ON AN EXTENSION OF THE DUBINS CONDITIONAL PROBABILITY AXIOMATIC TO COHERENT PROBABILITY OF FUZZY EVENTS

Fabrizio Maturo
Department of Management and Business Administration
University G. d'Annunzio of Chieti-Pescara
Italy
f.maturo@unich.it

Abstract. An approach to the concept of fuzzy event as an extension of conditional event is introduced. The probability of fuzzy events is presented as an extension of the de Finetti's probability of conditional events and depends on a score function subjectively assigned by an expert. It is shown that the introduced fuzzy probability extends in a fuzzy ambit the conditions considered by Dubins for finitely additive conditional probability. Possible applications for decision-making under uncertainty are sketched.

Keywords: conditional events, finitely additive conditional probability, fuzzy events, fuzzy probability, decision-making under uncertainty.

1. Introduction

According to the de Finetti's theory (1970), a conditional event $A|B$ is considered as a logical entity with three truth values: true, false, and a third truth value that can be: undeterminate or empty or undecidable or other term, depending occurring $A \cap B, A^c \cap B, B^c$. Many authors have studied the three-valued logic (Reichenbach, 1944: 269-297; Gentilhomme, 1968: 54; De Finetti, 1970:685-686; Fadini, 1979:41; Nguyen et al., 2003: 1061), attributing different terms and meanings to the third truth value. However, the logical operations do not depend on the term which is used. Following the literature regarding the three value logic (Gentilhomme, 1968; Fadini, 1979:45; Lane, 1999; Nguyen et al., 2003: 1061; Negarestani, 2012; 54), we represent the three logical values "true", "false", and "third truth value" with "1", "0", and "1/2", respectively.

The formalization of conditional probability of de Finetti has been largely studied in the literature (de Finetti, 1970; Dubins, 1975; Scozzafava, 1993, 2001; Yexin et al., 2005; Mundici, 2006; Montagna et al., 2013; Flaminio et al., 2015a; 2015b). In many of these studies the assessment of conditional probability is obtained by the concept of coherent bet. Various authors (e.g. Zadeh, 1968; Yager, 1999; Maturo, 2000; Mundici, 2006; Maturo and Doria, 2008) have considered fuzzy events and fuzzy probability. In this paper, we follow a different perspective; specifically, we consider the concept of fuzzy event as an extension of conditional event, propose a possible approach to the idea of probability of fuzzy events, and study their mathematical properties. Moreover, we introduce some possible meanings of fuzzy event and fuzzy probability (which are defined following our approach) in the context of decisional processes. Furthermore, we aim to discover formulas for the probability of fuzzy events, which satisfy, under broad conditions, the mathematical properties of conditional events, in particular the axiomatic conditions of Dubin (1976) for the finitely additive conditional probability.

2. Functional and algebraic representations of conditional events

In the subjective approach to the calculus of probability (e.g. de Finetti, 1970; Dubins, 1975; Scozzafava, 1993, 2001), a conditional event $A|B$, defined if B is a non-impossible event, is a statement assuming the following values: true if the intersection $A \cap B$ is true, false if the intersection $A^c \cap B$ is true (where A^c is the contrary of A), and undetermined if B is false. In this paper, for a better algebraic treatment, we also consider the conditional event $A|B$ with $B = \emptyset$, called the totally undetermined conditional event.

To show how fuzzy events, as will be defined in this paper, can be seen as extensions of conditional events, let's introduce an alternative (but logically equivalent) definition of conditional event $A|B$. Then we give the following definition.

Definition 2.1. Let Π be a partition of the certain event. We define *conditional event in functional form (briefly FFCE)*, with *domain* Π, every function $X : \Pi \to \{0, 1/2, 1\}$. The union of the elements $a \in \Pi$ such that $X(a) = 1$ (resp. $X(a) = 0, X(a) = 1/2$) is said to be the *true part* of X, denoted by T_X (resp. the *false part* of X, denoted by F_X, and the *undetermined part* of X, indicated with U_X). The event $D_X = T_X \cup F_X$ is called *the determined part* of X.

Definition 2.2. If X and Y are FFCE, we say that X is equivalent to Y, we write $X \sim Y$ if $(T_X, U_X, F_X) = (T_Y, U_Y, F_Y)$.

Evidently, a $FFCEX$, and all the $FFCEY$ equivalent to X, individuate the conditional event in normal form $T_X|D_X$ and vice versa, every conditional event $A|B$ represents the equivalence class of the $FFCE$ with $T_X = A \cap B$, $U_X = B^c$, $F_X = A^c \cap B$. In other words, an $FFCF$ equivalence class is a random number with codomain $\{0, 1/2, 1\}$.

The contrary of X is the conditional event $X^c : \Pi \to \{0, 1/2, 1\}$ having the same domain and determined part of X, but with false part equal to the true part of X.

The triplet (T_X, U_X, F_X) is said to be the algebraic representation of X. For indicating that X is a conditional event with algebraic representation (T_X, U_X, F_X), we write $X = (T_X, U_X, F_X)$.

Definition 2.3. Two conditional events X and Y are:

- disjoint, if $T_X \cap T_Y = \emptyset$;

• exhaustive, if $F_X \cap F_Y = \emptyset$;
• homogeneous, if $D_X = D_Y$.

Clearly, X and Y are contrary, if and only if, they are disjoint, exhaustive, and homogeneous.

In the subjective probability framework, a conditional event X is interpreted as a bet in which a decision-maker D wins if the true part occurs, loses if the false part happens, and the bet is canceled if the undetermined part is obtained.

Given two partitions Π_1 and Π_2 of the certain event, the product $\Pi_1 \bullet \Pi_2$ is the set of the not impossible intersections $a \cap b$, with $a \in \Pi_1$, $b \in \Pi_2$.

Definition 2.4. Let $*$ be an operation on $\{0, 1/2, 1\}$. For all conditional events $X : \Pi_X \to \{0, 1/2, 1\}, Y : \Pi_Y \to \{0, 1/2, 1\}$, the product of X and Y induced by $*$, or "$*$-product", is the conditional event $X * Y : \Pi_X \bullet \Pi_Y \to \{0, 1/2, 1\}$ such that:

$$(z \in \Pi_X \bullet \Pi_Y, z = a \cap b, a \in \Pi_X, b \in \Pi_Y) \Rightarrow (X * Y)(z) = X(a) * Y(b).$$

Important operations in $\{0, 1/2, 1\}$ are:
• the union (or disjunction) $\cup$, defined by: $a \cup b = \max\{a, b\}$;
• the intersection (or conjunction) $\cap$, defined by: $a \cap b = \min\{a, b\}$.

3. Fuzzy event as an extension of the concept of conditional event

We assume the following definition of fuzzy event as extension of conditional event.

Definition 3.1. Let Π be a partition of the certain event. A *fuzzy event* with *domain* Π is a function $X : \Pi \to [0, 1]$. The union of the elements $a \in \Pi$ such that $X(a) > 1/2$ (resp. $X(a) < 1/2$, $X(a) = 1/2$) is said to be the *true part* of X, denoted with T_X (resp. *false part* of X, denoted with F_X, and *undetermined part* of X, denoted with U_X). The event $D_X = T_X \cup F_X$ is the *determined* part of X.

The *contrary* of X is the fuzzy event $X^c : \Pi \to [0, 1]$ such that, $\forall a \in \Pi, X^c(a) = 1 - X(a)$.

The conditional event X_0, with domain Π and such that $X_0(a) = 1$, $1/2$, 0 depending on which $X(a) > 1/2, X(a) = 1/2, X(a) < 1/2$, is said to be the *crisp approximation* of X. In algebraic form $X_0 = (T_X, U_X, F_X)$.

Let V_X be the set of all the values assumed by the fuzzy event $X : \Pi \to [0, 1]$. For all $v \in V_X$, the union of the $a \in \Pi$ such that $X(a) = v$ is called *part at level* v of X, denoted X_v. The fuzzy event $X^* : \{X_v, v \in V_X\} \to [0, 1]$ that associates to X_v the number v is said to be the *normal form* of X. Using a slight modification of the notations introduced by Zadeh (Zadeh, 1965, 1968, 1975a, 1975b; Klir, Yuan, 1995; Zadeh at al., 1996), we write $X = \{v/X_v, v \in V_X\}$.

Two fuzzy events X and Y will be said equivalent if they have the same normal form, i.e. $V_X = V_Y$ and, $\forall v \in V_X, X_V = Y_V$.

With this notation, a $FFCEX = (T_X, U_X, F_X)$ is a fuzzy event such that $V_X \subseteq \{1, 1/2, 0\}$. Using the fuzzy notation, we can write $X = \{1/T_X, 0.5/U_X, 0/F_X\}$.

Extending to fuzzy events the basic ideas of subjective setting of probability theory, a fuzzy event X is interpreted as a bet in which a decision-maker D partially or totally wins if the true part occurs, partially or totally loses if the false part happens, and the bet is canceled if the undetermined part is obtained. Unlike the conditional event, the amount of winning (if $v > 1/2$) or loss (if $v < 1/2$) depends on the truth-value v, i.e. it decreases if v approaches 1/2, increases when v approaches the extremes of the interval $[0, 1]$, and is null if $v = 1/2$.

The concepts expressed by the definition 2.3 can be extended in various ways to fuzzy events. For this purpose, we introduce the following definition:

Definition 3.2. Two fuzzy events X and Y are called:
- quasi-disjoint, if $T_X \cap T_Y = \emptyset$;
- quasi-exhaustive, if $F_X \cap F_Y = \emptyset$;
- quasi-homogenous, if $D_X = D_Y$.

In general, for each fuzzy event X, we use the prefix "quasi" for indicating a property that holds for the crisp approximation X_0 of X.

We can extend the definition 2.4 to fuzzy events as follows:

Definition 3.3. Let $*$ be an operation in $[0, 1]$. If $X : \Pi_X \to [0, 1]$ and $Y : \Pi_Y \to [0, 1]$ are two fuzzy events, we define product of X and Y induced by $*$, or "$*$- product", the fuzzy event $X * Y : \Pi_X \bullet \Pi_Y \to [0, 1]$ such that:

$$(z \in \Pi_X \bullet \Pi_Y, z = a \cap b, a \in \Pi_X, b \in \Pi_Y) \Rightarrow (X * Y)(z) = X(a) * Y(b).$$

The most important operations in $[0, 1]$ we will use during the work are the following:
- the union (or disjunction) $\cup$, defined by: $a \cup b = \max\{a, b\}$;
- the intersection (or conjunction) $\cap$, defined by: $a \cap b = \min\{a, b\}$;

4. An approach to subjective probability of fuzzy events

Let Π be a finite partition of the certain event and let $X : \Pi \to \{0, 1/2, 1\}$ be a conditional event. The subjective conditional probability $p = p(X)$ of X is defined from a bet, with stake S and possible winning $V \geq S$. Three circumstances may happen:
- if T_X occurs, he receives the winning V;
- if U_X occurs, he gets back the stake S;
- if F_X happens, he loses the stake S.

The subjective conditional probability p is defined as the ratio $p = S/V$. Therefore, the gain of D is the random number:

$$G = [(1 - p)|T_X| - p|F_X|]V, \tag{4.1}$$

where, for any event E, we denote by $|E|$ its indicator.

The bet is coherent with respect a probability $p_1 : \Pi \to [0,1]$, if and only if the prevision of G is null, i.e.

$$[p_1(T_X) + p_1(F_X)]p = p_1(T_X), \tag{4.2}$$

where $p_1(T_X)$ and $p_1(F_X)$ are, respectively, the probabilities of the true part and the false part of X, and $p_1(T_X) + p_1(F_X) = p_1(D_X)$.

If $p_1(DX) \neq 0$, we get a single coherent value for p. Precisely, if we put $Q^1_X = p_1(D_X) = p_1(T_X) + p_1(F_X), Q^{+1}_X = p_1(T_X), Q^{-1}_X = p_1(F_X)$, we have:

$$p(X) = Q^{+1}_X / Q^1_X, p(X^c) = Q^{-1}_X / Q^1_X. \tag{4.3}$$

If $p_1(D_X) = 0$, but $D_X \neq \emptyset$, we can consider the set $\Pi_1 = \{E \in \Pi : p_1(E) = 0\}$ and we can assign (de Finetti, 1970) a subjective conditional probability $p_2(E) = p(E/\Pi_1)$ for every $E \in \Pi$ with the condition

$$\sum \{p_2(E) : E \in \Pi_1\} = 1. \tag{4.4}$$

The bet is coherent with respect the conditional probability $p_2 : \Pi \to [0,1]$, if and only if the prevision of G is null, i.e.

$$[p_2(T_X) + p_2(F_X)]p = p_2(T_X). \tag{4.5}$$

If $p_2(D_X) \neq 0$, we get a single coherent value for p. Precisely, if we put $Q^2_X = p_2(D_X) = p_2(T_X) + p_2(F_X), Q^{+2}_X = p_1(T_X), Q^{-2}_X = p_1(F_X)$, we have:

$$p(X) = Q^{+2}_X / Q^2_X, p(X^c) = Q^{-2}_X / Q^2_X. \tag{4.6}$$

Let us call 1st order probability and 2nd order probability, the probabilities given by (4.3) and (4.6), respectively. Proceeding for recurrence, as Π is finite, if $D_X \neq \emptyset$, we obtain an integer j and a j^{th} order probability for X and X^c given by formulae:

$$p(X) = Q^{+j}_X / Q^j_X, p(X^c) = Q^{-j}_X / Q^j_X. \tag{4.7}$$

where $Q^j_X = p_j(D_X) = p_j(T_X) + p_j(F_X), Q^{+j}_X = p_j(T_X)$, $Q^j_X = p_j(F_X)$.

We can get an extension of the subjective conditional probability to fuzzy events in various ways. We propose a possible approach, which is based on a suitable assessment of a "score function". The decision maker D assigns a "score function" $f : [0,1] \to [0,1]$, decreasing in $[0,1/2]$, increasing in $[1/2;1]$, and satisfying the conditions $f(0) = 1, f(1/2) = 0, f(1) = 1$.

Some desirable additional conditions are:

(a) $f(x) = f(1-x)$ (symmetry);

(b) $f(x) \neq 0$ for $x \neq 1/2$ (positivity);

(c) $f(x)$ is strictly decreasing in $[0,1/2]$ and strictly increasing in $[1/2,1]$;

(d) $f(x)$ is continuous.

The condition of symmetry has the following interesting implications:

(1) the number $2|x-1/2|$ can be interpreted as the degree of occurrence or non-occurrence of an event of the partition with value x;

(2) the meaning of $f(x)$ is that of a function that increases with the increase of the degree of occurrence or non-occurrence".

The other three conditions can be useful in many contexts. Hereafter, we assume that the conditions of symmetry and positivity hold.

Suppose, from now on, that the domains of fuzzy events considered are finite.

Let $X : \Pi = \{A_1, A_2, \ldots, A_n\} \to [0,1]$ be a fuzzy event, $X(A_i) = x_i$. Let us define the gain of the bettor D as the random number:

$$G = [(1-p)\Sigma\{f(x_i)|A_i| : x_i > 1/2\} - p\Sigma\{f(x_i)|A_i| : x_i < 1/2\}]V, \tag{4.8}$$

where $p = p(X)$ is the unknown probability of the fuzzy event X.

We assume that the bet is coherent with respect a probability $p_1 : \Pi \to [0,1]$ if and only if the prevision of G is null. Then the coherence condition implies the formula:

$$[\Sigma\{f(x_i)p_1(A_i) : x_i \neq 1/2\}]p = \Sigma\{f(x_i)p_1(A_i) : x_i > 1/2\}. \tag{4.9}$$

If $p_1(D_X) \neq 0$, because of the condition of positivity, we obtain a single coherent value for p. Precisely, if we put

$$Q_X^1 = \Sigma\{f(x_i)p_1(A_i) : x_i \neq 1/2\}, \tag{4.10}$$

$$Q_X^{+1} = \Sigma\{f(x_i)p_1(A_i) : x_i > 1/2\}, Q_X^{-1} = \Sigma\{f(x_i)p_1(A_i) : x_i < 1/2\}, \tag{4.11}$$

we have formulae:

$$p(X) = Q_X^{+1}/Q_X^1, p(X_c) = Q_X^{-1}/Q_X^1, \tag{4.12}$$

that extends to fuzzy events formulae (4.3). If X is a conditional event, then p, defined by (4.12), reduces to the conditional probability.

If $p_1(D_X) = 0$, but $D_X \neq \emptyset$, we can consider the set $\Pi_1 = \{E \in \Pi : p_1(E) = 0\}$ and we can assign (de Finetti, 1970) a subjective conditional probability $p_2(E) = p(E/\Pi_1)$ for every $E \in \Pi$ with the condition $\Sigma\{p_2(E) : E \in \Pi_1\} = 1$.

Formulae (4.10), (4.11), and (4.12) are replaced by:

$$Q_X^2 = \Sigma\{f(x_i)p_2(A_i) : x_i \neq 1/2\}, \tag{4.13}$$

$$Q_X^{+2} = \Sigma\{f(x_i)p_2(A_i) : x_i > 1/2\}, Q_X^{-2} = \Sigma\{f(x_i)p_2(A_i) : x_i < 1/2\}, \tag{4.14}$$

and we have formulae:

$$p(X) = Q_X^{+2}/Q_X^2, p(X^c) = Q_X^{-2}/Q_X^2. \tag{4.15}$$

Proceeding for recurrence, as Π is finite, if $D_X \neq \emptyset$, we obtain an integer j and a jth order probability for X and X^c given by formulae:

$$p(X) = Q_X^{+j}/Q_X^j, p(X^c) = Q_X^j/Q_X^{-j}, \tag{4.16}$$

where

$$(4.17)\quad Q_X^j = \Sigma\{f(x_i)p_j(A_i) : x_i \neq 1/2\},$$

$$(4.18)\quad Q_X^{+j} = \Sigma\{f(x_i)p_j(A_i) : x_i > 1/2\}, Q_X^{-j} = \Sigma\{f(x_i)p_j(A_i) : x_i < 1/2\}.$$

Let us call "*subjective fuzzy probability (briefly SFP) associated to the function f and the probability p_j*" the probability p given by (4.16).

In the theory of decisions under uncertainty, a fuzzy event X represents an alternative compared with a given alternative X_0, called the null alternative, Q_X^{+j} represents the positive change (improvement) compared to the X_0, Q_X^{-j} is the negative change (worsening), and Q_X^j is the total variation. Then, the fuzzy probability $p(X)$ is the ratio of the positive change to the total change, and can represent a significant index of the validity of the alternative X. In particular, if $p(X) > 1/2$, we can assume that X is better than the null strategy, while it is worst whether $p(X) < 1/2$.

The applications of fuzzy events to decision making may be particularly significant in multiagent decision-making (see, e.g., Maturo and Ventre, 2009) or in fuzzy statistical decisions related to, for example, fuzzy regression (see, e.g., Maturo, 2016; Maturo and Hošková-Mayerova, 2017; Maturo and Ventre, 2018).

An approximation of $p(X)$ is obtained by the probability of the crisp approximation X_0 of X. Precisely, if we put $Q_{X0}^j = p_j(D_X^j), Q_{X0}^{+j} = p_j(T_X^j), Q_{X0}^{-j} = p_j(F_X^j)$, we have:

$$(4.19)\qquad p(X_0) = Q_{X0}^{+j}/Q_{X0}^j, p(X_0^c) = Q_{X0}^{-j}/Q_{X0}^j.$$

Formula (4.16) reduces to (4.19) if $f(x) = 1$ for $x \neq 1/2$ and $f(1/2) = 0$. Such a function satisfies the conditions of symmetry and positivity but not the other conditions. Let us call it "the crisp score function".

5. A fuzzy extension of the axiomatic definition by Dubins

In (Dubins, 1975) the following axiomatic definition of finitely additive conditional probability is assumed.

Definition 5.1. Let **E** be an algebra of events. A finitely additive conditional probability on E is a function $p : (A, B) \in (\mathbf{E} \times (\mathbf{E} - \{\emptyset\})) \to p(A|B) \in [0, 1]$ such that:

(PC1) $\forall B \in (\mathbf{E} - \{\emptyset\})$, the partial function $p_B : A \in \mathbf{E} \to p(A|B)$ is a finitely additive probability on **E**;

(PC2) $\forall B \in (\mathbf{E} - \{\emptyset\}), p(B|B) = 1$;

(PC3) $\forall A \in \mathbf{E}, \forall B, C \in (\mathbf{E} - \{\emptyset\}), A \subseteq B \subseteq C \Rightarrow p(A|B)p(B|C) = p(A|C)$.

To extend the definition 5.1 to fuzzy events, it is necessary to rewrite it in the fuzzy (functional) notation, which has been introduced in Sec 3.

Let us denote with Ω (called the totally true event), $\emptyset$ (totally false), and U (totally undetermined), the fuzzy events with $X(\Pi) = \{1\}$, $X(\Pi) = \{0\}$, and $X(\Pi) = \{1/2\}$, respectively. The definition 5.1 is equivalent to the following one in functional notation:

Definition 5.2. Let C_E be the family of the conditional events $X : \Pi \to \{0, 1/2, 1\}$ with the domain Π contained in an algebra of events $\mathbf{E}$. A finitely additive conditional probability on C_E is a function $p : X \in C_E^* = C_E - \{U\} \to p(X) \in [0,1]$ such that:

(PA1) $\forall X, Y \in C_E^*, (T_{X \cap Y} = \emptyset, D_X = D_Y) \Rightarrow p(X \cup Y) = p(X) + p(Y);$

(PA2) $\forall X \in C_E^*, (F_X = \emptyset) \Rightarrow p(X) = 1;$

(PA3) $\forall X, Y, Z \in C_E^*, (T_X = T_Z, D_X = T_Y, D_Y = D_Z) \Rightarrow p(X)p(Y) = p(Z).$

Let us extend to fuzzy events the notions of disjoint, exhaustive, and homogeneous events. Without loss of generality, considering eventually refinements of the domains of the fuzzy events, we can assume that each pair of considered fuzzy events has the same domain.

Definition 5.3. Let $X : \Pi = \{E_1, E_2, \ldots, E_n\} \to [0,1], Y : \Pi = \{E_1, E_2, \ldots, E_n\} \to [0,1]$, be two fuzzy events, $X(E_i) = x_i, Y(E_i) = y_i$. We say that X and Y are:

- homogeneous, if $x_i = y_i$ or $x_i = 1 - y_i$;
- disjoint, if $x_i + y_i \leq 1$;
- exhaustive, if $x_i + y_i \geq 1$.

The homogeneity is equivalent to the fact that the values x_i and y_i are equidistant from 1/2. In particular (X and Y homogeneous) $\Rightarrow$ (X and Y quasi-homogeneous, i.e. $D_X = D_Y$). If X and Y are conditional events, then we have also (X and Y quasi-homogeneous) $\Rightarrow$ (X and Y homogeneous).

By previous definition, it follows:

Proposition 5.1. *Let X and Y be two fuzzy events. If they are homogeneous, then also $X^c, Y^c, X \cup Y, X \cap Y$ are homogeneous with X and Y.*

Moreover, the following properties are equivalent:

- X and Y are homogeneous;
- $X \cup X^c = Y \cup Y^c$;
- $X \cap X^c = Y \cap Y^c$.

For each fuzzy event $X : \Pi \to [0,1]$ with set of values $V_X, X = \{v/X_v, v \in V_X\}$, we put $X^+ = \{v/X_v, v \in V_X, v > 1/2\}, X^- = \{v/X_v, v \in V_X, v < 1/2\}$.

We note that, for conditional events, the condition $(T_X \cap T_Y = \emptyset, D_X = D_Y)$ is equivalent to (X and Y are disjoint and homogeneous); moreover, the condition $(T_X = T_Z, D_X = T_Y, D_Y = D_Z)$ is equivalent to $(X^+ = Z^+, (X \cup$

$X^c)^+ = Y^+, Y \cup Y^c = Z \cup Z^c)$. Then the definition 5.2 can be extended to fuzzy events in the following way.

Definition 5.4. Let $F_{\mathbf{E}}$ be the family of fuzzy events $X : \Pi \to [0,1]$, with the domain Π contained in an algebra of events $\mathbf{E}$. We define finitely additive fuzzy probability on $F_{\mathbf{E}}$ every function $p : X \in F^*_{\mathbf{E}} = F_{\mathbf{E}} - \{U\} \to p(X) \in [0,1]$ such that:

(PF1) $\forall X, Y \in F^*_{\mathbf{E}}, (T_{X\cap Y} = \emptyset, X \cup X^c = Y \cup Y^c) \Rightarrow p(X \cup Y) = p(X) + p(Y)$;

(PF2) $\forall X \in F^*_{\mathbf{E}}, (F_X = \emptyset) \Rightarrow p(X) = 1$;

(PF3) $\forall X, Y, Z \in F^*_{\mathbf{E}}, (X^+ = Z^+, (X \cup X^c)^+ = Y^+, Y \cup Y^c = Z \cup Z^c) \Rightarrow p(X)p(Y) = p(Z)$.

6. An extension theorem to subjective fuzzy probability

We prove that the fuzzy probability given by (4.16) satisfies the conditions $(PF1)$, $(PF2)$, and $(PF3)$ of the definition 5.4, and then, according to this definition, it is a finitely additive fuzzy probability on F_E.

Theorem 6.1 (Extension theorem)**.** Let F_E be the family of fuzzy events $X : \Pi \to [0,1]$ with the domain Π contained in a finite algebra of events $\mathbf{E}$. If p is a fuzzy probability associated to a function f that satisfies the conditions of symmetry and positivity, then p is a finitely additive fuzzy probability on $F_{\mathbf{E}}$.

Proof. Let X and Y be two fuzzy events of $F^*_{\mathbf{E}}$. Considering possibly a refinement, we can always assume that X and Y have the same domain $\Pi = \{A_1, A_2, \ldots, A_n\}$. Let $X(A_i) = x_i, Y(A_i) = y_i$.

$(PF1)$ Let $T_{X\cap Y} = \emptyset, X \cup X^c = Y \cup Y^c$. Then we have $x_i = y_i$, or $x_i = 1 - y_i$ and $\min\{x_i, y_i\} \leq 1/2$, $D_X = D_Y$. Using the notations of Section 4, if $D_X \neq \emptyset$, we have, for every j:

$$Q^j_X = \Sigma\{f(x_i)p_j(A_i) : x_i \neq 1/2\}; Q^j_Y = \Sigma\{f(y_i)p_j(A_i) : y_i \neq 1/2\}.$$

Since $f(x) = f(1-x)$, it follows $Q^j_X = Q^j_Y = Q^j_{(X\cup Y)}$.

Then from the procedure of Section 4 we obtain an integer j and the j^{th} order probability for X, Y and $X \cup Y$ given by formulae $p(X) = Q^{+j}_X / Q^j_X, p(Y) = Q^{-j}_Y / Q^j_Y, p(X \cup Y) = Q^{+j}_{(X\cup Y)} / Q^j_{(X\cup Y)}$, where denominators are equal and not null and

$$Q^{+j}_X = \Sigma\{f(x_i)p_j(A_i) : x_i > 1/2\}; Q^{+j}_Y = \Sigma\{f(y_i)p_j(A_i) : y_i > 1/2\}.$$

Since $T_{X\cap Y} = \emptyset$, it follows $Q^{+j}_{(X\cup Y)} = Q^{+j}_X + Q^{+j}_Y$. Therefore, we have:

$$p(X \cup Y) = Q^{+j}_{(X\cup Y)} / Q^j_{(X\cup Y)} = Q^{+j}_X / Q^j_X + Q^{+j}_Y / Q^j_Y = p(X) + p(Y).$$

Thus, (PF1) is verified.

(PF2) If $F_X = \emptyset$ then $Q_X^j = Q_X^{+j} \neq 0$, and we have $p(X) = Q_X^{+j}/Q_X^j = 1$ and so (PF2) holds.

(PF3) Let $X, Y, Z \in F_{\mathbf{E}}^*$, such that $X^+ = Z^+$, $(X \cup X^c)^+ = Y^+$, $Y \cup Y^c = Z \cup Z^c$. From formula (4.16), there exist integers i, j, t such that:

$$p(X) = Q_X^{+i}/Q_X^i, p(Y) = Q_Y^{+j}/Q_Y^j, p(Z) = Q_Z^{+t}/Q_Z^t.$$

Since $Y \cup Y^c = Z \cup Z^c, (X \cup X^c)^+ = Y^+$, we have $j = t$, $i \geq j$.

If $i > j$ then $Q_Y^{+j} = Q_X^j = 0$, $Q_Z^{+t} = Q_X^{+t} = 0$ and equality (PF3) reduces to $0 = 0$.

If $i = j$ then $Q_X^i = Q_Y^{+i}, Q_X^{+i} = Q_Z^{+i}, Q_Y^i = Q_Z^i$, and so $p(X)p(Y) = (Q_X^{+i}/Q_X^i)(Q_Y^{+i}/Q_Y^i) = Q_X^{+i}/Q_Y^i = Q_Z^{+i}/Q_Z^i = p(Z)$. Therefore, (PF3) is verified.

In conclusion, the theorem 6.1 extends the relationship between the subjective probability of de Finetti and finitely additive conditional probability of Dubins, to the more general relation between the fuzzy probability and finitely additive fuzzy probability, which have been introduced in the previous sections.

7. Conclusions and perspective of research

In this study, we have shown how the fuzzy probability introduced in Sec. 4 as an extension of the conditional probability can be significant in the case of decision-making under uncertainty.

This probability depends on a score function and appears to have good mathematical properties if this function satisfies at least the conditions of positivity and symmetry. In particular, we have shown that it is possible to obtain a fuzzy extension of Dubins conditions for finitely additive conditional probabilities.

An extension of the theory introduced can be made by replacing the crisp partitions of the certain event with fuzzy partitions. This may be useful for classification problems based on a given objective function.

In particular contexts, the imposition of suitable conditions to the score function can lead to very significant results both from the mathematical point of view and from the point of view of applications.

References

[1] B. De Finetti, *Teoria delle probabilità*, Vol. I, II, Einaudi Editore, Torino, 1970.

[2] L.E. Dubins, *Finitely Additive Conditional Probabilities*, Conglomerability and Disintegrations, Ann. Probab., 3 (1975), 89-99. doi:10.1214/aop/1176996451

[3] A. Fadini, *Introduzione alla teoria degli insiemi sfocati*, Liguori. Napoli, 1979.

[4] T. Flaminio, L. Godo, H. and Hosni, *Coherence in the aggregate: A betting method for belief functions on many-valued events*, International Journal of Approximate Reasoning, 58, 71-86, 2105a. doi:10.1016/j.ijar.2015.01.001

[5] T. Flaminio, L. Godo, H. Hosni, *On the Algebraic Structure of Conditional Events*, Symbolic and Quantitative Approaches to Reasoning with Uncertainty, 106-116, 2015b. doi:10.1007/978-3-319-20807-7_10

[6] Y. Gentilhomme, *Les ensembles flous en linguistique*, Cahiers de Linguistique Théorique et Appliquée (Bucarest), 5, 47-63, 1968.

[7] G.J. Klir, B. Yuan, *Fuzzy sets and fuzzy logic*, Prentice Hall PTR, Upple Saddle River, NJ, USA, 1995.

[8] R. Lane, *Peirce's Triadic Logic Revisited*, Transactions of the Charles S. Peirce Society, 35 (2), 284-311, 1999. Retrieved from http://www.jstor.org/stable/40320762

[9] A. Maturo, *Fuzzy events and their probability assessments*, Journal of Discrete Mathematical Sciences and Cryptography, 3(1-3), 83-94, 2000. doi:10.1080/09720529.2000.10697899.

[10] A. Maturo, S. Doria, *Coherent conditional probabilities and fuzzy implications*, In: Metodi, modelli e tecnologie dell'informazione a supporto delle decisioni. Parte I. Metodologia. Franco Angeli Editore. Milano, 2008, 268-274.

[11] A. Maturo, A.G.S. Ventre, *Aggregation and Consensus In Multiobjective And Multiperson Decision Making*, Int. J. Unc. Fuzz. Knowl. Based Syst., 17 (2009), 491-499. doi:10.1142/s021848850900611x

[12] F. Maturo, *Dealing with randomness and vagueness in business and management sciences: the fuzzy-probabilistic approach as a tool for the study of statistical relationships between imprecise variables*, Ratio Mathematica, 30 (2016), 45-58. doi:http://dx.doi.org/10.23755/rm.v30i1.8

[13] F. Maturo, Š. Hošková-Mayerov, *Fuzzy Regression Models and Alternative Operations for Economic and Social Sciences*, Studies in Systems, Decision and Control, 2017, 235-247. doi:10.1007/978-3-319-40585-8_21

[14] F. Maturo, V. Ventre, *Consensus in Multiperson Decision Making Using Fuzzy Coalitions*, Studies in Fuzziness and Soft Computing, 2018, 451-464. doi:10.1007/978-3-319-60207-3_26

[15] F. Montagna, M. Fedel, G. Scianna, *Non-standard probability, coherence and conditional probability on many-valued events*, International Journal of Approximate Reasoning, 54 (2013), 573-589. doi:10.1016/j.ijar.2013.02.003

[16] D. Mundici, *Bookmaking over infinite-valued events*, International Journal of Approximate Reasoning, 43 (2006), 223-240. doi:10.1016/j.ijar.2006.04.004

[17] R. Negarestani, *Leper Creativity: Cyclonopedia Symposium*, Ed Keller, 2012.

[18] H.T. Nguyen, V. Kreinovich, A. Di Nola, *Which truth values in fuzzy logics are definable?*, International Journal of Intelligent Systems, 18 (2003), 1057-1064. doi:10.1002/int.10131

[19] H. Reichenbach, *Philosophic Foundations of Quantum Mechanics*, University of California Press, Dover, 1998.

[20] R. Scozzafava, *La probabilità soggettiva e le sue applicazioni*, Masson, Milano, 1993.

[21] R. Scozzafava, *Incertezza e probabilità. Significato,valutazione, applicazioni della probabilità soggettiva*, Zanichelli Editore, Bologna, 2001.

[22] R.R. Yager, *Decision making with fuzzy probability assessments*, IEEE Transactions on Fuzzy Systems, 7 (1999), 462-467. doi: 10.1109/91.784209.

[23] S. Yexin, Y. Di, C. Mianyun, *Decision making with fuzzy probability assessments and fuzzy payoff*, Journal of Systems Engineering and Electronics, 16 (2005), 69-73.

[24] L. Zadeh, *Fuzzy sets*, Inf. Control, 8 (1965), 338-353.

[25] L. Zadeh, *Probability measures of fuzzy events*, J. Math. Anal. Appl., 23 (1968), 421-427.

[26] L. Zadeh, *The concept of a Linguistic Variable and its Application to Approximate Reasoning I and II,* Information Sciences, 8, 199-249 (1975), 301-357.

[27] L. Zadeh, *The concept of a Linguistic Variable and its Applications to Approximate reasoning III*, Information Sciences, 9 (1975), 43-80.

[28] L.A. Zadeh, G.J. Klir, B. Yuan, *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems*, Advances in Fuzzy Systems-Applications and Theory, 1996. doi:10.1142/2895

Accepted: 13.01.2018