot a)\in((a)_\sigma)^{-1}\cdot(aa^{-1})((a)_\sigma)\in A^{-1}\cdot(aa^{-1})A\subseteq E_S.$$

Hence it follows that S is E-unitary. □

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